

# Extreme cases of weak type interpolation

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## Abstract

We consider quasilinear operators  $T$  of *joint weak type*  $(a, b; p, q)$  (in the sense of [2]) and study their properties on spaces  $L_{\varphi, E}$  with the norm  $\|\varphi(t)f^*(t)\|_{\tilde{E}}$ , where  $\tilde{E}$  is arbitrary rearrangement-invariant space with respect to the measure  $dt/t$ . A space  $L_{\varphi, E}$  is said to be “close” to one of the endpoints of interpolation if the corresponding Boyd index of this space is equal to  $1/a$  or to  $1/p$ . For all possible kinds of such “closeness”, we give sharp estimates for the function  $\psi(t)$  so as to obtain that every  $T : L_{\varphi, E} \rightarrow L_{\psi, E}$ .

## 1. Introduction

Already the first theorems of real interpolation (e.g., the famous Marcinkiewicz theorem) have shown that various strong properties of linear operators on intermediate spaces can be derived from rather weak endpoint conditions. This fact appeared to be especially important for integral operators, which, in most cases, satisfy just these and not stronger conditions on endpoint spaces. Many powerful theorems on Sobolev type embedding, Fourier series and transforms, differential and integral equations were proved, using methods of weak type interpolation.

The initial theorems of weak type interpolation concerned the scales of spaces with numerical parameters like  $L_p$ ,  $L_{pq}$  etc. Every intermediate space of such scales automatically is sufficiently “distant” from the endpoint spaces so as to provide transformation of weak properties of operators into strong ones. The situation changed, when considering arbitrary rearrangement-invariant (r.i.) spaces which may be arbitrarily “close” to each other. In order to estimate the position of a r.i. space  $G$  among others, D. Boyd [3]

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proposed special indices

$$\pi_G = \lim_{t \rightarrow 0} \frac{\ln d_G(t)}{\ln t}, \quad \rho_G = \lim_{t \rightarrow \infty} \frac{\ln d_G(t)}{\ln t}, \quad \text{where} \quad d_G(t) = \sup_{f \in G} \frac{\|f(s/t)\|_G}{\|f(s)\|_G},$$

and proved that any linear operator with weak properties on the spaces  $L_{p_0}, L_{p_1}$  ( $p_0 < p_1$ ) is bounded on  $G$  if and only if

$$\frac{1}{p_1} < \pi_G \leq \rho_G < \frac{1}{p_0}.$$

Strict inequalities between the Boyd indices of intermediate and of the endpoint spaces are also required in many other, more general theorems on weak type interpolation (see, e.g., [7, Chapter II]).

Weak type interpolation theorems with strict inequalities between the Boyd indices turn out to be useless in the so-called “extreme” (or “limiting”) cases of various analytical problems, such as properties of integral transforms on Zygmund spaces  $L \log L$  and  $\exp L$ , embedding of Sobolev spaces  $W_p^k(\Omega)$ , when  $pk = n$  (dimension of  $\Omega$ ) etc. Generally speaking, we get here a problem of weak type interpolation on intermediate spaces with the same Boyd indices as for the endpoint spaces. These intermediate spaces, in some sense, are “too close” to the endpoints of interpolation, so that the (quasi)linear operators on them do not have “enough distance” for getting strong properties and remain somewhere “between” strong and weak estimates. For sharp results, we should correlate this “decay of strength” with the “position” of each space under consideration.

Since the Boyd indices of all “close” spaces are equal to the same value, we need a more delicate meter of their “positions”. Unfortunately, we do not have any general characteristic which is able to replace the Boyd indices for distinguishing spaces “near” the endpoints. That is why all known theorems in this direction concern only special classes of spaces which have an intrinsic parameterizations. Mostly they are the Lorentz spaces with the (quasi)norm  $\|f\|_G = \|w(t)f^*(t)\|_{L_r}$ , where  $1 \leq r \leq \infty$  and the weight  $w(t)$  depends on additional parameters. Actually sharp results were obtained in [1] and [6] for the cases of

$$w(t) = t^{1/p-1/r} \left(\ln \frac{e}{t}\right)^\varepsilon \quad \text{and} \quad w(t) = t^{1/p-1/r} \left(\ln \frac{e}{t}\right)^\varepsilon \left(\ln \ln \frac{e^2}{t}\right)^\sigma,$$

where  $t \in (0, 1)$ ,  $\varepsilon, \sigma \in \mathbb{R}$ . A next step of generalization was done in [5] and [9], where the exterior norm in  $L_r$  was replaced by the norm in arbitrary space  $\tilde{E}$ , which is rearrangement-invariant with respect to the measure  $dt/t$  on  $(0, 1)$ . This led to the spaces  $L_{p,\varepsilon,E}$  having the quasinorm

$$(1.1) \quad \|f\|_{L_{p,\varepsilon,E}} = \|t^{1/p} \left(\ln \frac{e}{t}\right)^\varepsilon f^*(t)\|_{\tilde{E}}$$

and being termed *spaces of Lorentz-Zygmund type*. Properties of all these spaces in weak type interpolation “near” the endpoint spaces may be described as follows.

Let a (quasi)linear operator  $T$  be of two weak types  $(a, b)$  and  $(p, q)$  with  $a < p$ ,  $b < q$  (all exact definitions will be given in the next section). Considering the range space of such operators on any of the above-mentioned spaces (except the cases of  $\varepsilon = 0, r = 1$  or  $\varepsilon = 0, \rho_E = 1$ ), we obtain one and the same phenomenon: *the exponent  $\varepsilon$  decreases by 1*. In particular,  $T : L_{p,\varepsilon,E} \rightarrow L_{q,\varepsilon-1,E}$ . At the same time, for any non-limiting value  $p_1$ ,  $a < p_1 < p$ , the corresponding value of  $\varepsilon$  does not change, i.e.,  $T : L_{p_1,\varepsilon,E} \rightarrow L_{q_1,\varepsilon,E}$ , where  $q_1$  is connected with  $p_1$  like in any classical interpolation theorem:

$$\frac{1}{p_1} = \frac{\tau}{a} + \frac{1-\tau}{p}, \quad \frac{1}{q_1} = \frac{\tau}{b} + \frac{1-\tau}{q}, \quad 0 < \tau < 1.$$

Thus “decay” of the weight function  $w(t)$  (as described above) should be regarded as a consequence of “closeness” of the space  $L_{p,\varepsilon,E}$  to the endpoint space  $L_p$  (as shown in [9], the Boyd indices of the space  $L_{p,\varepsilon,E}$  both are equal to  $1/p$ ). An analogous situation occurs when the right endpoint  $(p, q)$  of interpolation is replaced by the left endpoint  $(a, b)$ .

The author is not aware of any example in the literature, where weak type interpolation gives “decay” (change) of the weight function  $w(t)$  other than the factor  $(\ln e/t)^{-1}$  appearing in the norm of range spaces after interpolation. However, this constancy might be artificial, depending on the special (power-logarithmic) form of the weight  $w(t)$ . To our knowledge, the systematic study of arbitrary weights in extreme cases of weak type interpolation was never fulfilled before.

In the present paper we will investigate the spaces  $L_{\varphi,E}$  with the norm

$$(1.2) \quad \|f\|_{L_{\varphi,E}} = \|\varphi(t)f^*(t)\|_{\bar{E}}$$

for arbitrary positive increasing functions  $\varphi(t)$ . They generalize the notion of spaces of Lorentz-Zygmund type, since the choice  $\varphi(t) = t^{1/p} (\ln \frac{e}{t})^\varepsilon$  gives (1.1). The spaces  $L_{\varphi,E}$  with general  $\varphi$  were introduced in [11] as *ultrasymmetric spaces* and studied in detail. However, the main weak type interpolation theorem, proved in [11], also requires of these spaces to be “distant” from the endpoint spaces. The present paper is intended to fill up the existing gap in interpolation results, considering all possible cases of weak type interpolation.

In spite of the special form of the norm (1.2), the spaces  $L_{\varphi,E}$  comprise most of classical spaces such as  $L_p, L_{pq}, L^{pq}(\log L)^\varepsilon$  and many other kinds of Lorentz spaces. When the function  $\varphi(t)$  is “close” to  $t^{1/p}$ , these spaces also

include the corresponding Orlicz spaces. As shown in [11], the class of spaces  $L_{\varphi,E}$  coincides with the set of all r.i. spaces, which are interpolation between the extreme spaces  $\Lambda_{\varphi}, M_{\varphi}$ , so that  $\varphi(t)$  is the fundamental function of these spaces. The function  $\varphi(t)$  is quite suitable for measuring “proximity” of spaces  $L_{\varphi,E}$  to the endpoint spaces of interpolation.

For simplicity, we will consider only spaces of functions  $f(t)$  with  $t \in (0, 1)$  and study the change of parameter  $\varphi$  in those interpolation processes, where the space parameter  $\tilde{E}$  remains unchanged. The results for other situations can be easily derived then, using general methods and theorems from [11].

The paper is organized as follows. In the next section we give some needed information about ultrasymmetric spaces and weak type interpolation. Then, in Section 3, we prove general assertions about spaces which are “distant” from one of the endpoints of interpolation and show optimality of them. Section 4 is devoted to proving the necessary and sufficient conditions for the “decay factor” to be equivalent to classically known  $(\ln \frac{e}{t})^{-1}$ . In Section 5 we consider the “superclose” spaces, where this factor is stronger and attains the possible maximum. At last, in Section 6, we give some general sharp estimates for the “decay factor” and show how it becomes “slighter” up to complete absence. While the results of Section 4 generalize some known facts, the results of the last two sections have no predecessors.

Throughout the paper we write  $f \lesssim g$  instead of  $f \leq Cg$  with some constant  $C$  and  $f \approx g$  if  $g \lesssim f$ ,  $f \lesssim g$  (i.e.,  $f$  is equivalent to  $g$ ). We say “a function  $f$  is almost increasing (decreasing)” if it is equivalent to an increasing (decreasing) function  $g$ . We write  $X = Y$  for spaces with equivalent (quasi)norms and  $X \subset Y$  for continuous embedding, as well as  $T : X \rightarrow Y$  will stand only for continuous operator  $T$  acting from  $X$  to  $Y$ .

## 2. Preliminaries

For the main definitions and properties concerning rearrangement invariant (r.i.) spaces and interpolation theory, we refer the reader to the monographs [2] and [7]. We will consider only those r.i. spaces  $E$  which are exact interpolation between  $L_1$  and  $L_{\infty}$ , using measurable functions with respect to two kinds of measure: the standard Lebesgue measure  $dt$  and the homogeneous measure  $dt/t$ . We use letters with a tilde for spaces with the second measure. Moreover, we denote by the same letter the spaces  $E$  and  $\tilde{E}$  if they are obtained by the same interpolation functor from the basic couples:  $E = \mathcal{F}(L_1, L_{\infty})$ ,  $\tilde{E} = \mathcal{F}(\tilde{L}_1, \tilde{L}_{\infty})$ , where

$$\|f\|_{\tilde{L}_1} = \int_0^1 |f(t)| \frac{dt}{t}, \quad \|f\|_{\tilde{L}_{\infty}} = \|f\|_{L_{\infty}} = \sup_{0 < t < 1} |f(t)|.$$

The norms in the spaces  $E$  and  $\tilde{E}$  can be also connected without use of the functor  $\mathcal{F}$ , since  $\|f\|_{\tilde{E}} = \|g\|_E$  when  $g(u) = f(e^{-u})$ ,  $0 < u < \infty$ .

Let a function  $\varphi(t)$  be positive and almost increasing. Its *lower* and *upper extension indices* are defined as

$$\pi_\varphi = \lim_{t \rightarrow 0} \frac{\ln m_\varphi(t)}{\ln t}, \quad \rho_\varphi = \lim_{t \rightarrow \infty} \frac{\ln m_\varphi(t)}{\ln t}, \quad \text{where } m_\varphi(t) = \sup_s \frac{\varphi(st)}{\varphi(s)}.$$

It is easy to check that  $\rho_\varphi \geq \pi_\varphi \geq 0$ . In addition, we will always require of both indices to be finite. For example, if  $\varphi(t) = t^{1/p} (\ln \frac{e}{t})^\varepsilon$  then  $\pi_\varphi = \rho_\varphi = 1/p$  for any  $\varepsilon \in \mathbb{R}$ , thus we may admit all  $p > 0$ . The ratio  $\varphi(t)/t^\sigma$  is almost increasing for any  $\sigma < \pi_\varphi$  and almost decreasing for any  $\sigma > \rho_\varphi$ . Note also that both indices do not change after replacing  $\varphi(t)$  by arbitrary equivalent function.

The following properties of extension indices can be easily proved by the reader:

- i) if  $\varphi(t) = \alpha(t)t^{\pm 1/p}$  then  $\pi_\varphi = \pi_\alpha \pm 1/p$ ,  $\rho_\varphi = \rho_\alpha \pm 1/p$ ;
- ii) if  $\varphi(t) = \alpha(t)\psi(t)$  then  $\pi_\varphi \geq \pi_\alpha + \pi_\psi$ ,  $\rho_\varphi \leq \rho_\alpha + \rho_\psi$ ;
- iii) if  $\varphi(t) = \theta(\psi(t))$  then  $\pi_\varphi \geq \pi_\theta\pi_\psi$ ,  $\rho_\varphi \leq \rho_\theta\rho_\psi$ .

For arbitrary function  $\varphi$  as above and arbitrary r.i. space  $E$ , the corresponding *ultrasymmetric space*  $L_{\varphi,E}$  is defined as the space of all measurable functions  $f$  such that

$$(2.1) \quad \|f\|_{L_{\varphi,E}} = \|\varphi(t)f^*(t)\|_{\tilde{E}} < \infty,$$

where, as usual,  $f^*$  means the non-increasing rearrangement of  $f$ . The quantity (2.1) is a quasinorm which becomes equivalent to a norm if  $\rho_\varphi < 1$ ; this norm can be obtained from (2.1) via replacing  $f^*(t)$  by  $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds$ . Since we do not distinguish spaces with equivalent (quasi)norms, the parameter function  $\varphi$  can be replaced by any other equivalent function, that gives a possibility to transform this parameter into a strictly increasing and even smooth one.

For all needed properties of ultrasymmetric spaces, we refer the reader to the paper [11], where these spaces were introduced and studied in detail. In particular, we note that the dilation function  $d_G(t)$  for any  $G = L_{\varphi,E}$  is equivalent to  $m_\varphi(t)$ , so that the Boyd indices of  $G$  coincide with extension indices of  $\varphi$ . The spaces  $\Lambda_\varphi = L_{\varphi,L_1}$  and  $M_\varphi = L_{\varphi,L_\infty}$  are called respectively *Lorentz* and *Marcinkiewicz* spaces, and any other ultrasymmetric space  $L_{\varphi,E}$  is intermediate and interpolation between them.

For the simplicity, let us denote  $\Lambda_\varphi = \Lambda_p$  and  $M_\varphi = M_p$  if  $\varphi(t) = t^{1/p}$ . A (quasi)linear operator  $T$  is said to be of *weak type*  $(p, q)$  if  $T : \Lambda_p \rightarrow M_q$ , namely, if

$$(2.2) \quad t^{1/q}(Tf)^*(t) \lesssim \int_0^1 s^{1/p} f^*(s) \frac{ds}{s}$$

(of course, the right-hand side of this inequality can be finite for non-zero functions  $f$  only if  $p < \infty$ ). Taking another pair of numbers  $a, b$ , let us denote by  $W(a, b; p, q)$  the set of all quasilinear operators which are of two weak type  $(a, b)$  and  $(p, q)$  simultaneously. The problem of weak type interpolation is to describe those pairs of spaces  $G, H$ , for which any operator from  $W(a, b; p, q)$  is bounded as an operator from  $G$  to  $H$ .

In what follows we always assume that  $a < p$ . Since we consider all spaces defined on the finite interval  $(0, 1)$ , we should require also that  $b < q$  —otherwise we might take  $H = M_b$  for any  $G$ . From (2.2) and from the analogous inequality with  $a, b$  in place of  $p, q$ , we obtain immediately that, for any  $T \in W(a, b; p, q)$ ,

$$(Tf)^*(t) \lesssim \int_0^1 f^*(s) \min \left\{ \frac{s^{1/a}}{t^{1/b}}, \frac{s^{1/p}}{t^{1/q}} \right\} \frac{ds}{s}$$

or, explicitly,

$$(2.3) \quad (Tf)^*(t) \lesssim t^{-1/b} \int_0^{t^m} s^{1/a} f^*(s) \frac{ds}{s} + t^{-1/q} \int_{t^m}^1 s^{1/p} f^*(s) \frac{ds}{s},$$

$$m = \frac{1/b - 1/q}{1/a - 1/p}.$$

The right-hand side of this inequality itself is a quasilinear operator  $S(f^*)$  acting from  $\Lambda_a$  into  $M_b$  and from  $\Lambda_p$  into  $M_q$ ; it was introduced in [4] and is usually termed *Calderón operator* for the interpolation interval  $(a, b; p, q)$ . The inequality (2.3) means that this operator is maximal on the set of operators  $W(a, b; p, q)$ , i.e., it alone should be tested on any space  $G$  in the problem of weak type interpolation. We come to the following main assertion.

**Theorem 2.1 (A. P. Calderón)** *All operators  $T \in W(a, b; p, q)$  are bounded from  $G$  to  $H$  if and only if  $S : G \rightarrow H$ .*

An operator  $T$  satisfying (2.3) was said in [1] to be of *joint weak type*  $(a, b; p, q)$ . This definition is larger than two separate weak types  $(a, b)$  and  $(p, q)$ , because it admits the value  $p = \infty$ . We also will use this extension in what follows.

It is worth to note that due to (2.3) the boundedness of the operator  $S$  may be checked only on non-increasing functions (moreover,  $|Sf(t)| \leq S(f^*)(t)$  for any measurable function  $f$  if  $a \geq 1$ ). It is also convenient to represent  $S$  as the sum  $S = S_1 + S_2$ , where

$$(2.4) \quad S_1f(t) = t^{-1/b} \int_0^{t^m} s^{1/a} f(s) \frac{ds}{s}, \quad S_2f(t) = t^{-1/q} \int_{t^m}^1 s^{1/p} f(s) \frac{ds}{s}.$$

The functions  $Sf(t)$  and  $S_2f(t)$  are obviously decreasing for any nonnegative  $f(s)$ , but the function  $S_1f(t)$  may be not such even for decreasing  $f(s)$ . That is why we cannot consider the first summand in the relation

$$(2.5) \quad \|Sf(t)\|_{L_{\varphi,E}} = \|\varphi(t)Sf(t)\|_{\tilde{E}} \leq \|\varphi(t)S_1f(t)\|_{\tilde{E}} + \|\varphi(t)S_2f(t)\|_{\tilde{E}}$$

as the norm of  $S_1f(t)$  in the space  $L_{\varphi,E}$ .

### 3. Weak type interpolation for “distant” spaces

Due to the inequalities  $a < p, b < q$  we will mention the point  $(a, b)$  as the *left endpoint* and the point  $(p, q)$  as the *right endpoint* of the interpolation problem. The relation (2.5) allows us to consider operators  $S_1, S_2$  separately, and it turns out that each of them is connected with a space “closeness” to one of the endpoints only.

Further on we will use the notations

$$\varphi(t) = t^{1/p}\alpha(t) = \frac{t^{1/a}}{\tilde{\alpha}(t)}, \quad \psi(t) = t^{1/q}\beta(t) = \frac{t^{1/b}}{\tilde{\beta}(t)}$$

and put  $\gamma(t) = \beta(t^{1/m}), \tilde{\gamma}(t) = \tilde{\beta}(t^{1/m})$ . We set  $\sigma = 1/a - 1/p$  so that  $1/b - 1/q = \sigma m$ . We also require that all functions  $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$  be (almost) increasing in order to provide the spaces  $L_{\varphi,E}, L_{\psi,E}$  being intermediate in the couples  $\Lambda_a, \Lambda_p$  and  $M_b, M_q$  respectively<sup>1</sup>.

Recall that the Boyd indices of a space  $L_{\varphi,E}$  are equal to the extension indices of the parameter function  $\varphi(t)$ , thus this space is “close” to the right endpoint if  $\pi_\alpha = 0$  and to the left endpoint if  $\pi_{\tilde{\alpha}} = 0$ ; otherwise we say that the space  $L_{\varphi,E}$  is “distant” from the corresponding endpoint. In this section we study only the cases of “distant” spaces.

**Lemma 3.1** *Let  $\rho_\alpha < \sigma, \gamma(t) \lesssim \alpha(t)$  and  $f = f^*$ . Then*

$$\|\psi(t)S_1f(t)\|_{\tilde{E}} \lesssim \|f\|_{L_{\varphi,E}}.$$

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<sup>1</sup>As shown in [1] and [9], some ultrasymmetric spaces can be studied, not being intermediate, if to use their embedding into intermediate spaces of other types. Here we do not get out of the set of ultrasymmetric spaces.

**Proof.** By immediate substitution we obtain from (2.4) that

$$\begin{aligned} \|\psi(t)S_1f(t)\|_{\tilde{E}} &= \left\| \beta(t)t^{-\sigma m} \int_0^{t^m} \varphi(s)f(s) \frac{s^\sigma}{\alpha(s)} \frac{ds}{s} \right\|_{\tilde{E}} \\ &\approx \left\| \gamma(t)t^{-\sigma} \int_0^t \varphi(s)f(s) \frac{s^\sigma}{\alpha(s)} \frac{ds}{s} \right\|_{\tilde{E}}, \end{aligned}$$

since  $\|g(t^m)\|_{\tilde{E}} \approx \|g(t)\|_{\tilde{E}}$  for  $m > 0$  and any  $g \in \tilde{E}$  (see [9, Lemma 1.2]). Taking  $\varepsilon = (\sigma - \rho_\alpha)/2$ , we obtain that the function  $s^{\sigma-\varepsilon}/\alpha(s)$  is almost increasing, thus

$$\begin{aligned} (3.1) \quad \|\psi(t)S_1f(t)\|_{\tilde{E}} &\lesssim \left\| \gamma(t)t^{-\sigma} \cdot \frac{t^{\sigma-\varepsilon}}{\alpha(t)} \int_0^t \varphi(s)f(s) \frac{ds}{s^{1-\varepsilon}} \right\|_{\tilde{E}} \\ &\lesssim \left\| t^{-\varepsilon} \int_0^t \varphi(s)f(s)s^{\varepsilon-1} ds \right\|_{\tilde{E}} \end{aligned}$$

for arbitrary  $\gamma(t) \lesssim \alpha(t)$ .

Consider an operator

$$Ug(t) = t^{-\varepsilon} \int_0^t g(s)s^{\varepsilon-1} ds$$

and show that it is bounded in any space  $\tilde{E}$ . In virtue of interpolation properties of such spaces, it suffices to show that  $U$  is bounded in the “extreme” spaces  $\tilde{L}_1$  and  $L_\infty$ . For the first of them, we obtain (assuming  $g(t) \geq 0$ ):

$$\begin{aligned} \|Ug\|_{\tilde{L}_1} &= \int_0^1 t^{-\varepsilon} \left( \int_0^t g(s)s^{\varepsilon-1} ds \right) \frac{dt}{t} = \int_0^1 g(s)s^{\varepsilon-1} \left( \int_s^1 t^{-\varepsilon-1} dt \right) ds \\ &= \frac{1}{\varepsilon} \int_0^1 g(s)s^{\varepsilon-1}(s^{-\varepsilon} - 1) ds \lesssim \int_0^1 g(s) \frac{ds}{s} = \|g\|_{\tilde{L}_1}. \end{aligned}$$

Analogously, for the space  $L_\infty$ , we obtain that

$$\|Ug\|_{L_\infty} = \sup_t t^{-\varepsilon} \int_0^t g(s)s^{\varepsilon-1} ds \leq \|g\|_{L_\infty} \cdot \sup_t t^{-\varepsilon} \int_0^t s^{\varepsilon-1} ds = \frac{1}{\varepsilon} \|g\|_{L_\infty}.$$

Thus  $U$  is bounded in any space  $\tilde{E}$  and the relation (3.1) implies that

$$\|\psi(t)S_1f(t)\|_{\tilde{E}} \lesssim \|U(\varphi f)(t)\|_{\tilde{E}} \lesssim \|\varphi(t)f(t)\|_{\tilde{E}},$$

which coincides with  $\|f\|_{L_{\varphi,E}}$ . ■

An analogous assertion can be proved for the left endpoint of interpolation as well.



**Lemma 3.2** *Let  $\rho_{\tilde{\alpha}} < \sigma$ ,  $\tilde{\gamma}(t) \gtrsim \tilde{\alpha}(t)$  and  $f = f^*$ . Then*

$$\|\psi(t)S_2f(t)\|_{\tilde{E}} \lesssim \|f\|_{L_{\varphi,E}}.$$

**Proof.** As in the previous proof, we have

$$\begin{aligned} \|\psi(t)S_2f(t)\|_{\tilde{E}} &= \left\| \frac{t^{\sigma m}}{\tilde{\beta}(t)} \int_{t^m}^1 \varphi(s)f(s) \frac{\tilde{\alpha}(s)}{s^\sigma} \frac{ds}{s} \right\|_{\tilde{E}} \\ &= \left\| \frac{t^\sigma}{\tilde{\gamma}(t)} \int_t^1 \varphi(s)f(s) \frac{\tilde{\alpha}(s)}{s^\sigma} \frac{ds}{s} \right\|_{\tilde{E}}. \end{aligned}$$

For the same  $\varepsilon = (\sigma - \rho_{\tilde{\alpha}})/2$ , the function  $\tilde{\alpha}(s)/s^{\sigma-\varepsilon}$  is almost decreasing, thus

$$\begin{aligned} (3.2) \quad \|\psi(t)S_2f(t)\|_{\tilde{E}} &\lesssim \left\| \frac{t^\sigma}{\tilde{\gamma}(t)} \cdot \frac{\tilde{\alpha}(t)}{t^{\sigma-\varepsilon}} \int_t^1 \varphi(s)f(s) \frac{ds}{s^{1+\varepsilon}} \right\|_{\tilde{E}} \\ &\lesssim \left\| t^\varepsilon \int_t^1 \varphi(s)f(s) s^{-\varepsilon-1} ds \right\|_{\tilde{E}}. \end{aligned}$$

Consider a linear operator

$$Vg(t) = t^\varepsilon \int_t^1 g(s) s^{-\varepsilon-1} ds, \quad g(s) \geq 0.$$

Like for the operator  $U$  from the previous proof, we have

$$\begin{aligned} \|Vg\|_{\tilde{L}_1} &= \int_0^1 t^\varepsilon \left( \int_t^1 g(s) s^{-\varepsilon-1} ds \right) \frac{dt}{t} = \int_0^1 g(s) s^{-\varepsilon-1} \left( \int_0^s t^{\varepsilon-1} dt \right) ds \\ &= \frac{1}{\varepsilon} \int_0^1 g(s) s^{-\varepsilon-1} \cdot s^\varepsilon ds = \frac{1}{\varepsilon} \|g\|_{\tilde{L}_1} \end{aligned}$$

and

$$\|Vg\|_{L_\infty} = \sup_t t^\varepsilon \int_t^1 g(s) s^{-\varepsilon-1} ds \leq \|g\|_{L_\infty} \cdot \sup_t t^\varepsilon \int_t^1 s^{-\varepsilon-1} ds \leq \frac{1}{\varepsilon} \|g\|_{L_\infty}.$$

The interpolation properties of spaces  $\tilde{E}$  and the inequality (3.2) give as before that

$$\|\psi(t)S_2f(t)\|_{\tilde{E}} \lesssim \|V(\varphi f)(t)\|_{\tilde{E}} \lesssim \|f\|_{L_{\varphi,E}},$$

and the lemma is proved. ■

**Corollary 1** *If*

$$\varphi(t) = t^{1/p}\theta(t^{1/a-1/p}), \quad \psi(t) = t^{1/q}\theta(t^{1/b-1/q})$$

*for some function  $\theta$  with  $0 < \pi_\theta \leq \rho_\theta < 1$  then  $T : L_{\varphi,E} \rightarrow L_{\psi,E}$  for any (quasi)linear operator  $T \in W(a, b; p, q)$ .*

**Proof.** Obviously,  $\alpha(t) = \theta(t^{1/a-1/p})$  and  $\beta(t) = \theta(t^{1/b-1/q})$  whence  $\gamma(t^m) = \beta(t) = \alpha(t^m)$ . Using properties of extension indices, we easily find that

$$\rho_\alpha = \rho_\varphi - \frac{1}{p} = \rho_\theta \left( \frac{1}{a} - \frac{1}{p} \right) < \frac{1}{a} - \frac{1}{p} = \sigma,$$

thus the conditions of Lemma 3.1 are fulfilled and  $\|\psi(t)S_1f(t)\|_{\tilde{E}} \lesssim \|f\|_{L_{\varphi,E}}$ . Analogously,

$$\begin{aligned} \tilde{\gamma}(t^m) &= \tilde{\beta}(t) = t^{1/b-1/q} / \theta(t^{1/b} - t^{1/q}) = \tilde{\alpha}(t^m), \\ \rho_{\tilde{\alpha}} &= \frac{1}{a} - \pi_\varphi = \frac{1}{a} - \frac{1}{p} - \pi_\theta \left( \frac{1}{a} - \frac{1}{p} \right) < \frac{1}{a} - \frac{1}{p} \end{aligned}$$

and all conditions of Lemma 3.2 are fulfilled too, giving that

$$\|\psi(t)S_2f(t)\|_{\tilde{E}} \lesssim \|f\|_{L_{\varphi,E}}.$$

It remains to refer to the inequality (2.5). ■

**Remark.** For the case of finite  $p, q$ , this assertion can also be derived from some general interpolation theorems from [11].

In every assertion of this section we may take  $\gamma(t) = \alpha(t)$  (and thus  $\tilde{\gamma}(t) = \tilde{\alpha}(t)$ ), which is due to the space  $L_{\varphi,E}$  being “distant” from the corresponding endpoint of interpolation. Denoting the ratio  $\gamma(t)/\alpha(t) = \tilde{\alpha}(t)/\tilde{\gamma}(t)$  by  $\kappa(t)$ , we obtain that it may be taken equivalent to 1 in any “distant” case of interpolation. As we will see in the following sections, in the “close” cases this is not so and  $\kappa(t) = o(1)$  as  $t \rightarrow 0$ . That is why we will call  $\kappa(t)$  *decay factor* of interpolation.

At the last of section, let us show that the decay factor cannot be essentially bigger than 1, i.e., it is always bounded. For the operator  $S_1$  and arbitrary non-increasing function  $f$ , we have

$$\begin{aligned} \|\psi(t)S_1f(t)\|_{\tilde{E}} &\geq \left\| \gamma(t)t^{-\sigma} \int_{t/2}^t \varphi(s)f(s) \frac{s^{\sigma-1}}{\alpha(s)} ds \right\|_{\tilde{E}} \\ &\geq \left\| \kappa(t)\varphi\left(\frac{t}{2}\right) f(t)t^{-\sigma} \int_{t/2}^t s^{\sigma-1} ds \right\|_{\tilde{E}} \approx \|\kappa(t)\varphi(t)f(t)\|_{\tilde{E}}, \end{aligned}$$

since  $\varphi(t/2) \approx \varphi(t)$  for any function  $\varphi$  with finite extension indices. Now, if  $\kappa(t)$  is unbounded, the norm of  $\kappa\varphi f$  can be infinite for some  $f \in L_{\varphi,E}$ , while the norm of  $\psi S_1f$  must be finite by Lemma 3.1, and we come to contradiction. The operator  $S_2$  can be considered similarly.

### 4. General spaces of Lorentz-Zygmund type

The spaces of Lorentz-Zygmund type with norm (1.1) generalize the classical Lorentz-Zygmund spaces from [1] only in one direction: the exterior norm in  $L_r$  is replaced by the norm in arbitrary  $\tilde{E}$ . Thus they do not comprise even spaces from [6] with weights containing repeated logarithms. A more careful consideration of proofs from [5] and [9] shows that all results can be extended to the case, when a power of logarithm in the weight function is replaced by an arbitrary almost increasing function  $A(u)$ ,  $u = (\ln \frac{e}{t})^{-1}$ , satisfying the so-called  $\Delta_2$ -condition:  $A(2u) \approx A(u)$ . We obtain a partial case of ultrasymmetric spaces  $L_{\varphi,E}$  with parameter functions

$$(4.1) \quad \varphi(t) = t^{1/p}\alpha(t) = t^{1/p}A\left(\left(\ln \frac{e}{t}\right)^{-1}\right),$$

that will be called *general spaces of Lorentz-Zygmund type*. Note that, for the function  $\alpha(t)$  itself, the  $\Delta_2$ -condition for the function  $A(u)$  transforms into relation  $\alpha(t^2) \approx \alpha(t)$ .

It is easy to check that both extension indices of function (4.1) are equal to  $1/p$ , thus we do not get any new result in “distant” cases. For “close” cases, we have the following main assertion:

**Theorem 4.1** *The decay factor for general spaces of Lorentz-Zygmund type  $L_{\varphi,E}$  with  $\rho_E < 1$  is equivalent to  $(\ln \frac{e}{t})^{-1}$ .*

**Proof.** At first, let us show that  $S: L_{\varphi,E} \rightarrow L_{\psi,E}$  whenever  $\gamma(t) \lesssim (\ln \frac{e}{t})^{-1}\alpha(t)$ . Since all conditions of Lemma 3.1 are fulfilled, we have to consider only the operator  $S_2$ :

$$\begin{aligned} \|\psi(t)S_2f(t)\|_{\tilde{E}} &= \|\beta(t) \int_{tm}^1 \varphi(s)f(s)\frac{ds}{s\alpha(s)}\|_{\tilde{E}} \\ &= \|\gamma(t) \int_t^1 \varphi(s)f(s)\frac{ds}{s\alpha(s)}\|_{\tilde{E}} \lesssim \left\| \left(\ln \frac{e}{t}\right)^{-1} \int_t^1 \varphi(s)f(s)\frac{ds}{s} \right\|_{\tilde{E}}. \end{aligned}$$

Recall that  $\|g(t)\|_{\tilde{E}} = \|g(e^{-u})\|_E$  for any function  $g$ , thus

$$\|\psi(t)S_2f(t)\|_{\tilde{E}} \lesssim \left\| \frac{1}{1+u} \int_{e^{-u}}^1 \varphi(s)f(s)\frac{ds}{s} \right\|_E \leq \left\| \frac{1}{u} \int_0^u \varphi(e^{-v})f(e^{-v}) dv \right\|_E.$$

The last term of these inequalities is  $\|Pg\|_E$ , where  $g(v) = \varphi(e^{-v})f(e^{-v})$  and  $P$  is the usual Hardy operator, which is bounded in any r.i. space  $E$  with  $\rho_E < 1$ . Thus

$$\|Pg\|_E \lesssim \|g\|_E = \left\| g \left( \ln \frac{1}{t} \right) \right\|_{\tilde{E}} = \|\varphi(t)f(t)\|_{\tilde{E}},$$

and the required boundedness of the Calderón operator is proved.

Now we proceed to proving that the decay factor  $\kappa(t) = \gamma(t)/\alpha(t)$  cannot be made “slighter” than  $(\ln \frac{e}{t})^{-1}$ , namely, for  $\gamma(t) = \lambda(t) (\ln \frac{e}{t})^{-1} \alpha(t)$ , the operator  $S$  does not act from  $L_{\varphi,E}$  to  $L_{\psi,E}$  if the function  $\lambda(t)$  is unbounded. For simplicity, we exclude the case<sup>2</sup>  $p = \infty$ .

It was already mentioned before that the parameter function  $\varphi(t)$  may be always supposed to be continuous, smooth and strictly increasing; denote by  $\varphi^{-1}$  its converse function. By condition  $p < \infty$  we have  $\varphi(+0) = 0$  and we also may suppose that  $\varphi(1) = 1$ .

Let  $h(u) \in E$  be a decreasing function such that the product

$$\lambda(\varphi^{-1}(e^{-u})) h(u)$$

does not belong to  $E$  (this function necessarily exists, since the function  $\lambda(t)$  is unbounded). Without loss of generality, we may assume that the function  $uh(u)$  is increasing, replacing, if needed,  $h(u)$  by  $h^{**}(u)$ . (The last function has the same properties as  $h(u)$ , since it is greater than  $h(u)$ , but belongs to  $E$  whenever  $\rho_E < 1$ .) Define  $g(t) = h(\ln \frac{e}{t})$ , then  $g(t)$  belongs to  $\tilde{E}$  and is increasing, while the function  $g(t) \ln \frac{e}{t}$  is decreasing. All the more, the function  $g(t)/t$  is decreasing too.

Further on we will use the fact that the functions  $g(t)$  and  $g(\varphi(t))$  have equivalent norms in any space  $\tilde{E}$ . Indeed, the correspondence  $g(t) \mapsto g(\varphi(t))$  is a linear operator, which is an isometry on  $L_\infty$  and an isomorphism on  $L_1$ :

$$\int_0^1 g(t) \frac{dt}{t} = \int_0^1 g(\varphi(s)) \frac{\varphi'(s)}{\varphi(s)} ds \approx \int_0^1 g(\varphi(s)) \frac{ds}{s},$$

because  $s\varphi'(s) \approx \varphi(s)$  due to the condition  $p < \infty$ .

Let us check the action of operator  $S_2$  on the function  $f(t) = g(\varphi(t))/\varphi(t)$  which is known now as decreasing and belonging to  $L_{\varphi,E}$ . We obtain that

$$\|\psi(t)S_2f(t)\|_{\tilde{E}} \approx \|\gamma(t) \int_t^1 g(\varphi(s)) \frac{ds}{s\alpha(s)}\|_{\tilde{E}} \geq \|\gamma(t)g(\varphi(t)) \int_t^1 \frac{ds}{s\alpha(s)}\|_{\tilde{E}}.$$

But  $\alpha(t) \approx \alpha(t^2)$ , hence

$$\int_t^1 \frac{ds}{s\alpha(s)} \gtrsim \int_t^{\sqrt{t}} \frac{ds}{s\alpha(s^2)} \gtrsim \frac{1}{\alpha(t)} \ln \frac{e}{t}$$

and

$$\|\psi(t)S_2f(t)\|_{\tilde{E}} \gtrsim \|\lambda(t)g(\varphi(t))\|_{\tilde{E}} \approx \|\lambda(\varphi^{-1}(e^{-u})) h(u)\|_E.$$

This inequality contradicts to the action  $S : L_{\varphi,E} \rightarrow L_{\psi,E}$ , since the left-hand side here is infinite, while the function  $f$  belongs to  $L_{\varphi,E}$ . ■

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<sup>2</sup>This case requires a special care even when  $\alpha(t)$  is a power of logarithm. The corresponding proof was done in [10]; it can be extended to general case by standard modification.

Analogous results can be obtained if one considers the general spaces of Lorentz-Zygmund type “near” the left endpoint of interpolation. A space  $L_{\varphi,E}$  is of such a type if

$$\varphi(t) = t^{1/a}/\tilde{\alpha}(t) \quad \text{and} \quad \tilde{\alpha}(t^2) \approx \tilde{\alpha}(t);$$

the decay factor  $\kappa(t)$  is now equal to  $\tilde{\alpha}(t)/\tilde{\gamma}(t)$ .

**Theorem 4.2** *The decay factor for general spaces of Lorentz-Zygmund type  $L_{\varphi,E}$  “near” the left endpoint of interpolation is equivalent to  $(\ln \frac{e}{t})^{-1}$  whenever  $\pi_E > 0$ .*

**Proof.** This theorem can be derived from Theorem 4.1 by the use of some general methods from the paper [11] such as duality principle and special transformations of the measure on  $(0, 1)$ . We give here another (direct) proof in the general style of this paper, but only for the sufficiency part. Namely, we show that  $S : L_{\varphi,E} \rightarrow L_{\psi,E}$  whenever

$$\tilde{\gamma}(t) \gtrsim \tilde{\alpha}(t) \ln \frac{e}{t}.$$

We have

$$\begin{aligned} \|\psi(t)S_1 f(t)\|_{\tilde{E}} &= \left\| \frac{1}{\tilde{\beta}(t)} \int_0^{t^m} \varphi(s)f(s)\tilde{\alpha}(s) \frac{ds}{s} \right\|_{\tilde{E}} \\ &= \left\| \frac{1}{\tilde{\gamma}(t)} \int_0^t \varphi(s)f(s)\tilde{\alpha}(s) \frac{ds}{s} \right\|_{\tilde{E}} \lesssim \left\| \int_0^t \varphi(s)f(s) \left(\ln \frac{e}{s}\right)^{-1} \frac{ds}{s} \right\|_{\tilde{E}}. \end{aligned}$$

After change of variable  $s = e^{-v}$  in the integral and passing from norms in  $\tilde{E}$  to norms in  $E$ , we obtain that

$$\|\psi(t)S_1 f(t)\|_{\tilde{E}} \lesssim \left\| \int_u^\infty \varphi(e^{-v})f(e^{-v}) \frac{dv}{v} \right\|_E.$$

Denoting

$$g(v) = \varphi(e^{-v})f(e^{-v}),$$

we see that the last term here is the norm of  $Qg$ , where  $Q$  is the second (conjugate) operator Hardy. As known, this operator is bounded in any r.i. space  $E$  provided the lower Boyd index  $\pi_E > 0$ . In result

$$\|\psi(t)S_1 f(t)\|_{\tilde{E}} \lesssim \|g(u)\|_E = \|\varphi(t)f(t)\|_{\tilde{E}},$$

and the sufficiency part of the theorem is proved. ■

### 5. “Superclose” spaces

The condition  $\rho_E < 1$  in Theorem 4.1 and the condition  $\pi_E > 0$  in Theorem 4.2 are essential and, in general, cannot be omitted. This can be shown on the following example. Let  $\alpha(t) \equiv 1$  and  $E = L_1$ . Then  $L_{\varphi,E} = \Lambda_p$  and the best possible action of the Calderón operator is  $S : \Lambda_p \rightarrow M_q$ . We can obtain an action  $S : L_{\varphi,E} \rightarrow L_{\psi,E}$  only if  $M_q \subset L_{\psi,E}$ , which is equivalent to the condition  $t^{-1/q} \in L_{\psi,E}$ , i.e.,  $\|\psi(t)t^{-1/q}\|_{\tilde{L}_1} < \infty$ . The last inequality means that

$$\int_0^1 \beta(t) \frac{dt}{t} \approx \int_0^1 \gamma(t) \frac{dt}{t} < \infty,$$

so that the relation  $\gamma(t)/\alpha(t) \approx (\ln \frac{e}{t})^{-1}$  is impossible.

Nevertheless, it can be shown that Theorem 4.1 remains true even without the condition  $\rho_E < 1$ , if the space  $L_{\varphi,E}$  keeps some “minimal distance” from the right endpoint of interpolation. This “distance” can be described in terms of extension indices of the function  $A(u) = \alpha(e^{-1/u})$ , defined at the beginning of Section 4. In the proof of Theorem 4.1 we have used the  $\Delta_2$ -condition for this function that may be regarded as the inequality  $\rho_A < \infty$  for the upper extension index. Let us show that the inequality  $\pi_A > 0$  for the lower extension index allows us to omit the above mentioned restriction for the space  $E$ .

As in the proof of Theorem 4.1, we have that

$$\|\psi(t)S_2f(t)\|_{L_\infty} \lesssim \left\| \left( \ln \frac{e}{t} \right)^{-1} \int_t^1 \varphi(s)f(s) \frac{ds}{s} \right\|_{L_\infty} \leq \|\varphi(s)f(s)\|_{L_\infty},$$

while, for  $E = L_1$ ,

$$\begin{aligned} \|\psi(t)S_2f(t)\|_{\tilde{L}_1} &= \int_0^1 \gamma(t) \left( \int_t^1 \varphi(s)f(s) \frac{ds}{s\alpha(s)} \right) \frac{dt}{t} \\ &= \int_0^1 \frac{\varphi(s)f(s)}{s\alpha(s)} \left( \int_0^s \gamma(t) \frac{dt}{t} \right) ds. \end{aligned}$$

In order to get the norm of  $\varphi(s)f(s)$  in  $\tilde{L}_1$ , we should require that

$$(5.1) \quad \frac{1}{\alpha(s)} \int_0^s \gamma(t) \frac{dt}{t} \lesssim 1.$$

Taking  $\gamma(t) = \alpha(t) (\ln \frac{e}{t})^{-1}$ , we obtain the necessary and sufficient condition

$$\int_0^s \alpha(t) \left( \ln \frac{e}{t} \right)^{-1} \frac{dt}{t} \lesssim \alpha(s) \iff \int_0^u A(v) \frac{dv}{v} \lesssim A(u)$$

which is equivalent to the inequality  $\pi_A > 0$ .

The left endpoint of interpolation can be considered similarly, however, it is connected with another problem, which is absent at the right endpoint. Recall that we may consider only such spaces  $L_{\varphi,E}$  which are intermediate for the Banach couple  $\Lambda_a, \Lambda_p$ . At the right endpoint it is enough that  $\varphi(t) \lesssim t^{1/p}$ , and we may take any increasing function  $\alpha(t)$ . At the left endpoint we need an embedding  $L_{\varphi,E} \subset \Lambda_a$ . As shown in [11], this is equivalent to the condition  $\tilde{\alpha}(t) \in \tilde{E}'$  where  $E'$  is a Köthe dual (associate) space for  $E$ . This condition is always satisfied if  $\pi_E > 0$ , so we need not speak of the embedding problem in Theorem 4.2. But if we want to get all possible spaces  $E$  with  $\pi_E = 0$ , we should require that  $\tilde{\alpha}(t) \in \tilde{L}_1$ .

Considering the left endpoint of interpolation, we have to prove that the function  $\psi(t)S_1f(t)$  belongs to the space  $E = L_\infty$ , which leads to the condition

$$(5.2) \quad \frac{1}{\tilde{\gamma}(s)} \int_0^s \tilde{\alpha}(t) \frac{dt}{t} \lesssim 1$$

If we set  $\tilde{\alpha}(t) = u\tilde{A}(u)$ ,  $u = (\ln \frac{e}{t})^{-1}$ , we obtain for  $\tilde{A}(u)$  the same inequality as previously for  $A(u)$  and the condition  $\pi_{\tilde{A}} > 0$  will provide the action  $S_1 : L_{\varphi,E} \rightarrow L_{\psi,E}$  for any  $E$  without requiring  $\pi_E > 0$ .

All said above means that the decay factor can be stronger than  $(\ln \frac{e}{t})^{-1}$  only if  $\pi_A = 0$  and  $\rho_E = 1$  at the right endpoint or  $\pi_{\tilde{A}} = 0$  and  $\pi_E = 0$  at the left one; the corresponding spaces  $L_{\varphi,E}$  may be regarded as “superclose” to the endpoints of interpolation. In order to compute the decay factor for such spaces, we will use the inequalities (5.1) and (5.2) that remain necessary for the boundedness of Calderón operator. This leads to the following assertions.

**Theorem 5.1** *If “near” the right endpoint of interpolation*

$$\gamma(t) \lesssim \alpha(t) \left( \ln \frac{e}{t} \right)^{-1}$$

*and the inequality (5.1) holds then  $S : L_{\varphi,E} \rightarrow L_{\psi,E}$  for any r.i. space  $E$ . The same is true if “near” the left endpoint of interpolation*

$$\tilde{\gamma}(t) \gtrsim \tilde{\alpha}(t) \left( \ln \frac{e}{t} \right)$$

*and the inequality (5.2) holds.*

**Example.** Let us illustrate the last theorem by an example for the right endpoint of interpolation. Consider  $\alpha(t) \approx (\ln \ln \frac{e}{t})^\varepsilon$ ,  $\varepsilon < 0$ , for small  $t$ . In this case  $A(u) \approx (\ln \frac{e}{u})^\varepsilon$ , hence  $\pi_A = 0$  and the operator  $S$  does not act from  $L_{\varphi,E}$  to  $L_{\psi,E}$  with  $\gamma(t) \approx \alpha(t) (\ln \frac{e}{t})^{-1}$  if  $\rho_E = 1$ .

In order to get the best possible  $\gamma(t)$ , we use the relation (5.1) with the equality sign. This gives

$$\gamma(t) = t\alpha'(t) \approx \left(\ln \ln \frac{e}{t}\right)^{\varepsilon-1} \left(\ln \frac{e}{t}\right)^{-1}$$

and the decay factor will be

$$\kappa(t) = \gamma(t)/\alpha(t) \approx \left(\ln \ln \frac{e}{t}\right)^{-1} \left(\ln \frac{e}{t}\right)^{-1},$$

i.e., worse than classical  $(\ln \frac{e}{t})^{-1}$  by an additional factor  $(\ln \ln \frac{e}{t})^{-1}$ . Although the decay factor seems to be the same for all  $\varepsilon$ , the equivalence constant blows up when we take  $\varepsilon \rightarrow 0$ . For  $\varepsilon = 0$ , the equality in (5.1) is principally impossible; we only may take arbitrary  $\gamma(t)$  such that the ratio  $\gamma(t)/t$  is integrable at zero.

The space  $E = L_1$  is not only possible example of spaces with  $\rho_E = 1$ . As other examples we may take the Zygmund spaces  $E = L(\log L)^k$  with arbitrary  $k > 0$ .

### 6. General case

The first conditions on  $\gamma$  and  $\tilde{\gamma}$  in Theorem 5.1 prescribe for the decay factor  $\kappa$  to be not slighter than  $(\ln \frac{e}{t})^{-1}$ . Thus this theorem cannot be applied to arbitrary spaces  $L_{\varphi,E}$  “close” to the endpoints, although the second conditions, inequalities (5.1) and (5.2), are necessary. Instead of the prescribed restrictions for  $\kappa$ , we may take inequalities similar to (5.1) and (5.2) which give the boundedness of the norm of  $\psi(t)S_2f(t)$  in  $L_\infty$  for the right endpoint of interpolation and of the norm of  $\psi(t)S_1f(t)$  in  $\tilde{L}_1$  for the left endpoint.

**Theorem 6.1** *Let one<sup>3</sup> of the following sets of conditions is fulfilled:*

$$i) \quad \rho_\alpha < \frac{1}{a} - \frac{1}{p}, \quad \int_0^s \frac{\gamma(t)}{t} dt \lesssim \alpha(s), \quad \int_t^1 \frac{ds}{s\alpha(s)} \lesssim \frac{1}{\gamma(t)}$$

*for the right endpoint of interpolation;*

$$ii) \quad \rho_{\tilde{\alpha}} < \frac{1}{a} - \frac{1}{p}, \quad \int_0^t \frac{\tilde{\alpha}(s)}{s} ds \lesssim \tilde{\gamma}(t), \quad \int_s^1 \frac{dt}{t\tilde{\gamma}(t)} \lesssim \frac{1}{\tilde{\alpha}(s)}$$

*for the left endpoint of interpolation. Then  $S : L_{\varphi,E} \rightarrow L_{\psi,E}$ .*

---

<sup>3</sup>We assume here that the space  $L_{\varphi,E}$  is “close” only to one endpoint of interpolation. However, this is unnecessary, and the theorem can be readily modified to the “double-close” spaces.



**Proof.** For the right endpoint, we have (see the proof of Theorem 4.1)

$$\begin{aligned} \|\psi(t)S_2f(t)\|_{L_\infty} &= \left\| \gamma(t) \int_t^1 \varphi(s)f(s) \frac{ds}{s\alpha(s)} \right\|_{L_\infty} \\ &\leq \sup_t \gamma(t) \int_t^1 \frac{ds}{s\alpha(s)} \|\varphi(s)f(s)\|_{L_\infty}. \end{aligned}$$

Since the inequality (5.1) ensures analogous boundedness in  $\tilde{L}_1$ , the norm of  $\psi(t)S_2f(t)$  is bounded in any  $\tilde{E}$ . The boundedness of  $\|\psi(t)S_1f(t)\|_{\tilde{E}}$  follows from Lemma 3.1.

Analogously, for the left endpoint we have (see the proof of Theorem 4.2)

$$\begin{aligned} \|\psi(t)S_1f(t)\|_{\tilde{L}_1} &= \left\| \frac{1}{\tilde{\gamma}(t)} \int_0^t \varphi(s)f(s)\tilde{\alpha}(s) \frac{ds}{s} \right\|_{\tilde{L}_1} \\ &= \int_0^1 \varphi(s)f(s)\tilde{\alpha}(s) \left( \int_s^1 \frac{dt}{t\tilde{\gamma}(t)} \right) \frac{ds}{s} \lesssim \|\varphi(s)f(s)\|_{\tilde{L}_1}. \end{aligned}$$

Using the inequality (5.2) and Lemma 3.2, we get again the boundedness of  $\|\psi(t)S_1f(t)\|_{\tilde{E}}$  and  $\|\psi(t)S_2f(t)\|_{\tilde{E}}$  for any  $\tilde{E}$  as required. ■

It can be checked that this theorem is applicable even when  $\pi_\alpha$  (or  $\pi_{\tilde{\alpha}}$ ) is positive, giving the same result as the Corollary from Section 3. It is applicable also to most of the general spaces of Lorentz-Zygmund type, giving the same results as in Section 5. More interesting is to consider the situations when functions  $\alpha$  (or  $\tilde{\alpha}$ ) grow slower than any power function but faster than any power of logarithm. We will study the ultrasymmetric spaces  $L_{\varphi,E}$  with parameter  $\varphi$  such that

$$(6.1) \quad \alpha(t) \approx \exp \left\{ -\ln^\varepsilon \frac{e}{t} \right\}, \quad 0 < \varepsilon < 1.$$

For  $\varepsilon \geq 1$ , this function has a power growth, but for  $\varepsilon < 1$  it has exactly an intermediate character as desired. Note also that this function satisfies the Lorentz condition from [8] and plays an important role in the theory of Orlicz spaces.

As before, we try to take the best possible  $\gamma(t)$  giving equality in the condition (5.1). We obtain

$$(6.2) \quad \gamma(t) = t\alpha'(t) \approx \exp \left\{ -\ln^\varepsilon \frac{e}{t} \right\} \left( \ln \frac{e}{t} \right)^{\varepsilon-1}.$$

It remains to check that these  $\gamma$  and  $\alpha$  satisfy the second condition from Theorem 6.1, part i). Let us estimate the integral

$$I = \int_t^1 \frac{ds}{s\alpha(s)} = \int_t^1 \exp \left\{ \ln^\varepsilon \frac{e}{s} \right\} \frac{ds}{s}.$$

The change of variable  $u = \ln \frac{e}{s}$  gives

$$I = \int_1^{\ln \frac{e}{t}} e^{u^\varepsilon} du = k \int_1^z v^{k-1} e^v dv, \quad \text{where } z = \left( \ln \frac{e}{t} \right)^\varepsilon, \quad k = \frac{1}{\varepsilon}.$$

Integration by parts enables us to separate the principal part of this integral (for  $z \rightarrow \infty$ ) and to get an estimate

$$I \lesssim z^{k-1} e^z = \left( \ln \frac{e}{t} \right)^{1-\varepsilon} \exp \left\{ \ln^\varepsilon \frac{e}{t} \right\} \approx \frac{1}{\gamma(t)}$$

as desired. Thus (6.2) gives a proper value of  $\gamma(t)$  for  $\alpha(t)$ , defined by (6.1). The corresponding decay factor

$$\kappa(t) = \left( \ln \frac{e}{t} \right)^{\varepsilon-1},$$

i.e., it is slighter than the classical  $\left( \ln \frac{e}{t} \right)^{-1}$ .

The considered example leads to an interesting problem: for which functions  $\alpha(t)$  the function  $\gamma(t) = t\alpha'(t)$  satisfies also the second condition from Theorem 6.1? A necessary and sufficient condition for this is given by the inequality

$$\int_0^s \frac{dt}{t \int_t^1 \frac{d\tau}{\tau \alpha(\tau)}} \gtrsim \alpha(s).$$

This relation can be simplified, defining a function

$$\delta \left( \ln \frac{1}{t} \right) = \int_t^1 \frac{d\tau}{\tau \alpha(\tau)} \iff \alpha(t) = \frac{1}{\delta' \left( \ln \frac{1}{t} \right)},$$

that gives a simple condition on the function  $\delta$ :

$$(6.3) \quad \int_s^\infty \frac{dt}{\delta(t)} \gtrsim \frac{1}{\delta'(s)}.$$

In particular, this condition is fulfilled when the derivative  $\delta'(t)$  is such that  $\delta'(\infty) = \infty$  and the ratio  $\delta'(t)/\delta(t)$  is a decreasing function. Indeed, this gives

$$\left( \frac{\delta'(t)}{\delta(t)} \right)' = \frac{\delta''(t)\delta(t) - (\delta'(t))^2}{\delta^2(t)} \leq 0 \implies \frac{1}{\delta(t)} \geq \frac{\delta''(t)}{(\delta'(t))^2}$$

and after integration from  $s$  to  $\infty$  we come to (6.3).

The last observation allows us to construct examples of ultrasymmetric spaces with arbitrarily slight decay factor in weak type interpolation. The choice  $\gamma(t) = t\alpha'(t)$  gives that

$$\kappa(t) = \frac{t\alpha'(t)}{\alpha(t)} \implies \kappa(e^{-u}) = -(\ln \alpha(e^{-u}))' = (\ln \delta'(u))',$$

i.e.,

$$\delta'(u) = \exp \left\{ \int \kappa(e^{-u}) du \right\}.$$

For example, in order to get

$$\kappa(t) \approx \left( \ln \ln \frac{e}{t} \right)^{-1} \quad (\text{for small } t),$$

one could take

$$\alpha(t) \approx \exp \left\{ - \ln \frac{e}{t} / \ln \ln \frac{e}{t} \right\}$$

(both equivalences can be obtained if one starts with the function  $\delta(u) = e^{u/\ln u} \ln u$ ).

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