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Restricted Radon Transforms and Unions of Hyperplanes

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Abstract

We study $L^p(\mathbb{R}^n) \to L^{\alpha,\infty}_{d\mu(\sigma)}(L^\infty_{dt})$ estimates for the Radon transform in certain cases where the dimension of the measure μ on $\Sigma^{(n-1)}$ is less than $n-1$.

1. Introduction

If $\Sigma^{(n-1)}$ is the unit sphere in $\mathbb{R}^n$, the Radon transform Rf of a suitable function f on $\mathbb{R}^n$ is defined by

$$Rf(\sigma,t) = \int_{\sigma^\perp} f(p+t\sigma)\; dm_{n-1}(p) \qquad \sigma \in \Sigma^{(n-1)},\ t \in \mathbb{R},$$

where the integral is with respect to $(n-1)$-dimensional Lebesgue measure on the hyperplane $\sigma^\perp$. We also define, for $0<\delta<1$,

$$R_\delta f(\sigma,t) = \delta^{-1} \int_{[\sigma^\perp \cap B(0,1)]+B(0,\delta)} f(x+t\sigma)\; dm_n(x).$$

The paper [5] contains the sharp mapping properties of R from $L^p(\mathbb{R}^n)$ into the mixed norm spaces defined by the norms

$$\|g\|_{L^q(L^r)} = \Big(\int_{\Sigma^{(n-1)}} \Big[\int_{-\infty}^{\infty} |g(\sigma,t)|^r dt\Big]^{q/r} d\sigma\Big)^{1/q}.$$

Here $d\sigma$ denotes Lebesgue measure on $\Sigma^{(n-1)}$. The purpose of this paper is mainly to study the possibility of analogous mixed norm estimates when $d\sigma$ is replaced by measures $d\mu(\sigma)$ supported on subsets $S \subseteq \Sigma^{(n-1)}$ having

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dimension $< n-1$. We are usually interested in the case $r = \infty$ and will mostly settle for estimates of restricted weak type in the indices p and q and those only for f supported in a ball. The following theorem, which we regard as an estimate for a restricted Radon transform, is typical of our results here:

Theorem 1. *Fix $\alpha \in (1, n-1)$. Suppose μ is a nonnegative Borel measure on $\Sigma^{(n-1)}$ satisfying the Frostman condition*

$$\int_{\Sigma^{(n-1)}} \int_{\Sigma^{(n-1)}} \frac{d\mu(\sigma_1) d\mu(\sigma_2)}{|\sigma_1 - \sigma_2|^{\alpha}} < \infty.$$

Then, for some $C = C(n, \alpha, \mu)$,

$$\lambda\, \mu\Big(\Big\{\sigma \in \Sigma^{(n-1)} : \sup_{t \in \mathbb{R}} R\chi_E(\sigma, t) > \lambda\Big\}\Big)^{1/\alpha} \leq C\, |E|^{1/2} \tag{1}$$

for $\lambda > 0$ and Borel $E \subseteq B(0,1)$. That is,

$$\big\|R\chi_E\big\|_{L^{\alpha,\infty}_{\mu}(L^{\infty})} \leq C\, |E|^{1/2}.$$

Suppose that $\alpha \in (0, n-1)$. Say that a Borel set $E \subseteq \mathbb{R}^n$ satisfies the (Besicovitch) condition $B(n-1; \alpha)$ if there is a compact set $S \subseteq \Sigma^{(n-1)}$ having Hausdorff dimension α such that for each $\sigma \in S$ there is a translate of an $(n-1)$-plane orthogonal to σ which intersects E in a set of positive $(n-1)$-dimensional Lebesgue measure.

It is well-known that, given $\epsilon \in (0, \alpha)$, such an S supports a probability measure μ satisfying the hypothesis of Theorem 1, but with $\alpha - \epsilon$ in place of α. If $\alpha > 1$, Theorem 1, in conjunction with standard arguments, implies that such an E must have positive n-dimensional Lebesgue measure. That is, $B(n-1; \alpha)$ sets in $\mathbb{R}^n$ have positive Lebesgue measure if $\alpha > 1$.

As will be pointed out in §2, the next theorem implies that, for $\alpha \in (0,1)$ and in certain cases, $B(n-1; \alpha)$ sets have Hausdorff dimension at least $n - 1 + \alpha$. (Here is a notational comment: $|E|$ will usually denote the Lebesgue measure of E with the appropriate dimension being clear from the context.)

Theorem 2. *Suppose $\alpha \in (0,1)$. Suppose $\widetilde{\mu}$ is a nonnegative measure on a compact interval $J \subseteq \mathbb{R}$ which satisfies the condition*

$$\widetilde{\mu}(I) \leq C(\widetilde{\mu})\, |I|^{\alpha}$$

for subintervals $I \subseteq J$. Let μ be the image of $\widetilde{\mu}$ under a one-to-one and bi-Lipschitz mapping of J into $\Sigma^{(n-1)}$. If $0 < \gamma < \beta < \alpha$ and

$$\frac{1}{p} = \frac{1+\beta-\gamma}{1+2\beta-\gamma}, \quad \frac{1}{q} = \frac{1+\gamma}{1+2\beta-\gamma}, \quad \eta = \frac{1-\gamma}{1+2\beta-\gamma}$$

then there is the estimate

$$\|R_\delta \chi_E\|_{L^{q,\infty}_\mu(L^\infty)} \le C\ |E|^{1/p}\delta^{-\eta}$$

for $C = C(n, \mu, \alpha, \beta, \gamma)$ *and for all Borel* $E \subset B(0,1)$ *and* $\delta \in (0,1)$.

Contrasting with Theorems 1 and 2, the next result provides a global estimate for a restricted Radon transform:

Theorem 3. *Suppose* $n \ge 4$. *Let* S *be the* $(n-2)$*-sphere*

$$\{\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma^{(n-1)} : \sum_1^{n-1} \sigma_j^2 = \sigma_n^2\}$$

and let μ *be Lebesgue measure on* S. *Then there is an estimate*

$$\|R\chi_E\|_{L^{n-2}_\mu(L^\infty)} \le C\ |E|^{(n-1)/n}$$

for $C = C(n)$ *and for all Borel* $E \subseteq \mathbb{R}^n$.

Of course it follows, as in the remark after Theorem 1, that if a Borel set $E \subseteq \mathbb{R}^n$ has the property that for each σ in the $(n-2)$-sphere S there is a translate of an $(n-1)$-plane orthogonal to σ which intersects E in a set of positive $(n-1)$-dimensional Lebesgue measure, then E has positive n-dimensional Lebesgue measure. Theorem 3 is an analogue of (3) in [5] (which is a similar estimate but with μ replaced by Lebesgue measure on $\Sigma^{(n-1)}$ and $q = n$). The proof of Theorem 3 parallels the proof in [5] but requires the L^2 Fourier restriction estimates for the light cone in $\mathbb{R}^n$ in place of an easier L^2 estimate used in [5].

The main method employed in this paper is elementary and reasonably flexible, but it does not yield sharp results. For example, if $n \ge 3$ and if μ is Lebesgue measure on $\Sigma^{(n-1)}$, then Theorem 1 gives an $L^{2,1}(\mathbb{R}^n) \to L^{\alpha,\infty}_\mu(L^\infty)$ estimate for $1 \le \alpha < n-1$, while the sharp estimate (from [5]) is $L^{n/(n-1),1}(\mathbb{R}^n) \to L^n_\mu(L^\infty)$. In particular, it seems likely that Theorem 2 holds for general α-dimensional measures $(0 < \alpha < 1)$. (We have some unpublished partial results in this direction for measures of Cantor type.)

The remainder of this paper is organized as follows: §2 contains the proofs of Theorems 2 and 3 and the statement and proof of a similar result in the case when d is an integer strictly between 1 and $n-1$ and μ is Lebesgue measure on a suitable d-manifold in $\Sigma^{(n-1)}$; §3 contains the proof of Theorem 3; §4 contains some miscellaneous observations and remarks: an analogue for Kahane's notion of Fourier dimension of Theorem 2 when $n = 2$; three examples bearing on the question of whether $B(2;1)$ sets in $\mathbb{R}^3$ must have positive measure or only full dimension (the answer depends on the set S of directions); and some comments relating the size of $\cup_{P\in\mathcal{P}} P$ to the size of $\mathcal{P}$ when $\mathcal{P}$ is a collection of hyperplanes in $\mathbb{R}^n$.

2. Proofs of Theorems 1 and 2

As Theorem 1 is a consequence of its analogue, uniform in $\delta \in (0,1)$, for the operators R_δ, we will restrict our attention to these operators. A standard method for obtaining restricted weak type estimates is to estimate $|E|$ from below. We will do this with a particularly simple-minded strategy based on two observations and originally employed in [3] and [4]. (The paper [3] contains a not-quite-sharp estimate for the Radon transform when $n=2$ and was partial motivation for [5], while [4] contains estimates for a restricted X-ray transform in $\mathbb{R}^n$ for $n \geq 3$.) The first observation is that

$$| \cup_{n=1}^N E_n| \geq \sum\nolimits_{n=1}^N |E_n| - \sum\nolimits_{1\leq m<n\leq N} |E_m \cap E_n|.$$

The second is the well-known fact that if $\sigma \in \Sigma^{(n-1)}$ and if, for $t \in \mathbb{R}$, P_σ^δ denotes a plate $[\sigma^\perp \cap B(0,1)] + B(0,\delta) + t\sigma$, then

$$|P_{\sigma_1}^\delta \cap P_{\sigma_2}^\delta| \leq \frac{C(n)\delta^2}{|\,\sigma_1 - \sigma_2|}$$

(so long as σ_1 and σ_2 are not too far apart, an hypothesis we tacitly assume since it can be achieved by multiplying the measures μ appearing below by an appropriate partition of unity). Thus if, for $n = 1, \ldots, N$, we have plates $P_{\sigma_n}^\delta$ satisfying $|E \cap P_{\sigma_n}^\delta| \geq C_1 \lambda\delta$, it follows that

$$|E| \geq C_1 N\lambda\delta - C(n)\delta^2 \sum_{1\leq m<n\leq N} \frac{1}{|\,\sigma_m - \sigma_n|}\,. \tag{2}$$

Our strategy, then, will be to choose N and

$$\sigma_n \in \{\sigma \in \Sigma^{(n-1)} : \sup_{t\in\mathbb{R}} R_\delta \chi_E(\sigma,t) > \lambda\}$$

so that (2) gives, for example,

$$|E| \gtrsim \lambda^2 \mu\big(\{\sigma \in \Sigma^{(n-1)} : \sup_{t\in\mathbb{R}} R_\delta\chi_E(\sigma,t) > \lambda\}\big)^{2/\alpha},$$

which is the analogue of (1) for the operator R_δ. For Theorem 1 the following lemma will facilitate this choice:

Lemma 1. *Let μ be as in Theorem 1. There is $C = C(\mu)$ such that given $n \in \mathbb{N}$ and a Borel $S \subseteq \Sigma^{(n-1)}$ with $\mu(S) > 0$, one can choose $\sigma_n \in S$, $1 \leq n \leq N$, such that*

$$\sum_{1\leq m<n\leq N} \frac{1}{|\,\sigma_m - \sigma_n|} \leq \frac{CN^2}{\mu(S)^{2/\alpha}}.$$

Proof of Lemma 1. Suppose $\sigma_1, \dots, \sigma_N$ are chosen independently and at random from the probability space $(S, \frac{\mu}{\mu(S)})$. Then, for $1 \le m < n \le N$,

$$\mathbb{E}\Big(\frac{1}{|\,\sigma_m - \sigma_n|}\Big) = \frac{1}{\mu(S)^2}\int_S\int_S \frac{1}{|\,\sigma_m - \sigma_n|} d\mu(\sigma_m)\ d\mu(\sigma_n)$$
$$\le \frac{1}{\mu(S)^2}\Big(\int_S\int_S 1\ d\mu(\sigma_m)d\mu(\sigma_n)\Big)^{1-1/\alpha}\Big(\int_S\int_S \frac{1}{|\,\sigma_m - \sigma_n|^\alpha} d\mu(\sigma_m)d\mu(\sigma_n)\Big)^{1/\alpha}$$
$$\le \frac{C}{\mu(S)^{2/\alpha}},$$

by the hypothesis on μ. Thus

$$\mathbb{E}\Bigg(\sum_{1\le m<n\le N}\frac{1}{|\,\sigma_m - \sigma_n|}\Bigg) \le \frac{CN^2}{\mu(S)^{2/\alpha}}$$

and the lemma follows. ■

Proof of Theorem 1. Let S be the set

$$\Big\{\sigma \in \Sigma^{(n-1)} : \sup_{t\in\mathbb{R}} R_\delta \chi_E(\sigma, t) > \lambda\Big\}$$

so that if $\sigma \in S$ then there is $t \in \mathbb{R}$ such that if

$$P_\sigma^\delta = [\sigma^\perp \cap B(0,1)] + B(0,\delta) + t\sigma$$

then $|E \cap P_\sigma^\delta| \ge C_1\lambda\delta$. The conjunction of Lemma 1 and (2) yields

$$|E| \ge C_1 N\lambda\delta - C_2\delta^2 N^2 \mu(S)^{-2\alpha}. \tag{3}$$

We consider two cases (noting that $N = N_0 \doteq \lambda C_1 \mu(S)^{2/\alpha}/C_2\delta$ makes the RHS of (3) equal to 0):

Case I: Assume $N_0 > 10$.
In this case choose $N \in \mathbb{N}$ such that

$$\frac{\lambda C_1 \mu(S)^{2/\alpha}}{2C_2\delta} \ge N \ge \frac{\lambda C_1 \mu(S)^{2/\alpha}}{3C_2\delta}.$$

Then it follows from (3) that

$$|E| \ge C_1 \frac{\lambda C_1 \mu(S)^{2/\alpha}}{3C_2\delta}\lambda\delta - C_2\delta^2 \frac{\lambda^2 C_1^2 \mu(S)^{4/\alpha}}{4C_2^2\delta^2}\mu(S)^{-2/\alpha} = \kappa\lambda^2\mu(S)^{2/\alpha}$$

for $\kappa = C_1^2/(12C_2)$. This gives

$$\lambda\mu(S)^{1/\alpha} \lesssim |E|^{1/2}$$

as desired.

Case II: Assume $N_0 \leq 10$.
In this case (unless S is empty) we estimate

$$|E| \geq C_1 \lambda \delta \geq \frac{\lambda^2 C_1^2 \mu(S)^{2/\alpha}}{10 C_2}$$

which again yields $\lambda\ \mu(S)^{1/\alpha} \lesssim |E|^{1/2}$ and so completes the proof of Theorem 1. ■

The proof of Theorem 2 requires an analogue of Lemma 1:

Lemma 2. *Suppose μ is as in Theorem 2. Suppose $0 < \gamma < \beta < \alpha$. Then there is $C = C(\alpha, \mu, \beta, \gamma)$ such that given a Borel $S \subseteq \Sigma^{(n-1)}$ with $\mu(S) > 0$ and $N \in \mathbb{N}$, one can choose $\sigma_n \in S$, $1 \leq n \leq N$, such that*

$$\sum_{1 \leq m < n \leq N} \frac{1}{|\,\sigma_m - \sigma_n|} \leq \frac{C N^{(1+2\beta-\gamma)/\beta}}{\mu(S)^{(1+\gamma)/\beta}}.$$

Proof of Lemma 2. It suffices to show that there exists C such that if F is a measurable subset of J with $\widetilde{\mu}(F) > 0$ and if $N \in 2\mathbb{N}$, then there are $x_1, \ldots, x_{N/2}$ in F such that

$$\sum_{1 \leq m < n \leq N/2} \frac{1}{|\,x_m - x_n|} \leq \frac{C N^{(1+2\beta-\gamma)/\beta}}{\widetilde{\mu}(F)^{(1+\gamma)/\beta}}. \tag{4}$$

Note that because $\beta < \alpha$ it follows that $\widetilde{\mu}(I) \lesssim |I|^\beta$ for subintervals I of J. Now define η by $\eta^\beta = \widetilde{\mu}(F)/N$ and find $a_1 < b_1 \leq a_2 < \cdots < b_N$ in J such that $\widetilde{\mu}(F \cap [a_n, b_n]) = \eta^\beta$. Let $I_n = [a_n + \eta/L, b_n - \eta/L]$ where L is chosen large enough to guarantee that $\widetilde{\mu}(F \cap I_n) \geq \eta^\beta/2$ and then find intervals $\widetilde{I}_n \subseteq I_n$ satisfying $\widetilde{\mu}(F \cap \widetilde{I}_n) = \eta^\beta/2$. Choose Borel mappings

$$\tau_n : [0, \eta^\beta/2] \to F \cap \widetilde{I}_n$$

such that the equalities

$$\int_{F \cap \widetilde{I}_n} f \, d\widetilde{\mu} = \int_0^{\eta^\beta/2} f(\tau_n(s))\, dm_1(s)$$

hold for reasonable functions f on $F \cap \widetilde{I}_n$. Then

$$\int_0^{\eta^\beta/2} \int_0^{\eta^\beta/2} \sum_{n \neq m} \frac{dm_1(s)\, dm_1(t)}{|\tau_m(s) - \tau_n(t)|} = \sum_{n \neq m} \int_{F \cap \widetilde{I}_m} \int_{F \cap \widetilde{I}_n} \frac{d\widetilde{\mu}(x)\, d\widetilde{\mu}(y)}{|\,x - y|}.$$

Since $\gamma < 1$ and $d(\widetilde{I_m}, \widetilde{I_n}) \geq \eta/L$, the last sum is

$$\begin{aligned}
&\leq C\eta^{\gamma-1} \int_F \int_F \frac{d\widetilde{\mu}(x)\ d\widetilde{\mu}(y)}{|\,x-y|^{\gamma}} \\
&\leq C\eta^{\gamma-1} \Big(\int_F \int_F \frac{d\widetilde{\mu}(x)\ d\widetilde{\mu}(y)}{|\,x-y|^{\beta}} \Big)^{\gamma/\beta} \widetilde{\mu}(F)^{2(1-\gamma/\beta)} \\
&= C\eta^{\gamma-1} \widetilde{\mu}(F)^{2(1-\gamma/\beta)}
\end{aligned}$$

since

$$\int_J \int_J \frac{d\widetilde{\mu}(x)\ d\widetilde{\mu}(y)}{|\,x-y|^{\beta}} < \infty$$

follows from the hypothesis on $\widetilde{\mu}$ and the fact that $\beta < \alpha$. Thus

$$\begin{aligned}
\frac{1}{(\eta^{\beta}/2)^2} \int_0^{\eta^{\beta}/2} \int_0^{\eta^{\beta}/2} \sum_{n \neq m} \frac{dm_1(s)\ dm_1(t)}{|\tau_m(s) - \tau_n(t)|} &\leq C\eta^{-2\beta+\gamma-1} \widetilde{\mu}(F)^{2(1-\gamma/\beta)} \\
= C\Big(\frac{\widetilde{\mu}(F)}{N}\Big)^{(-2\beta+\gamma-1)/\beta} \widetilde{\mu}(F)^{2(1-\gamma/\beta)} &= CN^{(2\beta-\gamma+1)/\beta} \widetilde{\mu}(F)^{-(1+\gamma)/\beta}.
\end{aligned}$$

It follows that there are $s, t \in [0, \eta^{\beta}/2]$ such, for $m, n = 1, \ldots, N$, the points

$$x_n = \tau_n(s) \in F \cap \widetilde{I_n}, \quad y_m = \tau_m(t) \in F \cap \widetilde{I_m}$$

satisfy

$$\sum_{n \neq m} \frac{1}{|\,x_m - y_n|} \leq \frac{CN^{(2\beta-\gamma+1)/\beta}}{\widetilde{\mu}(F)^{(1+\gamma)/\beta}}.$$

Now either $x_n \leq y_n$ for at least $N/2$ n's or $y_n \leq x_n$ for at least $N/2$ n's. Without loss of generality, consider the first case and let

$$\mathcal{N} = \{n = 1, \ldots, N : x_n \leq y_n\}.$$

If $n_1, n_2 \in \mathcal{N}$ and $n_1 < n_2$ then (because $y_{n_1} \in I_{n_1}$ and $x_{n_2} \in I_{n_2}$), we have

$$x_{n_1} \leq y_{n_1} < x_{n_2} \leq y_{n_2}$$

and so

$$|\,x_{n_1} - x_{n_2}| > |y_{n_1} - x_{n_2}|.$$

Thus

$$\begin{aligned}
\sum_{n_1 \underset{\rightarrow}{<} n_2\, n_1, n_2 \in \mathcal{N}} \frac{1}{|\,x_{n_1} - x_{n_2}|} &< \sum_{n_1 \underset{\rightarrow}{<} n_2\, n_1, n_2 \in \mathcal{N}} \frac{1}{|y_{n_1} - x_{n_2}|} \\
&\leq \sum_{n \neq m} \frac{1}{|\,x_m - y_n|} \leq \frac{CN^{(2\beta-\gamma+1)/\beta}}{\widetilde{\mu}(F)^{(1+\gamma)/\beta}}.
\end{aligned}$$

Renumbering a subset of $\{x_n\}_{n \in \mathcal{N}}$ gives (4) and completes the proof of the lemma. ∎

Proof of Theorem 2. The proof is parallel to that of Theorem 1. Using Lemma 2 instead of Lemma 1, the analogue of (3) is

$$|E| \geq C_1 N\lambda\delta - C_2\delta^2 N^{(1+2\beta-\gamma)/\beta}\mu(S)^{-(1+\gamma)/\beta}. \tag{5}$$

The two cases are now defined by comparing

$$N_0 \doteq \left(\frac{C_1\lambda}{C_2\delta}\right)^{\beta/(1+\beta-\gamma)} \mu(S)^{\frac{1+\gamma}{1+\beta-\gamma}}$$

and 10. In case $N_0 > 10$, choosing N in (5) such that $N_0/2 \geq N \geq N_0/3$ gives

$$|E| \geq \lambda^{\frac{1+2\beta-\gamma}{1+\beta-\gamma}}\delta^{\frac{1-\gamma}{1+\beta-\gamma}}\mu(S)^{\frac{1+\gamma}{1+\beta-\gamma}}\kappa$$

where

$$\kappa = C_1^{\frac{1+2\beta-\gamma}{1+\beta-\gamma}} C_2^{\frac{-\beta}{1+\beta-\gamma}}\left(\frac{1}{3} - \frac{1}{2^{(1+2\beta-\gamma)/\beta}}\right) > 0.$$

This leads directly to the desired estimate $\lambda\mu(S)^{1/q} \lesssim |E|^{1/p}\delta^{-\eta}$ if $N_0 > 10$.

On the other hand, the inequality $N_0 \leq 10$ gives $\lambda\mu(S)^{(1+\gamma)/\beta} \lesssim \delta$ and so

$$\lambda^A\mu(S)^{A(1+\gamma)/\beta} \lesssim \delta^A \tag{6}$$

if $A > 0$. Since $|E| \geq C_1\lambda\delta$ (unless S is empty), there is also the inequality

$$\lambda^{1-A} \lesssim |E|^{1-A}\delta^{A-1} \tag{7}$$

as long as $0 < A < 1$. Multiplying (6) and (7) gives

$$\lambda\mu(S)^{A(1+\gamma)/\beta} \lesssim |E|^{1-A}\delta^{2A-1}.$$

Then the choice $A = \beta/(1+2\beta-\gamma)$ yields $\lambda\mu(S)^{1/q} \lesssim |E|^{1/p}\delta^{-\eta}$ again, completing the proof of Theorem 2. ■

It follows from a small modification of the proof of Lemma 2.15 in [1] that the estimate

$$\|R_\delta\chi_E\|_{L^{q,\infty}_\mu(L^\infty)} \lesssim |E|^{1/p}\delta^{-\eta}$$

implies a lower bound of $n - p\eta$ for the Hausdorff dimension of a Borel set containing positive-measure sections of hyperplanes associated with each of the directions σ in the support of μ. Plugging in the values for p and η which are given in Theorem 2 yields first the lower bound $n - (1-\gamma)/(1+\beta-\gamma)$ and then, since that is valid for $0 < \gamma < \beta < \alpha$, the desired lower bound of $n - 1 + \alpha$. A subset $S \subseteq \Sigma^{(n-1)}$ of Hausdorff dimension $\alpha \in (0,1)$ and located on a curve as in the hypotheses of Theorem 2, will, for each $\epsilon \in (0, \alpha)$, support a measure μ satisfying the hypotheses of Theorem 2, but with $\alpha-\epsilon$ instead of α. It follows that the $B(n-1;\alpha)$ sets associated with such sets of directions S will all have Hausdorff dimension at least $n-1+\alpha$. Finally, note that if $n=2$ then the hypothesis that μ be supported on a curve is no restriction and so all $B(1;\alpha)$ sets in $\mathbb{R}^2$ have dimension at least $1+\alpha$.

The next result gives, in certain special situations, an improvement over Theorem 1 on the index q in the bound $\|R\chi_E\|_{L^{q,\infty}_\mu(L^\infty)} \lesssim |E|^{1/2}$.

Proposition 1. *Suppose $d\in\mathbb{N}$, $1<d<n-1$. Suppose that μ is the image of Lebesgue measure on a closed ball in $\mathbb{R}^d$ under a bi-Lipschitz mapping of that ball into $\Sigma^{(n-1)}$. Then for Borel $E\subseteq B(0,1)$ there is the estimate*

$$\|R\chi_E\|_{L^{2d,\infty}_\mu(L^\infty)} \leq C\ |E|^{1/2}$$

for some $C=C(n,d,\mu)$.

Proof of Proposition 1. The proof is again analogous to the proof of Theorem 1. The required analogue of Lemma 1 is

Lemma 3. *Suppose μ is as in Proposition 1. Then there is C such that given a Borel $S\subseteq\Sigma^{(n-1)}$ with $\mu(S)>0$ and given $N\in\mathbb{N}$, one can choose $\sigma_n\in S$, $1\leq n\leq N$, such that*

$$\sum_{1\leq m<n\leq N}\frac{1}{|\,\sigma_m-\sigma_n|}\leq\frac{CN^2}{\mu(S)^{1/d}}.$$

Proof of Lemma 3. Letting $\eta>0$ be defined by $\eta^d=\mu(S)/(CN)$, where C is sufficiently large, choose N η-separated points $\sigma_1,\dots,\sigma_N$ from S. Then, for fixed m,

$$\sum_{n\neq m}\frac{1}{|\,\sigma_m-\sigma_n|}\lesssim\eta^{-d}\int_{\cup_n B(\sigma_n,\eta/2)}\frac{d\sigma}{|\,\sigma_m-\sigma|}.$$

The function $\sigma\mapsto|\,\sigma_m-\sigma|^{-1}$ is in $L^{d,\infty}(d\mu)$. So, still for fixed m,

$$\sum_{n\neq m}\frac{1}{|\,\sigma_m-\sigma_n|}\lesssim\eta^{-d}(N\eta^d)^{1-1/d}.$$

The lemma follows from the choice of η by summing on m. ■

Returning to the proof of Proposition 1, the analogue of (3) is now

$$|E|\geq C_1N\lambda\delta-C_2\delta^2N^2\mu(S)^{-1/d},$$

the choice for N_0 is $\lambda C_1\mu(S)^{1/d}/(C_2\delta)$, and the remainder of the proof of Proposition 1 is completely parallel to that of Theorem 1. ■

3. Proof of Theorem 3

As previously mentioned, the proof is an adaptation of the proof of (3) in [5]. We begin by noting that

$$\widehat{Rf(\sigma,\cdot)}(y) = \int_{-\infty}^{\infty} e^{-2\pi i y t} \int_{\sigma^{\perp}} f(p+t\sigma) dm_{n-1}(p)\ dm_1(t) = \widehat{f}(y\sigma).$$

Thus

$$\|Rf\|^2_{L^2_{d\mu}(L^2)} = \int_S \int_{-\infty}^{\infty} \left|\widehat{f}(y\sigma)\right|^2 dm_1(y)\ d\mu(\sigma) = \int_{\mathbb{R}^{(n-1)}} \left|\widehat{f}(\xi,|\xi|)\right|^2 \frac{d\xi}{|\xi|^{n-2}}$$

and so estimates for R as a mapping into $L^2_\mu(L^2)$ are just Fourier restriction estimates for the light cone in $\mathbb{R}^n$. More generally, we have

$$\left\|\left(\frac{\partial}{\partial t}\right)^{\beta} Rf\right\|^2_{L^2_\mu(L^2)} = \int_{\mathbb{R}^{(n-1)}} \left|\widehat{f}(\xi,|\xi|)\right|^2 \frac{d\xi}{|\xi|^{n-2-2\beta}}.$$

Thus the results of 5.17(b) on p. 367 in [6] give the estimate

$$\left\|\left(\frac{\partial}{\partial t}\right)^{\beta} Rf\right\|_{L^2_\mu(L^2)} \lesssim \|f\|_p \tag{8}$$

whenever

$$-\frac{1}{2} < \beta \le \frac{n-3}{2} \text{ and } \frac{1}{p} = \frac{2n-2\beta-1}{2n}.$$

Estimate (8) will lead to a mixed norm estimate in which the "inside" norm is a Lipschitz norm. The proof of Theorem 3 is simply an interpolation of this estimate with the trivial $L^1 \to L^\infty(L^1)$ estimate for R. The following generalization of an observation from [5] allows this interpolation.

Lemma 4. *Fix $\alpha > 0$ and $m \in \mathbb{N}$ with $m > \alpha$. For a Borel function g on $\mathbb{R}$ and for $t \in \mathbb{R}$, write Δ_t for the usual difference operator given by $\Delta_t g(x) = g(x+t) - g(x)$, $x \in \mathbb{R}$. Let $\|g\|_\alpha$ be the Lipschitz norm given by*

$$\|g\|_\alpha = \sup_{x\in\mathbb{R}, t\neq 0} \frac{|\Delta_t^m g(x)|}{|\,t|^\alpha}.$$

Then, for $1 \le r < \infty$, we have

$$\|g\|_{L^\infty} \lesssim \|g\|_{L^{r,\infty}}^{\alpha r/(1+\alpha r)} \|g\|_\alpha^{1/(1+\alpha r)}.$$

Proof of Lemma 4. Write

$$\Delta_t^m g(x) = \sum_{j=1}^{m} c_j g(x+jt) \pm g(x).$$

Assume that $|g(x)| \geq \lambda$ for some fixed $x \in \mathbb{R}$ and some $\lambda > 0$. If $|t|$ is so small that

$$|t|^\alpha \|g\|_\alpha \leq \frac{\lambda}{2}$$

then

$$|\sum_{j=1}^{m} c_j g(x+jt)| \geq \frac{\lambda}{2}.$$

Thus

$$\frac{\lambda}{2}\Big(2\big(\frac{\lambda}{2\|g\|_\alpha}\big)^{1/\alpha}\Big)^{1/r} \leq \|\sum_{j=1}^{m} c_j g(x+jt)\|_{L_t^{r,\infty}} \lesssim \|g\|_{L^{r,\infty}}$$

and so

$$\lambda \lesssim \|g\|_{L^{r,\infty}}^{\alpha r/(1+\alpha r)} \|g\|_\alpha^{1/(1+\alpha r)}.$$

Since $x \in \mathbb{R}$ and $\lambda \leq |g(x)|$ were arbitrary, the desired inequality follows and the proof of Lemma 4 is complete. ■

For the remainder of this section, the "outside" norms $\|\cdot\|_{L^s}$ will refer to the measure μ on S while $\|\cdot\|_p$ will be the norm on $L^p(\mathbb{R}^n)$ (or on $L^p(\mathbb{R})$) and $\|\cdot\|_\alpha$ will be the Lipschitz norm of Lemma 4. Taking $r=1$ in Lemma 4 gives

$$\|Rf\|_{L^{n-2}(L^\infty)} \lesssim \| \, \|Rf\|_1^{\alpha/(1+\alpha)}\|_{L^\infty} \; \| \, \|Rf\|_\alpha^{1/(1+\alpha)}\|_{L^{n-2}}. \tag{9}$$

Since

$$\|Rf(\sigma,\cdot)\|_1 \leq \|f\|_1,$$

for all $\sigma \in \Sigma^{(n-1)}$, (9) gives

$$\|Rf\|_{L^{n-2}(L^\infty)} \lesssim \|f\|_1^{\alpha/(1+\alpha)} \; \| \, \|Rf\|_\alpha^{1/(1+\alpha)}\|_{L^{n-2}}. \tag{10}$$

To bound the second term of the RHS of (10), we note that the estimate

$$\big\| \, \|Rf\|_\alpha \big\|_{L^2} \lesssim \left\| \Big(\frac{\partial}{\partial t}\Big)^{1/2+\alpha} Rf \right\|_{L^2(L^2)}$$

follows from Lemma 1 in [5]. Thus if

$$\alpha = \frac{n-4}{2} \quad \text{and} \quad \frac{1}{p} = \frac{n-1-\alpha}{n} = \frac{n+2}{2n},$$

then (8) with $\beta = 1/2 + \alpha$ yields

$$\begin{aligned}
\| \, \|Rf\|_\alpha^{1/(1+\alpha)} \|_{L^{n-2}} &= \Big\| \, \|Rf\|_\alpha \Big\|_{L^2}^{1/(1+\alpha)} \\
&\lesssim \Big\| \Big(\frac{\partial}{\partial t}\Big)^{1/2+\alpha} Rf \Big\|_{L^2(L^2)}^{1/(1+\alpha)} \\
&\lesssim \|f\|_{2n/(n+2)}^{1/(1+\alpha)} \\
&= \|f\|_{2n/(n+2)}^{2/(n-2)}.
\end{aligned}$$

With (10), this gives

$$\Big\| \, \|R\chi_E\|_{L^\infty} \Big\|_{L^{n-2}} \lesssim |E|^{(n-1)/n},$$

which is the desired result.

4. Miscellany

Fourier dimension

As introduced by Kahane in [2], the Fourier dimension of a compact set $E \subseteq \mathbb{R}^n$ is twice the least upper bound of the set of nonnegative β's for which E carries a Borel probability measure λ satisfying $|\widehat{\lambda}(\xi)| = o(|\xi|^\beta)$ for large $|\xi|$. It is observed in [2] that the Hausdorff dimension of E is always at least equal to the Fourier dimension of E and is generally strictly larger, since the Hausdorff dimension of $E \subseteq \mathbb{R}^n$ does not change if $\mathbb{R}^n$ if embedded in $\mathbb{R}^{n+1}$ while the Fourier dimension of E now considered as a subset of $\mathbb{R}^{n+1}$ will be 0. The next result is an analogue for Fourier dimension of the $n = 2$ case of Theorem 2:

Proposition 2. *Suppose $\alpha \in (0,1)$ and $S \subseteq \Sigma^{(1)}$ has Hausdorff dimension α. Suppose that E is a compact subset of $\mathbb{R}^2$ containing a unit line segment in each of the directions $\sigma \in S$. Then the Fourier dimension of E is at least 2α.*

Since Fourier dimension is generally strictly smaller than Hausdorff dimension, it is not surprising that our lower bound 2α for the Fourier dimension of E is strictly smaller than the lower bound $1+\alpha$ for the Hausdorff dimension of E which follows from Theorem 2. Still, it follows from Proposition 2 that Kakeya sets in $\mathbb{R}^2$ have Fourier dimension 2, providing a different proof of the well-known fact that such sets have Hausdorff dimension 2. It would be interesting to have examples, for $\alpha \in (0,1)$, of sets E as in the proposition and having Fourier dimension equal to 2α.

Proof of Proposition 2. The heuristic is simple: for each $\beta < \alpha$, S carries a Borel probability measure μ satisfying

$$\mu(J) \le C\ |J|^{\beta} \tag{11}$$

for intervals $J \subseteq \Sigma^{(1)}$ (where C depends on β and $|J|$ denotes the "length" of J).

For each $\sigma \in S$ find $x_\sigma \in \mathbb{R}^2$ such that $x_\sigma + t\sigma \in E$ if $|t| \le 1/2$. Let $\varphi \in C_0^\infty([-1/2, 1/2])$ be a nonnegative function with integral 1 and define the measure λ on E by

$$\int_E f\ d\lambda = \int_S \int_{-1/2}^{1/2} f(x_\sigma + t\sigma)\ \varphi(t)\ dt\ d\mu(\sigma). \tag{12}$$

Then

$$|\widehat{\lambda}(\xi)| \le \int_S \left| \int_{-1/2}^{1/2} e^{-2\pi i \xi\cdot(x_\sigma + t\sigma)} \varphi(t)\ dt \right| d\mu(\sigma) = \int_S \left|\widehat{\varphi}(\xi\cdot\sigma)\right|\ d\mu(\sigma). \tag{13}$$

For each $p \in \mathbb{N}$ there is $C(p)$ such that

$$\left|\widehat{\varphi}(\xi\cdot\sigma)\right| \le \frac{C(p)}{|\xi\cdot\sigma|^p}.$$

Thus for any $\xi \in \mathbb{R}^2$ there are two intervals $J_1, J_2 \subset \Sigma^{(1)}$ of length $\eta > 0$ such that for $\sigma \in \Sigma^{(1)} - (J_1 \cup J_2)$ we have

$$\left|\widehat{\varphi}(\xi\cdot\sigma)\right| \le \frac{C(p)}{(|\xi|\eta)^p}.$$

With (11) and (13) this leads to

$$|\widehat{\lambda}(\xi)| \lesssim \eta^\beta + \frac{1}{(|\xi|\eta)^p}.$$

Optimizing with the choice $\eta = |\xi|^{-p/(\beta+p)}$ then gives

$$|\widehat{\lambda}(\xi)| \le C(\beta, p)|\xi|^{-\beta p/(\beta+p)}, \tag{14}$$

and this implies the lower bound $2\beta p/(\beta+p)$ for the Fourier dimension of E. As that bound should hold for $0 < \beta < \alpha$ and for $p \in \mathbb{N}$, the desired lower bound 2α follows.

The problem with this heuristic argument lies, of course, in the measurability of the selection $\sigma \mapsto x_\sigma$. A standard approximation procedure circumvents this: for each $N \in \mathbb{N}$, partition $\Sigma^{(1)}$ into N intervals $J_1, \ldots, J_N$ of length $2\pi/N$. Choose (if possible) $\sigma_n \in J_n \cap S$ and define

$$\mu_N = \sum_{n=1}^{N} \mu(J_n)\ \delta_{\sigma_n}.$$

Define λ_N as in (12) but with μ replaced by μ_N. Then the argument above shows that

$$|\widehat{\lambda_N}(\xi)| \leq C(\beta, p)|\xi|^{-\beta p/(\beta+p)}$$

for $|\xi| \leq N^{1+\beta/p}$. Thus some weak* limit point λ of the sequence $\{\lambda_N\}$ will satisfy (14). This completes the proof of Proposition 2. ■

Examples of $B(2;1)$ sets

Recall that $E \subseteq \mathbb{R}^n$ is a $B(n-1;1)$ set if there is a compact set $S \subseteq \Sigma^{(n-1)}$ having Hausdorff dimension 1 such that for each $\sigma \in S$ there is a hyperplane orthogonal to σ which intersects E in a set of positive $(n-1)$-dimensional Lebesgue measure. Although we have not proved it unless S sits on a nice curve in $\Sigma^{(n-1)}$, one expects that $B(n-1;1)$ sets should have Hausdorff dimension n. Here are some examples in dimension 3:

Example 1. Suppose that $\widetilde{E}$ is a (Kakeya) subset of $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ having 2-dimensional Lebesgue measure 0 and containing a line segment in each direction. If E is the product of $\widetilde{E}$ and a line segment orthogonal to $\mathbb{R}^2$, then E is a measure-zero $B(2;1)$ set having full dimension and associated with the 1-sphere of directions

$$S_1 \doteq \{\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \Sigma^{(2)} : \sigma_3 = 0\}.$$

Example 2. Suppose that $S \subseteq \Sigma^{(2)}$ is a compact set of Hausdorff dimension 1 which supports a Borel probability measure μ satisfying the condition

$$\int_S \int_S \frac{d\mu(\sigma_1) d\mu(\sigma_2)}{|\sigma_1 - \sigma_2|} < \infty.$$

(It is not too difficult to construct such an S and μ using a Cantor set with variable ratio of dissection.) The proof of Theorem 1 yields in this case the estimate

$$\|R\chi_E\|_{L^{1,\infty}_\mu(L^\infty)} \lesssim |E|^{1/2}$$

for Borel $E \subseteq B(0,1)$. Thus any $B(2;1)$ set associated with the set of directions S must have not only full dimension but also positive measure.

Example 3. Consider the 1-sphere of directions

$$S_2 \doteq \{\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \Sigma^{(2)} : \sigma_1^2 + \sigma_2^2 = \sigma_3^2\}.$$

As with S_1 in Example 1, it follows from Theorem 2 that the $B(2;1)$ sets associated with S_2 have full dimension. A difference between S_1 and S_2 appears when considering the possibility of

$$L^p \to L^2_{\mu_j}(L^2) \tag{15}$$

estimates for R (here μ_j is Lebesgue measure on the circle S_j). For $j = 2$ there will be such an estimate for $p = 6/5$. This follows from (8) and, as mentioned in the proof of Theorem 3, is just the Tomas-Stein restriction theorem for the light cone in $\mathbb{R}^3$. On the other hand, there is no estimate (15) for μ_1 (because there are no Fourier restriction theorems for hyperplanes). It would be interesting to know whether, in contrast to the situation in Example 1, the $B(2;1)$ sets associated with S_2 must actually have positive measure.

Unions of collections of hyperplanes

The ideas in the proofs of Theorems 1 and 2 can be used to give some answers to special cases of the following question: if $\mathcal{P}$ is a collection of hyperplanes, what can be said about the size of

$$\bigcup_{P \in \mathcal{P}} P \tag{16}$$

given information about the size of $\mathcal{P}$? To illustrate, we will consider one case by indicating why (16) must have positive measure if the dimension of $\mathcal{P}$ exceeds 1. Parametrize the set of hyperplanes in $\mathbb{R}^n$ as $\Sigma^{(n-1)} \times [0, \infty)$ by writing $P = (\sigma, t)$ if $P = \sigma^\perp + t\sigma$ and say that a compact set $\mathcal{P}$ of hyperplanes has dimension $\alpha > 0$ if, for each $\epsilon \in (0, \alpha)$, $\mathcal{P}$ carries a Borel probability measure μ such that

$$\int_{\mathcal{P}} \int_{\mathcal{P}} \frac{d\mu(P_1)\ d\mu(P_2)}{(|\,\sigma_1 - \sigma_2| + |\,t_1 - t_2|\,)^{\alpha - \epsilon}} < \infty.$$

Fix such a $\mathcal{P}$ and μ. Writing $P^\delta_{\sigma,t}$ for the plate $[\sigma^\perp \cap B(0,1)] + B(0,\delta) + t\sigma$, one can check that if

$$P^\delta_{\sigma_1,t_1} \cap P^\delta_{\sigma_2,t_2} \neq \emptyset$$

then $|\,t_1 - t_2| \lesssim |\,\sigma_1 - \sigma_2| + \delta$. This leads to the bound

$$|P^\delta_{\sigma_1,t_1} \cap P^\delta_{\sigma_2,t_2}| \leq \frac{C(n)\delta^2}{|\,\sigma_1 - \sigma_2| + |\,t_1 - t_2|}$$

if σ_1 and σ_2 are not too far apart. Let R_0 be the truncated Radon transform given by

$$R_0 f(\sigma, t) = \int_{\sigma^\perp \cap B(0,1)} f(p + t\sigma)\ dm_{n-1}(p).$$

If $\alpha - \epsilon > 1$, the proof of Theorem 1 now gives the estimate

$$\|R_0 \chi_E\|_{L_\mu^{\alpha-\epsilon,\infty}} \lesssim |E|^{1/2}$$

for Borel $E \subseteq \mathbb{R}^n$. It follows that if

$$\bigcup_{P \in \mathcal{P}} P \subseteq E$$

then $|E| > 0$.

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