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THE ALGEBRA OF LOGIC: WHAT BOOLE REALLY STARTED*

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Abstract. Although the concept of a Boolean algebra has its roots in the algebra of logic, an algebra of logic of the nineteenth century was a scheme for symbolizing logical relationships as algebraic ones in such a way that logical deductions could be accomplished by algebraic manipulations. Boole wrote three works on logic, but it is in the first of these, the 1847 *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Logic*, that one finds the most careful algebraic development. None of the three, however, dealt adequately with existential statements and, in fact, none of Boole's successors dealt adequately with them either, although some of the later versions of the algebra of logic improved substantially on Boole's treatment. Despite these improvements, during the approximately fifty years that constituted the period in which the algebra of logic was the mainstream of mathematical research in logic, logicians never agreed upon a single notation system. Finally, around the turn of the century, the term "algebra of logic" began to be used in the modern sense of Boolean algebra and, because of that, what Boole started in 1847 is now essentially hidden.

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What George Boole really started was not Boolean algebra but the algebra of logic. Although the concept of a Boolean algebra has its roots in the algebra of logic, an algebra of logic of the nineteenth century was not an algebraic structure defined in terms of operations and axioms that the operations satisfy. Rather, an algebra of logic was a scheme for symbolizing logical relationships as algebraic ones in such a way that logical deductions

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could be accomplished by algebraic manipulations. Boole's development of such a scheme was motivated by work in the calculus of operations in early nineteenth century Great Britain. Boole himself discussed Duncan Gregory's work on the calculus of operations in his 1844 paper, "On a General Method in Analysis." In that paper he quotes Gregory describing the fundamental principle of the calculus of operations as follows:

> "There are a number of theorems in ordinary algebra, that, though apparently proved to be true only for symbols representing numbers, admit of a much more extended application. Such theorems depend only on the laws of combination to which the symbols are subject, and are therefore true for all symbols, whatever their nature may be, which are subject to the same laws of combination." [Boole *1844*, 225]

Boole refers again to this principle three years later in the introduction to his first work on logic, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*. He states:

> We might justly assign it as the definitive character of a true Calculus, that it is a method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation. That to the existing forms of Analysis a quantitative interpretation is assigned, is the result of the circumstances by which those forms were determined, and is not to be construed into a universal condition of Analysis. It is upon the foundation of this general principle, that I purpose to establish the Calculus of Logic, and that I claim for it a place among the acknowledged forms of Mathematical Analysis, regardless that in its object and in its instruments it must at present stand alone. [Boole *1847*, 4]

In attempting to establish this calculus of reasoning, Boole defines his symbols so that 1 represents his universe, that is, "every conceivable class of objects whether actually existing or not" [Boole *1847*, 15]; upper case letters, which appear only in the text and not in the formulas, represent generic members of classes; and lower case letters, which he calls elective symbols, operate on classes. He states that "the symbol x operating upon any subject comprehending individuals or classes, shall be supposed to select from that subject all the Xs which it contains.... When no subject is expressed, we shall suppose 1 (the Universe) to be the subject understood" [Boole *1847*, 15]. Except for the universal class, Boole uses no symbol to explicitly denote a class; the symbol x only represents a class as an abbreviation for $x1$, that is to represent all Xs. The juxtaposition xy, read as multiplying x and y, again operates on classes and only represents a class as an abbreviation for $xy1$, that is x operating on $y1$ or, in modern terms, the intersection of $x1$ and $y1$. Interpreting addition (in effect) as disjoint union, Boole shows that his interpretation requires three rules, the first two of which, distribution of multiplication over addition and commutativity of multiplication, are rules that symbols of quantity also satisfy. The third rule, which he calls the index (exponent) law, is $x^n = x$. He states that these rules "are sufficient for the basis of a Calculus" [Boole *1847*, 18]. Although in his two later works on logic (1848 and 1854) it is again these three algebraic rules upon which Boole bases his development, in neither does one find as careful an algebraic development as one does in his first work on logic.

In *The Mathematical Analysis of Logic,* Boole considers the four categorical propositions with which the classical syllogisms are concerned: ALL XS ARE YS, NO XS ARE YS, SOME XS ARE YS, and SOME XS ARE NOT YS. He interprets them as statements involving elements of classes and represents them as equations in his calculus of classes as in the following table:

ALL XS ARE YS ALL YS ARE XS	$x = y$	
ALL XS ARE YS	$x(1-y) = 0$	
NO XS ARE YS	$xy = 0$	
ALL YS ARE XS SOME XS ARE YS	$y = vx$	vx = SOME XS $v(1-x) = 0$
NO YS ARE XS SOME NOT-XS ARE YS	$y = v(1-x)$	$v(1-x)$ = SOME NOT-XS $vx = 0$
SOME XS ARE YS	$v = xy$ OR $vx = vy$ OR $vx(1-y) = 0$	v = SOME XS; OR SOME YS vx = SOME XS; vy = SOME YS $v(1-x) = 0$; $v(1-y) = 0$
SOME XS ARE NOT YS	$v = x(1-y)$ OR $vx = v(1-y)$ OR $vxy = 0$	v = SOME XS OR SOME NOT-YS vx = SOME XS; $v(1-y)$ = SOME NOT-YS $v(1-x) = 0$; $vy = 0$

[Boole *1847*, 25]

While the universal propositions ALL XS ARE YS and NO XS ARE YS are readily represented respectively as $x(1-y) = 0$ and $xy = 0$, the particular propositions SOME XS ARE YS and SOME XS ARE NOT YS require some expansion of the algebraic language. Boole accomplishes this by introducing a separate elective symbol v that roughly represents the operation of selecting all elements, V, of a *nonempty* subset of appropriate terms. For example, when Boole represents SOME XS ARE YS as $v = xy$, he is selecting all elements, V, that are members of the class of XS that are also YS. In addition to listing three interpretations for each particular proposition, Boole introduces auxiliary conditions on the interpretation of v as well as auxiliary equations. He calls the entries of the third column "the conditions of final interpretation... [and notes] ...that the auxiliary equations which are given in this column are not independent: they are implied either in the equations of the second column, or in the condition for the interpretation of v" [Boole *1847*, 24-25]. In effect, when a logical premise is interpreted algebraically according to this scheme, the auxiliary equations or conditions become additional premises that carry existential interpretations. For example, the auxiliary equations for the expression SOME XS ARE NOT YS carry the existential implications that there are Xs and there are NOT-YS. When an algebraic conclusion is to be reinterpreted in logical terms, the auxiliary conditions, and equations, must be satisfied.

Boole algebraically derives syllogisms by eliminating the variable representing the middle term from the two equations representing the premises. For example, the premises ALL YS ARE XS and ALL ZS ARE Ys are translated as $y(1-x) = 0$ and $z(1-y) = 0$. Using the usual elimination procedures for algebraic expressions, he gets $z(1-x) = 0$, which is then reinterpreted as ALL ZS ARE XS. For those syllogisms involving particulars, the manipulations are less straightforward and choosing the algebraic translation of the premises at the outset is a crucial step. For example, in order to derive the conclusion SOME ZS ARE NOT XS from the premises ALL XS ARE YS and SOME ZS ARE NOT YS, Boole translates the first premise as $x(1-y) = 0$ and the second as $vz = v(1-y)$. Using standard algebraic manipulations Boole transforms these equations into $x = xy$ and $vy = v(1-z)$, which give $vx = vx(1-z)$ and, finally, $vxz = 0$. In order to interpret this equation as SOME ZS ARE NOT XS, Boole needs to interpret vz as SOME ZS, which he can do since it "is implied...in the equation $vz = v(1-y)$ considered as representing the proposition SOME ZS ARE NOT YS" [Boole *1847*, 37].

In deriving some syllogisms, Boole uses the symbol v to solve equations representing universal premises for the variable representing the middle term. However, in doing so he introduces existential premises. For example, his derivation of the syllogism Darapti — ALL YS ARE XS, ALL YS ARE ZS, therefore SOME ZS ARE XS — involves solving the equation representing the first premise, $y(1-x) = 0$, to find $y = vx$. This adds to the premises the assertion SOME XS ARE YS and carries the existential premises, there are XS and there are YS. Because the existence of YS is necessary to ensure its validity, Darapti is no longer considered a valid syllogism.

Boole states later in this work that this technique could have been used in all instances of a universal premise in a syllogism and lists general forms for the four categorical propositions employing the symbol v but not listing any auxiliary equation or condition. Although the apparatus of auxiliary equations is somewhat cumbersome, it does provide a consistent algebraic treatment of syllogistic logic. This approach, however, was abandoned by Boole in all of his later work in favor of one that starts with the representation of both universal and particular propositions in terms of v and suppresses explicit reference to auxiliary conditions and equations.

After discussing the syllogism, Boole extends his 1847 system to deal with a calculus of propositions. As before, the lower case letters that appear in the formulas are given an operational interpretation, x representing the operation of selecting those cases for which the proposition X is true and $1-x$ selecting those for which it is false. In this situation, "the hypothetical Universe, 1, shall comprehend all conceivable cases and conjunctions of circumstances" [Boole *1847*, 49]. Although the expressive possibilities of this system are precisely analogous to those of the calculus of classes, Boole considers only statements that are universal in form, X IS TRUE and X IS NOT TRUE, rather than those that are particular in form, X IS SOMETIMES TRUE and X IS SOMETIMES NOT TRUE. He therefore does not need to use the complicated apparatus he created to overcome the difficulties associated with the translation of particulars. On the other hand, he does analyze a series of hypothetical syllogisms having as premises and conclusions hypothetical propositions with conditional forms such as IF X IS TRUE, THEN Y IS TRUE, and disjunctive forms such as EITHER X IS TRUE OR Y IS TRUE. The conditional is translated $x(1-y) = 0$, while the disjunction is translated $x + y - 2xy = 1$ if OR is interpreted exclusively and is translated $x + y - xy = 1$ if OR is interpreted inclusively. In general, the translations and the algebraic derivations are quite straightforward even though this approach does not produce a calculus of propositions that would be considered successful in modern terms.[1] However, as he abandoned his first approach to defining a calculus of classes so, too, did Boole abandon his first approach to defining a calculus of propositions.

Boole ends *The Mathematical Analysis of Logic* by considering elective functions and equations, that is functions and equations involving elective symbols, x, y, v, etc. In solving elective equations he uses techniques of the algebra of quantity, such as Maclaurin's theorem and division, including division by 0. Although we now do not consider division by 0 a valid algebraic process, in the first half of the nineteenth century such division was viewed differently. For example, in an 1831 treatise published by the Society for Diffusion of Useful Knowledge, Augustus De Morgan writes that in treating "fractions of the form $\frac{0}{0}$, etc., I have followed the method adopted by several of the most esteemed continental writers, of referring the explanation to some particular problem" [De Morgan *1831*, vi]. In

[1]A discussion of the problems in Boole's approach appears in [Prior *1949*]. Boole's basic ideas are reworked in [Hailperin *1984*].

particular, he writes the following in connection with what we would now call a word problem:

> Such an equation as $x = \frac{a}{0}$ indicates that the supposition from which x was deduced can never hold good. Nevertheless in the common language of algebra it is said that ... $\frac{a}{0}$ is infinite. This phrase is one which in its literal meaning is an absurdity.... But as the use of the phrase is very general, the only method is to attach a meaning which shall not involve absurdity or confusion of ideas. The phrase used is this: When $c = b$, $\frac{a}{c-b} = \frac{a}{0}$ and is infinitely great. [De Morgan *1831*, 123–124]

For the case $a = 0$, De Morgan states:

> It is now evident that ... any value of x whatever is an answer to the question, ... [thus] when the value of any quantity appears in the form $\frac{0}{0}$ that quantity admits of an infinite number of values, and this indicates that the conditions given to determine that quantity are not sufficient. [De Morgan *1831*, 126–127]

Since the expression $\frac{0}{0}$ can appear in other circumstances, such as $\frac{a^2 - b^2}{a - b}$, De Morgan cautions his reader to establish the reason for its occurrence.

Boole encounters expressions of the form $\frac{0}{0}$ and $\frac{1}{0}$ when, for example, he finds the solution of the equation $\phi(xy) = 0$ to be

$$y = \frac{\phi(10)}{\phi(10) - \phi(11)}x + \frac{\phi(00)}{\phi(00) - \phi(01)}(1 - x). \text{ [Boole } \mathit{1847}\text{, 73]}$$

Boole deals with coefficients that have 0 in the denominator as follows: "the indefinite symbol $\frac{0}{0}$ must be replaced by an arbitrary elective symbol v [while] the term, which is multiplied by a factor $\frac{1}{0}$ (or by any numerical constant except 1) must be separately equated to 0, and will indicate the existence of a subsidiary Proposition" [Boole *1847*, 74]. Although in his subsequent work Boole abandons the use of auxiliary conditions and equations and his initial approach to the calculus of proposition, he does not abandon the techniques that caused the introduction of the symbols $\frac{0}{0}$ and $\frac{1}{0}$.

Boole's second work on logic, "The Calculus of Logic," was published a year after *The Mathematical Analysis of Logic*. Although this paper presents the calculus of classes very differently from the earlier essay, there are some similarities to the later portions of that work. Specifically, he represents all four categorical propositions by equations that involve the symbol v and once again considers the solutions of elective equations. Representing ALL XS ARE YS and NO XS ARE YS as $y = vx$ and $y = v(1 - x)$, he derives the equations $x(1 - y) = 0$ and $xy = 0$, calling them subsidiary relations. Although the vast majority of the paper deals with categorical propositions, Boole does note that he can transfer his results on categorical propositions to hypothetical propositions and deduce "the one from the other by mere analytic process" [Boole *1848*, 197].

It is in this 1848 paper that Boole first makes it clear that it is very important to him that his calculus is based on the single relation of equality. Other relational symbols, or copulæ as they were called, were used by logicians of the time; Boole, however, specifically rejected the suggestion, made to him in 1848, that he use the copula $>$ and thus avoid using the symbol v [Smith *1983*, 32]. A hint of Boole's reasoning appears in his discussion of why he represents NO YS ARE XS in such a way as to make the predicate rather than the subject negative.

> There are but two ways in which the proposition, NO XS ARE YS, can be understood. 1st, In the sense of ALL XS ARE NOT-Y. 2nd, In the sense of IT IS NOT TRUE THAT ANY XS ARE YS, *i.e.* the proposition SOME XS ARE YS is false. The former of these is a single categorical proposition. The latter is *an assertion respecting a proposition*, and its expression belongs to a distinct part of the elective system. It appears to me that it is the latter sense, which is really adopted by those who refer the negative, *not*, to the copula. To refer it to the predicate is not a useless refinement, but a necessary step, in order to make the proposition truly a *relation between classes*. [Boole *1848*, 187n]

The Laws of Thought, which appeared in 1854, again addresses both categorical and hypothetical propositions, this time referring to the categorical propositions, ones that express relations between things, as primary, and to the hypothetical propositions, ones that express relations between propositions, as secondary. In this work, Boole is once again developing a calculus in which to symbolically represent the laws of reasoning; he continues to use only one copula and continues to represent all propositions by equations that involve v. Although the development of the calculus is similar to his 1848 paper, this book is much longer and, in addition to dealing with logic, Boole applies his method of logic to probability.

It is also in *The Laws of Thought* that one first sees a suggestion of the theory of Boolean algebras.

> Instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity *admitting only of the values* 0 *and* 1. Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. [Boole *1854*, 37–38]

The formalistic tone of this paragraph is reinforced by the following later extract from the same work:

> The formal processes of reasoning depend only upon the laws of the symbols, and not upon the nature of their interpretation... *We may in fact lay aside the logical interpretation of the symbols in the given equation; convert them into quantitative symbols, susceptible only of the values* 0 *and* 1; *perform upon them as such all the requisite processes of solution; and finally restore to them their logical interpretation.* [Boole *1854*, 69–70]

Despite these statements, even at this time Boole is not engaged either in constructing a Boolean algebra or in constructing a formal theory of Boolean algebras and applying it to the calculus of sets and propositions.[2] His manipulations continue to involve division, including division by 0, and, in particular, the expression $\frac{0}{0}$ is still very common. Boole's statement regarding $\frac{0}{0}$ makes it clear that his work is not axiomatic in the current sense and, futhermore, it echoes De Morgan's caution that $\frac{0}{0}$ can occur for many different reasons in the algebra of quantity: "Its actual interpretation...as an indefinite class symbol, cannot, I conceive, except upon the ground of analogy, be deduced from its arithmetical properties, but must be established experimentally" [Boole *1854*, 91–92]. As division occurs frequently in the chapter in which Boole reconsiders the rules of syllogism, this treatment is in many ways less satisfactory than the one he had given seven years earlier.

Boole's aim in developing his system in 1847 had been to extend to logic what he believed to be the fundamental principle of algebra, i.e. "that the validity of the processes of analysis does not depend upon the interpretation of the symbols which employed, but solely upon the laws of their combination" [Boole *1847*, 3]. By 1854 he claimed he was

[2] [Hailperin *1981*] considers in modern day terminology what Boole meant by his algebra of 0 and 1.

"exhibit[ing] Logic, in its practical aspect, as a system of processes carried on by the aid of symbols having a definite interpretation, and subject to laws founded upon that interpretation alone. But at the same time they exhibit those laws as identical in form with the laws of the general symbols of algebra, with this single addition, viz., that the symbols of Logic are further subject to a special law...to which the symbols of quantity, as such, are not subject" [Boole *1854*, 6].

The logicians who followed Boole also relied heavily on the interpretations in which they were working. In general their aim was to find better sets of symbols and rules so that the formal manipulations on these symbols were clearer, yet produced significant logical results. Some of them did not use the symbols of the algebra of quantity; some explicitly stated their axioms and some did not. However, even when they did state their axioms, their work, like Boole's, tended to contain manipulations that relied for justification on the intended application to logic rather than on any previously stated axiomatic foundation. Thus in the development and presentation of these algebras of logic, there was an ongoing interaction between the formulation of the calculus and its intended interpretation that is absent in the more modern formalistic view.

In developing a multiplicity of algebras of logic, Boole's successors were strongly motivated by the difficulty of representing particular statements. None of these systems was so successful as to command widespread adoption. In fact, John Venn commenting on the situation of about 1880 wrote:

> Particular propositions, in their common acceptation, are of a somewhat temporary and unscientific character. Science seeks for the universal, and will not be fully satisfied until it has attained it. Indefiniteness indeed in respect of the predicate cannot, or need not, always be avoided; but the indefiniteness of the subject, which is the essential characteristic of the particular proposition, mostly can and should be avoided. For we can very often succeed at last in determining the SOME; so that instead of saying vaguely that SOME *A* IS *B*, we can put it more accurately by stating that THE *A* WHICH IS *C* IS *B*, when of course the proposition becomes universal. Propositions which resist such treatment and remain incurably particular are comparatively rare: *their* hope and aim is to be treated statistically, and so to be admitted into the theory of Probability. [Venn *1881*, 169–170]

Venn's remarks were intended on a purely philosophical level and were not reflected in any special way in his own presentation of the algebra of logic. His views on particulars differ greatly from those of Boole: Venn wanted to give particular statements a universal interpretation, while Boole, in his later works, gave his universal statements an existential import. Moreover, Venn's suggestion that a particular statement should be replaced by a reference to a specific object that has the required property bears some resemblance to the modern constructivist development of mathematics.

Although the algebra of logic was never entirely successful in dealing with particulars, some of the later versions improved substantially on Boole's treatment. One of the more successful systems was developed by Christine Ladd, a student of C. S. Peirce, and was published in 1883. In her paper, "On the Algebra of Logic," Ladd prefaces the description of her system with some analysis of the systems of several of her predecessors. The following comments show a much less rigid view than Boole's of the relationship between the algebra of logic and the algebra of quantity:

> The addition of logic has small connection with the addition of mathematics, and the multiplication has no connection at all with the process whose name it has taken. The object in borrowing the words and the signs is to utilize the familiarity which one has already acquired with processes which obey somewhat similar laws.... The essential processes of symbolic logic are either addition or multiplication (for greater convenience, both are used), and negation. The latter process renders any inverse processes which might correspond to subtraction and division quite unnecessary, and it is only on account of a supposed resemblance between the logical and the mathematical processes that an attempt to introduce them has been made. [Ladd *1883*, 18–19]

Even though Ladd and others were less insistent than Boole on maintaining a strict analogy between the algebra of logic and the algebra of quantity, their analyses blur some crucial distinctions. For example, the logic of sets and propositions are often not distinguished from one another. At a more fundamental level, the algebra of logic never achieved separation between syntax and semantics, that is, between the rules of formation of formulas and their manipulation on the one hand, and the interpretation of the symbols on the other. However, this distinction was hinted at by Augustus De Morgan who, in 1839, describes algebra of quantity as follows:

> Algebra now consists of two parts, the technical, and the logical. Technical algebra is the art of using symbols under regulations which, when this part of the subject is considered independently of the other, are prescribed as the definitions of the symbols. Logical algebra is the science which investigates the method of giving meaning to the primary symbols, and of interpreting all subsequent symbolic results. [De Morgan *1841*, 173]

George Peacock, whom De Morgan credits with first making the distinction between these two parts of algebra, writes of arithmetical and symbolical algebra in the Preface to his

1830 *Treatise on Algebra.* For Peacock, symbols are restricted in arithmetical algebra so that, for example, one does not subtract a larger number from a smaller. In symbolical algebra, the operations coincide with those of arithmetical algebra but the symbols are not restricted. Although it is clear that De Morgan was also thinking of a restriction on symbols in technical algebra, his logical algebra is not merely an unrestricted version of technical algebra. One can see this again when De Morgan comments on his names for the two parts of algebra: he does not like his term logical algebra, but he likes even less Peacock's term symbolical because it "does not distinguish the use of symbols from the explanation of symbols" [De Morgan *1841*,177n]. While De Morgan's description of this difference is not as polished as that which we now make between syntax and semantics, it is a distinction not made by Boole and his followers.

As we mentioned above, Boole placed great emphasis on using no relational symbol other than equality, and specifically rejected the use of a copula for the relation of inequality. On the other hand, some of the systems developed by Boole's successors use such copulæ. Ladd writes about them:

> Algebras of Logic may be divided into two classes, according as they assign the expression of the "quantity" of propositions to the copula or to the subject. Algebras of the latter class have been developed with one copula only,— the sign of equality; for an algebra of the former class two copulas are necessary, — one universal and one particular. [Ladd *1883*, 23–24]

The word "quantity" in the above quotation refers to the distinction between universal and existential statements, that is between statements that involve the concept all and the concept some.

Ladd accompanies her discussion of copulæ with a chart displaying some of the variety of notations for propositional forms used by her predecessors [Ladd *1883*, 24]. Her brief discussion of symbols refers to only five different forms and gives only a small glance at the multiplicity of notations in use in the algebra of logic by 1882. The use of many different notations reflects not only the then current question of whether the quantity of a proposition should be expressed by the subject or the copula, but also reflects a variety of judgments as to which relationships should be regarded as fundamental. Two years before Ladd published her essay, John Venn's book, *Symbolic Logic*, was published and includes twenty-five different symbolic expressions for NO *S* IS *P*, reflecting not merely differing notational preferences but, in many cases, actual conceptual differences [Venn *1881*, 407]. By 1894, when Venn published the second edition of *Symbolic Logic*, he was able to cite eight more, although only four had been introduced after the publication of the first edition [Venn *1894*, 481]. Of the total, more than one third predate Boole's earliest contribution.

These lists appear in the "Historical Notes" section of Venn's books along with detailed descriptions of the symbols.

Boole's system i the first systematic development of an algebra of logic, and Ladd's is one of the later ones. However, the work that is generally considered to have presented the algebra of logic in its most mature form is Ernst Schröder's three volume *Vorlesungen über die Algebra der Logik*, published between 1890 and 1905. Volume one of the *Vorlesungen* develops operations inverse to logical addition and multiplication, and more than one copula is used in order to derive many of Boole's results without recourse to the kind of *ad hoc* manipulation to which Boole frequently resorted. The two later volumes introduce a theory of quantifiers and relations that, like some of Schröder's earlier work, is based on ideas of C. S. Peirce.

Although the work of both Peirce and Schröder has some elements in common with Gottlob Frege's *Begriffsschrift*, which appeared in 1879, their work remained within the Boolean tradition of devising essentially algebraic symbolism for logical relationships. In fact, in 1880 Schröder wrote a lengthy review of the *Begriffsschrift* in which he criticizes Frege for ignoring the Boolean tradition and attempts to reformulate Frege's ideas within that algebraic tradition.

Both Peirce and Schröder, however, had a more sophisticated view than Boole of what axioms are, and both recognized that the algebra of logic is not really an axiomatic system. In a paper published in 1870, Peirce explains his views on the nature of the algebra of logic as follows:

> If the question is asked, What are the axiomatic principles of this branch of logic, not deducible from others? I reply that whatever rank is assigned to the laws of contradiction and excluded middle belongs equally to the interpretations of [various equations]... But these axioms are mere substitutes for definitions of the universal logical relations, and so far as these can be defined, all axioms may be dispensed with. The fundamental principles of formal logic are not properly axioms, but definitions and divisions; and the only *facts* which it contains relate to the identity of the conceptions resulting from those processes with certain familiar ones. [Peirce *1870*, 378]

Essentially the same view is expressed by Schröder in 1877 in his first work in logic:

> Sämtliche Theoreme unserer Disciplin sind *intuitiv*; sie erscheinen, sobald sie zum Bewusstsein gebracht werden, als unmittelbar einleuchtend, und deshalb könnten auch die als Axiome hier angeführten Behauptungen mit einer gewissen Berechtigung als Folgerungen hingestellt werden, welche durch die Definitionen unmittelbar mit gegeben seien. [All the theorems of our discipline are *intuitive*; as soon as

they are noted they appear immediately obvious, and therefore the statements cited here as axioms could, with a certain justification, also be presented as conclusions based directly on the definitions.] [Schröder *1877*, 4]

Although the symbolical representation of logic has a long history, the period in which the algebra of logic constituted the mainstream of mathematical research in logic lasted only from mid-nineteenth century, that is from the time of Boole's work, until the turn of the century. During these approximately fifty years, those who were working in the algebra of logic concerned themselves with the solution of a number of logical problems, some of which, such as the problem of eliminating a variable from an algebraic expression or set of expressions, had analogues in the algebra of quantity. Although the solution of such problems was important to those who worked in the algebra of logic, the fact that the subject never achieved notational stability has greatly influenced how we view it.

Although Schröder's algebraic notation remained in use during the early twentieth century, the work of the logicians who used it was not directed to the goal, characteristic of the algebra of logic, of interpreting logical relationships in terms of algebraic ones. Nonetheless, in 1915 Leopold Löwenheim used results and techniques of the Peirce-Schröder logic of relatives to prove the model-theoretic result that bears his name.

Around the turn of the century, Alfred North Whitehead (1898) and E. V. Huntington (1904) used the expression "algebra of logic" to denote the formal calculus that can be abstracted from the propositional calculus and naive set theory and that forms the basis of the theory of Boolean algebras. The term "Boolean algebra" was not used in this sense until H.M. Sheffer coined the term in his 1913 paper, "A Set of Five Independent Postulates for Boolean Algebras, with Application to Logical Constants." Although the term had been used by C. S. Peirce around 1880, Peirce used the term to refer to techniques of symbol manipulation, not to algebraic structures. In fact, Peirce used the terms "Boolian [sic] algebra" and "Boolian calculus" interchangeably in the same way that Boole used the terms "algebra of logic" and "calculus of logic" interchangeably. In 1933, Huntington adopted the more modern terminology, Boolean algebra, even though the title of his paper, "New Sets of Postulates for the Algebra of Logic," still contains the older terminology. This changing meaning and interchangeable use of the two expressions, "algebra of logic" and "Boolean algebra," has tended to lead to an over-simplified historical picture of the nineteenth century algebra of logic, which was both less abstract and more ambitious than the theory of Boolean algebras.

REFERENCES

BOOLE, George. 1844. *On a general method in analysis*, Philosophical Transactions of the Royal Society of London 134, 225-282.

—. 1847. *The mathematical analysis of logic*. Reprint: New York, Barnes & Noble, 1948. Reprinted with Later Notes in [Boole *1952*, 1, 45–124].

—. 1848. *The calculus of logic*, Cambridge and Dublin Mathematical Journal 3, 183–198. Reprinted in [Boole *1952*, 1, 125–140].

—. 1854. *An investigation of the laws of thought*. Reprint with corrections: New York, Dover Publications, 1958. Reprinted as [Boole *1952*, volume 2].

—. 1952. *Collected logical works*, 2 vols. La Salle, Ill., Open Court. (Volume 1 includes Boole *1847* and *1848*; Volume 2 is Boole *1854*.)

DE MORGAN, Augustus. 1831. *On the study and difficulties of mathematics*. 4th reprint ed.: La Salle, Ill., Open Court,1943.

—. 1841. *On the foundation of algebra*, Transactions of the Cambridge Philosophical Society 7, 173–187. (Read 9 Dec. 1839.)

FREGE, Gottlob. 1972. *Conceptual notation and related articles* (edited and translated by Terrell Ward Bynum), Oxford, Clarendon Press.

HAILPERIN, Theodore. 1981. *Boole's algebra isn't Boolean algebra*, Mathematics Magazine 54, 172–184.

—. 1984. *Boole's abandoned propositional logic*, History and Philosophy of Logic 5, 39–48.

HUNTINGTON, Edward V. 1904. *Sets of independent postulates for the algebra of logic*, Transactions of the AMS 5, 288–309.

— . 1933. *New sets of postulates for the algebra of logic, with special reference to Whitehead and Russell's "Principia mathematica,"* Transactions of the AMS 35, 274–304. *Corrections, ibid*, 557–558, 971.

LADD, Christine. 1883. *On the algebra of logic*, in [Peirce *1883*, 17–71].

LÖWENHEIM, Leopold. 1915. *Über Möglichkeiten im Relativkalkül*, Mathematische Annalen 76, 447–470. English translation in [van Heijenoort *1967*, 232–251].

PEACOCK, George. 1830. *A treatise on algebra*, Cambridge, J. & J.J. Deighton.

PEIRCE, C. S. 1870. *Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic*, Memoirs of the American Academy of Arts and Sciences (n.s.) 9, 317–378. Reprinted in [Peirce *1931-1935*, 3, 27-98] and [*1982–*, 2, 359–429].

—. ca. 1880. *A Boolian algebra with one constant*, in [Peirce *1931-1935*, 4, 13–18] and [*1982-*, 4, 218–221].

—. 1883. (editor), *Studies in logic, by members of the Johns Hopkins University*, Boston, Little, Brown, and Co. Reprint: Amsterdam and Philadelphia, John Benjamins Publishing Co., 1983.

—.1931-1935. *Collected papers of Charles Sanders Peirce* (Charles Hartshorne and Paul Weiss, editors), 6 vols., Cambridge, Mass., Harvard University Press. Reprint: Cambridge, Mass., Belknap Press of Harvard University Press, 1960.

—. 1982–. *Writings of Charles S. Peirce: a chronological edition* (Max H. Fisch, *et. al.* editors), Bloomington, Indiana University Press, 4 vols. to date.

PRIOR, Arthur N. 1949. *Categoricals and hypotheticals in George Boole and his successors*, Australasian Journal of Philosophy 27, 171–196.

SCHRÖDER, Ernst. 1877. *Der Operationskreis des Logikkalkuls*, Leipzig, B. G. Teubner. Reprint: Stuttgart, B. G. Teubner, 1966.

—. 1880. *Review of Frege's "Begriffsschrift,"* Zeitschrift für Mathematik und Physik 25, 81–94. English translation in [Frege *1972*, 218–232.]

—. 1890-1905. *Vorlesungen über die Algebra der Logik*, Leipzig, Eugen Müller. Reprint: Bronx, N.Y., Chelsea Publishing Co., 1966, 3 vols.

SHEFFER, Henry Maurice. 1913. *A set of five independent postulates for Boolean algebras, with application to logical constants*, Transactions of the AMS 14, 481–488.

SMITH, G. C. 1983. *Boole's annotations on "The Mathematical Analysis of Logic,"* History and Philosophy of Logic 4, 27-39.

VAN HEIJENOORT, Jean. 1967. (editor), *From Frege to Gödel: a source book in mathematical logic, 1879 – 1931*. Cambridge, Mass., Harvard University Press.

VENN, John. 1881. *Symbolic logic*, London, Macmillan and Co.

—. 1894. *Symbolic logic*, 2d ed., London, Macmillan and Co. Reprint: Bronx, N.Y., Chelsea Publishing Co., 1971.

WHITEHEAD, Alfred North. 1898. *A treatise on universal algebra with applications*, Cambridge, Cambridge University Press. Reprint: New York, Hafner Publishing Co., 1960.