ℵ Modern Logic ω

ON THE FOOTHILLS

Review of Stuart Hollingdale, *Makers of Mathematics* (Penguin, 1989), John Stillwell, *Mathematics and Its History* (Springer, 1989), and William Dunham, *Journey Through Genius: The Great Theorems of Mathematics* (Wiley, 1990)

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The author of a popular book on the history of mathematics has to balance conflicting obligations. In the first place, the book is supposed to be popular history. What characterizes the writings of Antonia Fraser or Barbara Tuchman is a strength of narrative flow that carries the reader along in a way that more scholarly history is unlikely to. Partly as a result the popular historian presents an easy target to the specialist ready to point out the need for precisely those qualifications and restrictions that would kill the interest of a reader.

On the other hand, popular history of mathematics usually involves a popularization of the mathematics as well as the history, and there the author has an even harder task. Some readers ('broadly-educated and lively-minded', as Stuart Hollingdale calls them) are willing to admit to an interest in history, even if they regret the way in which the subject was taught in their schooldays. Fewer still are those who are willing to risk reopening their mathematical wounds. The audience is out there, as witnessed by the success of Hofstadter's *Gödel, Escher, Bach* [*1979*] and Penrose's *The Emperor's New Mind* [*1989*], but it takes an extraordinary book to find them. In addition, the preparation appropriate for doing mathematics and expounding it does not bring historical sensitivity in its wake. It is hard to tell whether 'mathematician' or 'non-mathematician' is a more pejorative term when applied by historians of mathematics. The non-mathematician is accused of mathematical incompetence, of misreading the text because of an inability to see the deeper mathematical ideas underlying it. The mathematician is accused of taking ideas out of historical context, of anachronism on a large scale in reading works from other centuries, and frequently of ignorance of factors outside of mathematics.

Despite the tempting nature of the target being critical, popular histories of mathematics continue to appear. Recent examples include Stuart Hollingdale, *Makers of Mathematics*; John Stillwell, *Mathematics and Its History*; and William Dunham, *Journey Through Genius: The Great Theorems of Mathematics*. This is not intended to be a full-scale review of any of these three, but only an examination of what they have to say about the history of mathematical logic with reference to the history of mathematics in general. The two questions to be addressed are how well each lives up to the standards of the genre and what the value of the genre is when executed to perfection.

All three volumes talk about mathematical logic, primarily in association with foundational questions rather than processes of mathematical reasoning. The latter requires close reading of the primary sources in order to figure out how the argument is being conducted and not just where it is heading. It is easiest to look at logic historically when it is addressed explicitly in the text. In Greek times the presentation of mathematics was done by Euclid, the discussion of methods by Aristotle. Even Archimedes' *The Method*, although from the pen of a mathematician, does not examine styles of reasoning at length.

Despite some reference to logical works of earlier periods, all three books take up mathematical logic explicitly only when they come to the 1870's and 1880's. Stillwell argues that it is legitimate to take the ideas of set, logic, and computation as typical for the mathematics of the last century. On the one hand, he claims, those topics are more accessible than any other modern topics of importance. On the other hand, they throw light on the question of what mathematics is. As the volumes present themselves, it is legitimate to ask how far their discussions of those ideas live up to Stillwell's claims for them.

Hollingdale avows that his work is in no sense a balanced history of mathematics, but an informal, personal, idiosyncratic attempt to recount 'the main features of a long story through the lives and achievements of some of its great men' (p. xi). His intention, as he states in the introduction, is to counter the common view of the mathematician as 'desiccated' with pictures of the eventful lives and colourful personalities of those who have played major roles on the mathematical stage (p. xii). If this sounds reminiscent of Eric Temple Bell's *Men of Mathematics* [*1937*], there is less ground for surprise in how Hollingdale carries out his task. His bibliography includes few primary or even secondary sources, as he tends to follow tertiary sources like Morris Kline's *Mathematical Thought from Ancient to Modern Times* [*1972*], which he calls a 'more specialist or advanced' book. Although Kline does deal with some sophisticated mathematics, it is misleading not to find it in the general section. Bell's book also makes an appearance, as do Carl Boyer's [*1968*] and Dirk Struik's [*1967*] histories.

The chapters before the final one on Einstein are entitled 'Hamilton and Boole' and 'Dedekind and Cantor'. Each proceeds by posing a mathematical question, giving a biographical sketch, describing how the subject of the sketch contributed to solving the problem, and so on. Both Hamilton and Boole have been the subject of full-length biographies in recent years ([Hankins *1980*], [MacHale *1985*]) and there is no trace of them in Hollingdale's bibliography. From the discussion of quaternions and Boolean algebras, Hollingdale turns to the work of De Morgan and of the Peirces on algebra, even though they do not receive biographical attention; The flavour of the biographical treatment can be gauged from the line about Boole: 'Boole died in 1864 in his fiftieth year, honoured and famous at last' (p.344). Bell was willing to report unsubstantiated gossip, even if it was to the discredit of the subject, if it was lively enough. Hollingdale is inclined to be too respectful even to cite the misdeeds of his subjects.

In the chapter that follows, Weierstrass is given credit for finally 'banishing' infinitesimals from mathematics (p. 354), and there is no mention of nonstandard analysis in the book. The biographical treatment of Cantor makes do without reference to Joseph Dauben's [*1979*] biography and, once again, reads something like a pale image of Bell. The most outstanding technical error of the chapter is the definition of $\aleph_1$ as the cardinality of the power set of $\aleph_0$. As a specimen of the confusion to which this gives rise, the reader is confronted with the following: 'Cantor also proved that $\aleph_1$ is exactly equal to $\mathfrak{c}$, the transfinite number of the real numbers' (p. 363). Even though his accounts of Cantor's proofs of the denumerability of the rationals and of the algebraic numbers and of the uncountability of the reals are easy to follow, his confusion of the aleph and beth hierarchies will confuse the reader who turns to other sources.

After talking about Cantor, Hollingdale turns to the work of Gödel, following on a remarkably brief summary of the view of 'Logicists, Intuitionists, and Formalists'. Listing the leading proponents of each school and stating their 'philosophies' takes up two paragraphs, the sort of summary gratefully welcomed by students preparing for a multiple-choice examination but not likely to be helpful to a reader wondering what brought these differences into the open. Both here and in talking about Cantor Poincaré's name receives the prefix 'great' without any mention of Poincaré's work.

In the last paragraph on logic (p.366), Hollingdale announces that Gödel's work dealt a 'devastating blow' to the "optimists" among the 'contending parties'. Since the three points of view were presented as opinions rather than as programmes, it is hard to tell what 'optimism' with regard to mathematical progress involved. Hollingdale ends, 'It was all very disturbing, especially to the Formalists. However, after the dust had settled most mathematicians took a deep breath and continued to pursue their specialized researches as before.' The picture this gives of the extent to which mathematicians before Gödel took up one of the philosophical positions sketched and then stood in trepidation after the publication of Gödel's work misrepresents history rather than just simplifying it.

Stillwell writes his history in the interests of the student who would like to see mathematics woven together rather than presented as separate strands. His bibliography includes primary sources as well as the relevant secondary sources and (a healthy sign) does not include Bell's *Men of Mathematics*. As a general indication of his seriousness on historical issues, he uses Tony Rothman's [*1982*] article on Galois to help cut through the tissue of legend that surrounds the French mathematician. In fact, he even goes to the length of trying to explain the attractions of a conspiracy theory for mathematical historians more interested in canonization than in the historical figure. 'The police agent theory seems rather to reflect twentieth-century bafflement over dueling, something which we no longer understand or sympathize with (though we still applaud successful duelists, such as Bolyai and Weierstrass)' (p. 290). Stillwell's virtues as expositor and stylist come out effectively in such passages.

He also recognizes his use of anachronism in notation and interpretation and seeks to justify this practice on the ground that mathematical ideas arise before there is notation to express them clearly (p. viii). If this is so, he argues, then the historian, 'who is presumably trying to be both clear and explicit', must overstep the language and notation contemporary with the work described. There, of course, speaks the mathematician (Stillwell has written on combinatorial group theory), for whom there is no limit to the degree of clarity to be achieved. The historian, by contrast, is frequently forced to make a declaration of ignorance, even at the risk of disappointing the reader. Since Stillwell's volume is in the series 'Undergraduate Texts in Mathematics' rather than 'Undergraduate Texts in History', his allegiance is explicit. Where there are differences between the mathematician and the historian (for example, Weil [*1973*] and Mahoney [*1973*] on Fermat), Stillwell sides with the mathematician.

Each of Stillwell's chapters ends with biographical notes, with a mild amount of documentation. He observes that in addition to other sources the *Dictionary of Scientific Biography* (DSB) was generally called into play. As is characteristic of any reference work, the articles in the DSB are of uneven quality, but the best of the articles are remarkable, brief distillations of biography and scientific accomplishment. In some cases, Stillwell could easily have sent the reader to the relevant DSB article without being limited to the space in his book.

Stillwell addresses the question of infinity in Greek mathematics, but is concerned more with how it was used than with philosophical discussions of its status. In particular, he addresses the question of rigour as tending to preserve mathematics rather than to help it forward. He points with enthusiasm to Archimedes' *The Method* as probably the first place 'to explain that there is a difference between the way theorems are discovered and the way they are proved' (p. 39). This is, no doubt, a salutary lesson for the student of mathematics, but it is not clear how necessary it was at the time Archimedes wrote.

Stillwell also praises Eudoxus' theory of proportions with zest, but observes that 'it delayed the development of a theory of real numbers for 2000 years' (p. 39). This claim is unfair to Eudoxus, whose theory of proportion was lost for many of the centuries of those two millennia. The tangled complications of the mediaeval theory of ratios arose from not having the Eudoxan theory and making do with the discussion in Book VII of Euclid instead. It was not that subsequent commentators defended the Eudoxan view with rigidity, it was that they did not have the Eudoxan definition to defend until the Renaissance.

The notion of the infinite also arises in Stillwell's discussion of infinite series back to Greek times. He states, 'There is no question that Zeno's paradox of the dichotomy..., for example, concerns the decomposition of the number 1 into the infinite series $1/2 + 1/2^2 + 1/2^3 + 1/2^4 + \ldots$ and that Archimedes found the are aof the parabolic segment ... essentially by summing the infinite series $1 + 1/4 + 1/4^2 + 1/4^3 + \ldots = 4/3$' (p.118). The use of the words 'concerns' and 'essentially' indicates the extent to which some reading of modern attitudes back into the texts has taken place. It is not surprising that the work of Swineshead and Oresme in the fourteenth century is also appraised favourably and anachronistically.

When Stillwell turns to Cantor's work, however, he is careful to place it in the historical context of Fourier series. Thanks to the discussion of limit points, Stillwell can make Cantor's concern for infinite totalities arise out of mathematical issues. This leaves Cantor's theological attitudes out of account, but that may have been the result of not having a biographical sketch of Cantor.

Stillwell is also careful to distinguish questions about measure from those about sets in looking at work subsequent to Cantor's. He indicates how axioms about 'measurable' cardinals arose out of questions about measure and paradoxes resulting from the axiom of choice. (His failure to include Moore's book on Zermelo [*1982*] is one of the few conspicuous omissions from the bibliography.) It is not surprising that his historical antennae are most sensitive for the most recent material.

After talking about sets, Stillwell turns to computability. He gives Turing and Post independent credit for arriving at the notion of a Turing machine and suggests why the diagonal argument was not immediately applied to demonstrate the existence of noncomputable functions. It was not clear in the period between Cantor and Turing that the notion of 'computable' was a mathematical one, he argues, but once computability could be expressed mathematically the availability of appropriate arguments in the literature made simultaneous discovery less unlikely (p. 321). His observations about computability are put in an algebraic context by quoting Higman's result that a finitely generated group has a computable set of relations if and only if it is a subgroup of a finitely generated group with a finite set of relations.

When Stillwell comes to Gödel's theorem itself, his analysis of both first and second incompleteness theorems is cogent. From the historical point of view, however, what sticks out most is the statement, 'Gödel's theorem created a sensation when it first appeared' (p. 324). Stillwell argues for this claim by examining the mathematical consequences of the theorem without noticing that the claim itself is a historical one. As John Dawson's [*1991*] article on the reception of the incompleteness theorems shows, a sensation is precisely what the theorems did not create.

Stillwell's biographical sketch of Gödel is heavily dependent on Kreisel's obituary [*1980*], but he does note that there is much to be hoped for in the publication of Gödel's manuscripts. Part of the attraction for Stillwell in Gödel's work is the consequence drawn by Post that 'these developments will result in a reversal of the entire axiomatic trend of the late nineteenth, and early twentieth centuries, with a return to meaning and truth' (p. 328). Stillwell feels that the official view within mathematics is that the subject consists in the formal deduction of theorems from fixed axioms. He does not indicate whether he finds much support for Post's view in the work since Gödel. It is possible to argue that work in automated deduction systems is more likely to do something to justify Post's argument.

Dunham's purpose is to attract students to mathematics by pointing to particularly choice examples of the mathematician's art. It is like a collection of ideas from chess, illustrated by especially brilliant combinations. The task he set himself was to pick the most attractive arguments for important results 'rather than merely clever little tricks or puzzles' (p. vii). By limiting himself to twelve 'great theorems' he avoids too much debate over exactly what constitutes importance.

Dunham explicitly rejects certain kinds of criticism of his book by saying, 'This book, after all, is meant for the popular, not the scientific, press' (p. viii). As mentioned above, this projected audience puts a strain on both the mathematics and the history being presented. Dunham is ready to admit that he has to make sacrifices with regard to the latter and argues that the modifications are no more serious than performing Mozart on modern instruments. Since he is writing as a mathematician, however, he feels that he is leaving the mathematics intact, as though with Mozart it is the score that counts.

Dunham's chapters involve the setting of the historical stage, the presentation of the 'Great Theorem', and following subsequent developments in an epilogue. His bibliography includes both Bell and Dauben [*1979*], as though trying to steer a middle course. He refers to Morris Kline as a 'mathematics historian', while Bell is 'a popular writer on the history of mathematics' (pp. 247, 266). Since one of Dunham's avowed aims is to present mathematicians as 'flesh-and-blood human beings', the attractions of Bell's style of narration were hard to resist.

One of the entertaining features of Dunham's exposition is his willingness to bring in comparisons with art. He compares the revolution initiated by Cantor's [*1874*] paper showing the reals to be nondenumerable with the impressionism of Monet as a revolution following on Delacroix and Ingres. Dunham admits that the article by Cantor would not have had as dramatic an impact on the viewer as Monet. Perhaps it would have been fairer to try to identify the stylistic predecessors against whom Cantor was revolting (p. 257).

Dunham does not describe the origins of Cantor's work in the same detail as Stillwell did. It is perhaps confusing that he quotes the title of Cantor's 1874 paper having to do with 'the collection of all real algebraic numbers', indicates that it is the locus of the proof of nondenumerability of the reals, and then mentions elsewhere that Cantor shows that the algebraic numbers were denumerable. Dunham's eagerness to get to the great theorem does not allow him time to look at the structure of the paper.

Dunham (who devotes two chapters to Cantor, one to the diagonal argument applied to real numbers and the other to the argument applied to sets) hails Cantor as a revolutionary and compares him to Van Gogh. The use of Dauben enables Dunham to look at Cantor's background and personality but somehow his vision of Cantor as hero does not square with the more detailed biographical picture he presents. That, perhaps, is ultimately the weakness of a 'Great Theorems' approach to history of mathematics; even when it is accompanied by accurate biographical sketches of the creators of mathematics, the practitioners become intellectual heroes. The biographer succumbs to hero-worship.

In their presentations of modern logic, all three authors compare the discovery of the independence of the continuum hypothesis to the discovery of non-Euclidean geometries. However, the point they are making is not historical, as none of them examine the ways in which the twentieth-century result was received differed from that of the nineteenth century. Stillwell is the only one to ask whether 'the notion of "set" is open to different natural interpretations, like the notion of "straight line"...' (p. 316). The question at least seeks to obtain something from the comparison drawn with geometry.

In general, popular history of mathematics designed for mathematicians does not seek to recreate exactly what earlier generations did. The best that the popularizer can do is to take current scholarship and try to be faithful to the main currents detectable in the work of mathematical predecessors. They are bound to make history of mathematics look less difficult than it is, since their work is usually bound up with making mathematics itself more attractive. It is, however, the popular history of mathematics that will determine the general image among mathematicians of the character of their predecessors. The improvement in books like that by Stillwell is possible because of the detailed scholarship in history of mathematics over the last fifty years. Even with that scholarly foundation, however, the popularizer has no easy task and any measure of success deserves recognition.

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