Mary Tiles, *The Philosophy of Set Theory: A Historical Introduction to Cantor's Paradise*. viv + 239pp., figures, bibliography, index. New York: Basil Blackwell, 1989. $49.95.

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This is a greatly expanded version of my review appearing in *Isis* ([Anellis *1991a*]).

Eric Temple Bell [*1986*, p. 556] reports on a quotation from Russell, dated 1901, which he found in R.E. Moritz's (*1914*) book *Memorabilia Mathematica* (and not identified any further), according to which Russell said that:

> Zeno was concerned with three problems. ...These are the problem of the infinitesimal, the infinite, and continuity. ...From his day to our own, the finest intellects of each generation in turn attacked these problems, but achieved, broadly speaking, nothing. ...Weierstrass, Dedekind, and Cantor... have completely solved them. Their solutions...are so clear as to leave no longer the slightest doubt of difficulty. This achievement is probably the greatest of which the age can

> boast. ...The problem of the infinitesimal was solved by Weierstrass, the solution of the other two was begun by Dedekind and definitely accomplished by Cantor.

The quotation – inaccurately presented – is actually from Russell's popular article "Recent Work on the Principles of Mathematics," which appeared in the magazine *International Monthly* [*1901*] and was reprinted as "Mathematics and the Metaphysicians" in Russell's book *Mysticism and Logic*.

To determine whether Cantor created or discovered infinity in developing set theory, Tiles investigates the history of the role of infinity in mathematics and logic. She begins with Zeno's paradoxes about the impossibility of motion due to an infinite regress and Aristotle's criticisms of Zeno, and then considers not only Cantor's work, but also the mathematics that fostered and clarified the concepts leading to the creation of Cantor's theory. The book, a mixture of history and philosophy, with the philosophical aspect preponderant, is more concerned to argue against finitism and show that Cantor discovered infinity than to understand the historical development of set theory or the mathematical issues involved. The entire first chapter is devoted to a characterization of finitism.

The logical analysis of Zeno's paradoxes introduced to Greek and medieval science and philosophy the concepts of finitude and infinity and a distinction between the actual and the potential infinite. Aristotelian logic, treating terms as classes, provided definitions of *class* which were ambiguous, permitting classes to be taken either as finite collections of entities sharing some property, or as names whose references can be potentially infinite. Calculus, developed in the seventeenth and eighteenth centuries to study continuous motion as presented by early modern physics, reintroduced the infinite. Georg Cantor (1845–1918), motivated by his study of infinite series, developed set theory to provide a foundation for analysis in which transfinite numbers are the mathematical objects that metaphysically ground infinite series, and defined the real numbers. Contemporary axiomatic set theories are treated as elaborations of Cantor's theory.

After Tiles discusses the philosophical differences between actual and potential infinities and between the two definitions of a class, she shows how these ambiguities led to difficulties for Cantor's theory, and how his theory compounded those difficulties by extending classes to the actual infinite (Cantor's "transfinite"), including the real numbers. The results for Cantorian set theory were the paradoxes of the infinite, in which infinite sets have as many elements as one of their proper subsets. Tiles is principally concerned with paradoxes of the infinite, for example that there are as many real numbers between zero and one as there are in the entire set of reals. But the set-theoretic paradoxes are a problem only when attempting to understand the arithmetic properties of transfinite numbers from the viewpoint of a finitism which rejects not only the existence of infinite numbers, but which assimilates both the Euclidean view that the whole is greater than the part and the atomistic view that space is composed of discrete, indivisible, points and that numbers are discrete and indivisible. (The difficulty with an atomistic and finitist analysis of continuous motion and of the number continuum is seen in Bertrand Russell's earliest attempts to understand real and infinitesimal analysis and Cantorian set theory, exhibited in [Russell *1990*] and discussed in [Anellis, *1984*; *1986*; *1987*].)

A fundamental weakness in Tiles's account is that she often sacrifices historical and technical accuracy to make a philosophical point. Thus, she states (p. 62) that a full characterization of infinite sets is "blocked" by results in which an infinite set has as many elements as one of its proper subsets. It is not said whether this "blockage" is historical or technical, leaving unmentioned until p. 157 that the difficulties with Cantor's theory, and in particular the paradoxes of the infinite that arose from Cantor's naive theory, led precisely to the development of various new formal set theories which permitted study of the properties of such sets, and (pp. 97, 120-121) that the paradoxes of the infinite result from the intuitive notion that a set cannot be larger than any of its proper subsets. The "blockage" of which Tiles speaks applies to Cantorian set theory, but not, for example, to **ZF**. The problem is that Tiles examines difficulties encountered in Cantor's set theory even before giving an account of Cantor's work, and without taking into consideration the historical context. It is therefore not always immediately clear from reading her account why or how the paradoxes of the infinite had arisen, or that axiomatic set theories were developed to clarify and restrict the intuitive notion of set that created the difficulties encountered by Cantor's naive set theory. It would have been appropriate, for example, for Tiles to mention specifically the Russell paradox on p. 151, where she speaks of the correspondence between Frege's universe of objects and the "logical category" 'object' which is the extension of the concept '$x = x$', and where she noted that one of the meanings of this is "that the universe itself is an object and is thus a member of itself."

Historical insensitivity, disregard for technical intricacies, and carelessness, lead to other errors, e.g. (on p. 34), of incorrectly claiming that "Euler, and later Venn, introduced diagrams to illustrate the[se] relationships" among classes and for testing the validity of syllogisms. In fact, Euler merely simplified and popularized Leibniz's diagrams and we can even find such logical diagrams as far back as Lully's *Ars Magna*.

Serious technical errors occur because Tiles does not distinguish between the most important standard axiomatization of set theory from related, but different, axiomatizations (e.g. she does not appear to notice that **ZFC** is an *extension* of **ZF** formed by adjoining the Axiom of Choice to **ZF** or that Gödel-Bernays theory is a conservative extension of **ZF**, and in any case does not explain these differences). The failure to keep clear the distinction between a theory and an extension of that theory leads her to forget that theorems in one theory are not always theorems in another. She treats axiom schemata, rules for formulating infinitely many axioms, as axioms, and misstates or omits crucial conditions in presenting specific set-theoretic results or their proofs. Her most egregious error consists of misquoting statements. She states, e.g., on p. 193 that "Drake (1974, p. 66) expresses the feeling that GCH is just too simple to be right," when in fact he actually reports that *some mathematicians* feel that GCH is too simple to be right as an answer to the question of whether the cumulative type structure is real, after which he gives examples to show why some mathematicians might harbor this feeling.

An indication of lack of care on Tiles's part is that she consistently misspells (as "van Heijenhoort", on each occurrence – on pp. 225, 229, 231, and 232) the name of Jean van Heijenoort, one of the leading historians of logic of our time.

An example of a misleading statement born of over-generalization occurs on p. 33, where it is said that:

It was as a result of the challenge to provide a coherent,

> paradox-free account of numbers along Cantorian lines, based on classes, that the traditional Aristotelian logic came to be superceded [sic] by that of Frege (in the forms popularized by Russell, Hilbert, and others).

At least one possible reading of this passage suggests that the logic of Frege (in the forms presented by Russell, Hilbert, et. al.) would not have been developed except for the presence of the set-theoretic paradoxes in Cantorian theory. But we know, of course, that Frege, for example, carried out much his work before any of the set-theoretic paradoxes were detected in Cantor's theory. Indeed, although in *Die Grundlagen der Arithmetik*, Frege [*1884*, 97-99] expresses some problems which he sees with Cantor's treatment of the definition of successor and with his principles for generating ordinals in *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* [Cantor *1883*], he generally expresses his approval of Cantor's treatment of the infinite and in any case, does not detect any paradoxes. Frege reiterated his view eight years later, in his review of part 1 of Cantor's "Beiträge...", saying there [Frege *1892*, 270] that Cantor's definitions lead to difficulties – without being very specific about these are – which his own work in the *Grundlagen der Arithmetik* avoided, but still failing to detect any paradoxes in Cantorian set theory. Indeed, the standard history says that the set-theoretic paradoxes were not detected until 1896-1899. In their revised history of the discovery of the paradoxes, [Moore and Garciadiego *1981*] argued that the Burali-Forti paradox, commonly dated from 1896 and the Cantor paradox, usually dated to 1897 or 1899, were not recognized at the time by either of their purported discoverers; that the paradoxes took "recognizable form" only in 1903 in Russell's *Principles of Mathematics*. And as everyone knows, if Frege's goal was to create a theory of number free of paradoxes and based on Cantorian set theory, then the attempt was a failure, as Russell pointed out in his famous letter to Frege of 16 June 1902. (On the confused accounts of the history of the discovery of the Russell paradox, see [Anellis *1991*], especially pp. 34-35). Moreover, the difficulties which Frege attributed to Cantor's definitions were seen to be rooted precisely in the "account of numbers along Cantorian lines, based on classes" as opposed to Frege's own course-of-values semantic for his function-theoretic syntax; this course-of-values semantics is clearly extensional, but hardly set-theoretic, "along Cantorian lines, based on classes." Thus, Tiles's over-generalization on p. 33 leads to the false assertion on p. 139 that "Frege accepted Cantor's basic idea that the concept of number should be elucidated by reference to sets." (Indeed, van Heijenoort went so far as to suggest that there exist in Frege's *Universum* only two objects, *The True* and *The False*.)

Another instance of a misleading over-generalization occurs on p. 139, where Tiles makes the inaccurate claim that it was through the achievements of Frege that logic "became capable of handling relations and functions," thereby (apparently) suggesting that algebraic logic, as developed from Boole to Schröder, was "incapable" of "handling" relations. It is also asserted (p. 140) that Frege introduced quantifier notation into logic. Strictly speaking, this is false, since Charles S. Peirce began developing a quantification theory as early as 1867. Admittedly, the attempt of [*1867*], in the paper "On an Improvement in Boole's Calculus of Logic," was unsuccessful, and it was only in [*1885*], in the article "On the Algebra of Logic: A Contribution to the Philosophy of Notation" that Peirce was successful in his efforts to introduce quantifiers into Boole's logic. It would be much more accurate to say that, whereas Frege was clearly the first to *successfully* introduce quantifiers and the bound variable notation into logic, Peirce clearly preceded Frege in the attempt to do so, and may therefore be said to be the first to actually *introduce* quantifiers into logic (see, e.g., [Anellis & Houser, *1991*]). This is a crucial historical distinction, of the kind which Tiles glosses over or ignores.

Examples of some of Tiles's technical errors that a careful reading single out include:

p. 180: Tiles defines an *accessible cardinal* as "an initial ordinal number which can be 'reached from below'; that is, it can be reached either by adding together a smaller number of smaller ordinals, or be expressed as a value of one of the functions 2^{α} or $2^{\aleph_\alpha}$ for some smaller ordinal α."

– Since Tiles relies so heavily on Drake's [*1974*] *Set Theory* for information on modern set theory, let us observe that Drake [*1974*, 67] defines an *accessible cardinal* $\mathfrak{n}$ as a cardinal which "can be attained from below, either by adding a smaller number of smaller cardinals (where $\mathfrak{n}$ is singular), or by using the functions or $\aleph(\mathfrak{m})$ or $2^{\mathfrak{m}}$ on smaller cardinals (in the second case possibly only to get something larger than $\mathfrak{n}$)." Thus, Tiles confuses cardinals with ordinals, and fails to note the condition that, to obtain $\mathfrak{n}$ by adding a smaller number of smaller cardinals, $\mathfrak{n}$ must be singular.

p. 180: Referring specifically to [Drake *1974*, 109-110], Tiles asserts that it can be proved that if μ is an inaccessible cardinal, then the sets in the cumulative hierarchy of rank $\leq \mu$ form a model for ZF+AC;

– Drake (p. 109) presents the proof of the theorem (**ZFC**) that *If* κ *is strongly inaccessible, then* V_κ *is a model of* **ZFC** and (pp. 109-110) the corollary that *unless* **ZFC** *is inconsistent, we cannot prove the existence of strongly inaccessible cardinals in* ZFC; here, Tiles has conflated the theorem and the corollary and neglected to notice that the cardinals in question are *strongly inaccessible* rather than (*weakly*) *inaccessible.*

p. 181: Tiles defines *hyper-inaccessible cardinals* as inaccessible cardinals μ which have μ inaccessible cardinals below them; whereas hyper-inaccessible cardinals must be *strongly* inaccessible and have *strongly* inaccessible cardinals below them. This goes back to the general problem which Tiles has of the definition of *inaccessibility*.

There is clearly a failure here to accurately present the material in Drake's textbook. We see from these few examples the more general problem that Tiles is not always particularly careful about considering technical details any more than she is about precise historical distinctions.

Despite Tiles's expositions of logic, set theory, and some of the geometry and calculus behind the history of set theory, mathematical concepts are lost in philosophical discussions which do not include the historical settings required to give them perspective and significance, so that the book is poorly organized. In fairness to Tiles, the book is described in the "Preface" (p. ix) as the result of attempts to prepare a text that would be suitable for teaching "undergraduate classes and seminars in the philosophy of mathematics, sometimes to a mixture of mathematicians and philosophers, sometimes just to interested philosophers," and the author readily concedes that her attempt was not wholly successful. As its main title suggests, this book cannot in any standard sense be regarded as a history of set theory.

Tiles's *Philosophy of Set Theory* nevertheless is one of the few recent works in philosophy of mathematics attempting to deal with technical material in set theory, and with sufficient caution may provide an informal introduction, for philosophers without mathematical training, both to set theory and to some of the mathematics that comprises its historical background. Tiles's sloppiness is most likely due to her consideration of the aims and targeted audience for this book – simply to the desire to accommodate the needs of students without mathematical background. Tiles' book will not, however, satisfy the needs of historians of mathematics who are looking for a history of set theory. Joseph Dauben's work *Georg Cantor: His Mathematics and Philosophy of the Infinite* [*1979*; reprinted in 1990] remains the *ne plus ultra* on Cantor's work, with Walter Purkert's and Hans Joachim Ilgauds's largely biographical *Georg Cantor, 1845–1918* [*1987*] and Michael Hallett's *Cantorian Set Theory and Limitation of Size* [*1984*] both preferable, as second and third choices, to Tiles's book, even despite the just and cogent criticisms levelled by Moore [*1987*] against the latter. Phillip Johnson's *A History of Set Theory* [*1972*], which concentrated on Cantor's work, was a non-technical history, and in any case has long been out of print. Maria J. Frápolli has dealt with various aspects of the historical development of Cantor's set theory, largely from the standpoint of philosophy: Frápolli's recent piece "Is Cantorian Set Theory an Iterative Conception of Set?" [*1991*] argues against the notion that Cantor's theory is iterative; Frápolli's latest paper, "The Status of Cantorian Numbers" [*1992*], treats the evolution of Cantor's set-theoretic conception of the nature of *number*.

It is evident, despite the subtitle of her book, that Tiles's account is not restricted to Cantorian set theory. This is really a philosophical introduction to the history of infinity, from Zeno and Aristotle to Gödel and Drake, but focussing on the question of whether Cantor created or discovered the transfinite. This question is itself somewhat artificial, philosophically motivated from the problem of whether infinite sets are real or ideal. We cannot help notice, however, that Tiles ignores the constructivist view, which would tend to argue in favor of the infinite being a construct, hence ideal, and therefore invented rather than discovered; this suggests that she is more concerned to present her own philosophical answer to this question that to examine every side of the history of the philosophy of infinite sets. Moreover, there is no evidence that Tiles has considered in any detail Cantor's work, or she would have noticed that Cantor in his more philosophical and historical moments carefully dealt with the philosophical literature on the continuum and the on distinctions between the actual infinite and the potential infinite by Aristotle, Aquinas, and Spinoza, among many others, and that an important use of the term "transfinite" occuring in Cantor's writings (in *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (*1883*); see p. 176 of the 1962 reprint of Zermelo's edition of Cantor's collected works [*1932*]). Here, Cantor informs us he found the concept of the *Transfinitum* in association with his consideration of Aristotle's, Spinoza's and Leibniz's views, adding that the *Transfinitum* could also be called the *Suprafinitum*.

As we have seen, Tiles's consideration of axiomatic set theory is technically weak and, in many cases, inaccurate. Moreover, it ignores many important contributions to the historical and philosophical development of set theory. We do not hear from Tiles of the work of Cantor's contemporaries, such as Peirce and Dedekind, for example. Among the more notable of the studies of work in set theory by Cantor's contemporaries is Josep Pla i Carrera's extensive consideration in Spanish of Dedekind's work [*1991*], planned for appearance in THIS JOURNAL; for Peirce's contributions to set theory, there are several papers by Dauben, including *C.S. Peirce's Philosophy of Infinite Sets* [*1977*], and Paul Shields's doctoral thesis, *Charles S. Peirce on the Logic of Number* [*1981*]. The are also important topical studies. On the history of the axiom of choice, Gregory H. Moore's book *Zermelo's Axiom of Choice: Its Origins, Development, and Influence* (Springer, [*1982*], recently reviewed by Thomas Drucker [*1991*]), and F.A. Medvedev's *Ранняя история аксиомы выбора* [Early history of the axiom of choice; *1982*] are in many respects similar, although, as the title of Medvedev's book indicates, it does not bring the history as far as does Moore's book, stopping at the first world war, whereas Moore takes the history up to 1963. The best recommendation for Moore's book comes from Medvedev, who told me in conversation (in 1987) that he much prefers it to his own.

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