A Parabolic Quasilinear Problem for Linear Growth Functionals

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Dedicated to Philippe Bénilan on the Occasion of His 60th Birthday

Abstract

We prove existence and uniqueness of solutions for the Dirichlet problem for quasilinear parabolic equations in divergent form for which the energy functional has linear growth. A typical example of energy functional we consider is the one given by the nonparametric area integrand $f(x,\xi) = \sqrt{1 + \|\xi\|^2}$, which corresponds with the time-dependent minimal surface equation. We also study the asymptotic behaviour of the solutions.

1. Introduction and preliminaries

Let Ω be a bounded set in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial\Omega$. We are interested in the problem

$$(1.1) \quad \begin{cases} \dfrac{\partial u}{\partial t} = div \ \mathbf{a}(x, Du) & \text{in} \quad Q = (0,\infty) \times \Omega \,, \\[2mm] u(t,x) = \varphi(x) & \text{on} \quad S = (0,\infty) \times \partial\Omega \,, \\[2mm] u(0,x) = u_0(x) & \text{in} \quad x \in \Omega \,, \end{cases}$$

where $u_0 \in L^2(\Omega)$ and $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$, f being a function with linear growth as $\|\xi\| \to \infty$.

A typical example of a function $f(x,\xi)$ satisfying the conditions we need is the nonparametric area integrand $f(x,\xi) = \sqrt{1 + \|\xi\|^2}$. Problem (1.1) for this particular f, that is, the time-dependent minimal surface equation,

has been studied in [12] and [17]. Another examples of problems of type
(1.1) included in our case, are the following: The evolution problem for
plastic antiplanar shear, studied in [21], which corresponds to the plasticity
functional f given by

$$f(\xi) = \begin{cases} \frac{1}{2}\|\xi\|^2 & \text{if} \quad \|\xi\| \leq 1, \\[2mm] \|\xi\| - \frac{1}{2} & \text{if} \quad \|\xi\| \geq 1, \end{cases}$$

and the evolution problems associated with the Lagrangians:

$$f(x, \xi) = \sqrt{1 + a_{ij}(x)\xi_i\xi_j}\,,$$

where the functions a_{ij} are continuous and satisfy $a_{ij}(x) = a_{ji}(x)$, $\|\xi\|^2 \leq a_{ij}(x)\xi_i\xi_j \leq C\|\xi\|^2$ for all $\xi \in \mathbb{R}^N$; and the Lagrangian

$$g(x, \xi) = \sqrt{1 + x^2 + \|\xi\|^2}\,,$$

which was considered by S. Bernstein ([8]). On the other hand, problem
(1.1) is studied in [14] for some Lagrangians f, which do not include the
nonparametric area integrand, but instead include the plasticity functional
and the total variation flow, that is, the case $f(\xi) = \|\xi\|$. Now, the concept of
solution given in [14] is the one obtained by considering the abstract Cauchy
problem in $L^2(\Omega)$ associated to the relaxed energy, but the subdifferential
of the energy functional is not characterized. For the particular case of
the total variation flow, we give in [4] a different approach to the Dirichlet
problem. There, we studied the problem in the framework of the L^1- theory,
and we characterized the subdifferential in $L^2(\Omega)$ of its relaxed energy (we
refer also to [3] where we treated the L^1-theory for the Neumann problem
for the total variational flow).

In general, problem (1.1) does not have a classical solution. The aim
of this paper is to introduce a concept of solution of the Dirichlet problem
(1.1), for which existence and uniqueness for initial data in $L^2(\Omega)$ is proved.
To do that we characterize the subdifferential of the energy associated with
the problem and we use the nonlinear semigroup theory. In a forthcoming
paper we will study the same problem in the framework of the L^1-theory, as
we did with the Dirichlet problem for the total variational flow ([4]).

In order to consider the relaxed energy we recall the definition of func-
tion of measure (see for instance, [6] or [12]). Let $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a
Carathéodory function such that

$$(1.2) \qquad |g(x, \xi)| \leq M(1 + \|\xi\|) \qquad \forall\, (x, \xi) \in \Omega \times \mathbb{R}^N,$$

for some constant $M \geq 0$. Furthermore, we assume that g possesses an asymptotic function, *i.e.* for almost all $x \in \Omega$ there exists the finite limit

$$(1.3) \qquad \lim_{t \to 0^+} tg\left(x, \frac{\xi}{t}\right) = g^0(x, \xi).$$

It is clear that the function $g^0(x, \xi)$ is positively homogeneous of degree one in ξ, *i.e.*

$$g^0(x, s\xi) = sg^0(x, \xi) \qquad \text{for all } x, \xi \text{ and } s > 0.$$

We denote by $\mathcal{M}(\Omega, \mathbb{R}^N)$ the set of all $\mathbb{R}^N$-valued bounded Radon measures on Ω. Given $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$, we consider its Lebesgue decomposition

$$\mu = \mu^a + \mu^s,$$

where μ^a is the absolutely continuous part of μ with respect to the Lebesgue measure λ_N of $\mathbb{R}^N$, and μ^s is singular with respect to λ_N. We denote by $\mu^a(x)$ the density of the measure μ^a with respect to λ_N and by $(d\mu^s/d|\mu|^s)(x)$ the density of μ^s with respect to $|\mu|^s$.

Given $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$, we define $\tilde{\mu} \in \mathcal{M}(\Omega, \mathbb{R}^{N+1})$ by

$$\tilde{\mu}(B) := \big(\mu(B), \lambda_N(B)\big),$$

for every Borel set $B \subset \mathbb{R}^N$. Then, we have

$$\tilde{\mu} = \tilde{\mu}^a + \tilde{\mu}^s = \tilde{\mu}^a(x)\lambda_N + \tilde{\mu}^s = (\mu^a(x), \mathbf{1})\lambda_N + (\mu^s, 0).$$

Hence, we have

$$|\tilde{\mu}^s| = |\mu^s|, \qquad \frac{d\tilde{\mu}^s}{d|\tilde{\mu}^s|} = \left(\frac{d\mu^s}{d|\mu|^s}, 0\right) \quad |\mu^s| - \text{a.e.}$$

For $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$ and g satisfying the above conditions, we define the measure $g(x, \mu)$ on Ω as

$$(1.4) \qquad \int_B g(x, \mu) := \int_B g(x, \mu^a(x)) \, dx + \int_B g^0\left(x, \frac{d\mu^s}{d|\mu|^s}(x)\right) d|\mu|^s$$

for all Borel set $B \subset \Omega$. In formula (1.4) we may write $(d\mu/d|\mu|)(x)$ instead of $(d\mu^s/d|\mu|^s)(x)$, because the two functions are equal $|\mu|^s$-a.e.

Another way of writing the measure $g(x, \mu)$ is the following. Let us consider the function $\tilde{g} : \Omega \times \mathbb{R}^N \times [0, +\infty[\longrightarrow \mathbb{R}$ defined as

$$(1.5) \qquad \tilde{g}(x, \xi, t) := \begin{cases} g\left(x, \dfrac{\xi}{t}\right)t & \text{if} \quad t > 0, \\[2mm] g^0(x, \xi) & \text{if} \quad t = 0. \end{cases}$$

As it is proved in [6], if g is a Carathéodory function satisfying (1.2), then one has

$$(1.6) \qquad \int_B g(x,\mu) = \int_B \tilde{g}\left(x, \frac{d\mu}{d\alpha}(x), \frac{d\lambda_N}{d\alpha}(x)\right) d\alpha,$$

where α is any positive Borel measure such that $|\mu| + \lambda_N \ll \alpha$.

Due to the linear growth condition on the Lagrangian, the natural energy space to study (1.1) is the space of functions of bounded variation. Let us recall several facts concerning functions of bounded variation (for further information concerning functions of bounded variation we refer to [13], [22] or [2]).

A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, \ldots, \mu_N$ defined in Ω with finite total mass in Ω and

$$(1.7) \qquad \int_\Omega u D_i \varphi \, dx = -\int_\Omega \varphi \, d\mu_i$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \ldots, N$. Thus the gradient of u is a vector valued measure with finite total variation

$$(1.8)$$
$$\| Du \| = \sup\left\{ \int_\Omega u \, div \, \varphi \, dx : \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), |\varphi(x)| \le 1 \text{ for } x \in \Omega \right\}.$$

The space $BV(\Omega)$ is endowed with the norm

$$(1.9) \qquad \| u \|_{BV} = \| u \|_{L^1(\Omega)} + \| Du \|.$$

For $u \in BV(\Omega)$, the gradient Du is a Radon measure that decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \, \lambda_N$ where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure λ_N. There is also the polar decomposition $D^s u = \overrightarrow{D^s u}|D^s u|$ where $|D^s u|$ is the total variation measure of $D^s u$.

We shall need several results from [5] (see also [16]). Following [5], let

$$(1.10) \qquad X(\Omega) = \{z \in L^\infty(\Omega, \mathbb{R}^N) : div(z) \in L^1(\Omega)\}.$$

If $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ we define the functional $(z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$(1.11) \qquad < (z, Dw), \varphi >= -\int_\Omega w \, \varphi \, div(z) \, dx - \int_\Omega w \, z \cdot \nabla\varphi \, dx.$$

Then (z, Dw) is a Radon measure in Ω,

$$(1.12) \qquad \int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \, dx \qquad \forall \, w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$$

and

$$(1.13) \qquad \left| \int_B (z, Dw) \right| \le \int_B |(z, Dw)| \le \|z\|_\infty \int_B \|Dw\|$$

for any Borel set $B \subseteq \Omega$. Moreover, (z, Dw) is absolutely continuous with respect to $\|Dw\|$ with Radon-Nikodym derivative $\theta(z, Dw, x)$ which is a $\|Dw\|$ measurable function from Ω to $\mathbb{R}$ such that

$$(1.14) \qquad \int_B (z, Dw) = \int_B \theta(z, Dw, x) \|Dw\|$$

for any Borel set $B \subseteq \Omega$. We also have that

$$(1.15) \qquad \|\theta(z, Dw, .)\|_{L^\infty(\Omega, \|Dw\|)} \le \|z\|_{L^\infty(\Omega, \mathbb{R}^N)}.$$

By writing

$$z \cdot D^s u := (z, Du) - (z \cdot \nabla u) \, d\lambda_N,$$

we see that $z \cdot D^s u$ is a bounded measure. Furthermore, in [16] it is proved that $z \cdot D^s u$ is absolutely continuous with respect to $|D^s u|$ (and, thus, it is a singular measure with respect to λ_N), and

$$(1.16) \qquad |z \cdot D^s u| \le \|z\|_\infty |D^s u|.$$

As a consequence of Theorem 2.4 of [5], we have:

$$(1.17) \qquad \text{If} \ \ z \in X(\Omega) \cap C(\Omega, \mathbb{R}^N), \ \ \text{then} \ \ z \cdot D^s u = (z \cdot \overrightarrow{D^s u}) \, d|D^s u|.$$

In [5], a weak trace on $\partial\Omega$ of the normal component of $z \in X(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator $\gamma : X(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \le \|z\|_\infty$$

$$\gamma(z)(x) = z(x) \cdot \nu(x) \quad \text{for all } x \in \partial\Omega \ \text{ if } \ z \in C^1(\overline{\Omega}, \mathbb{R}^N).$$

We shall denote $\gamma(z)(x)$ by $[z, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[z, \nu]$ and the measure (z, Dw), for $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, is established:

$$(1.18) \qquad \int_\Omega w \, div(z) \, dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, \nu] w \, dH^{N-1}.$$

Let g be a function satisfying (1.2). Then for every $u \in BV(\Omega)$ we have the measure $g(x, Du)$ defined by

$$\int_B g(x, Du) = \int_B g(x, \nabla u(x))\ dx + \int_B g^0(x, \overrightarrow{D^s u}(x))\ d|D^s u|$$

for all Borel set $B \subset \Omega$. If we assume that Ω has a Lipschitz boundary, and that $g(x, \xi)$ is defined also for $x \in \partial\Omega$, we may consider the functional G in $BV(\Omega)$ defined by

$$(1.19) \qquad G(u) := \int_\Omega g(x, Du) + \int_{\partial\Omega} g^0\big(x, \nu(x)[\varphi(x) - u(x)]\big)\ dH^{N-1},$$

where $\varphi \in L^1(\partial\Omega)$ is a given function and ν is the outer unit normal to $\partial\Omega$. It is proved in [6] that, if $\tilde{g}(x, \xi, t)$ is continuous on $\overline{\Omega} \times \mathbb{R}^N \times [0, +\infty[$ and convex in (ξ, t) for each fixed $x \in \overline{\Omega}$, then G is the greatest functional on $BV(\Omega)$ which is lower-semicontinuous with respect to the $L^1(\Omega)$-convergence and satisfies

$$G(u) \leq \int_\Omega g(x, \nabla u(x))\ dx$$

for all functions $u \in C^1(\Omega) \cap W^{1,1}(\Omega)$ with $u = \varphi$ on $\partial\Omega$.

The paper is organized as follows: in Section 2 we give the definition of solution for the Dirichlet problem and we state the existence and uniqueness result for this type of solutions. Section 3 is devoted to prove the existence and uniqueness result. To do that, we study the problem from the point of view of nonlinear semigroup theory. We characterize the subdifferential in $L^2(\Omega)$ of the relaxed energy functional associated with the problem. In Section 4 we give a weakened form of the maximum principle and we study the asymptotic behaviour of solutions proving that they stabilize as $t \to \infty$ by converging to a solution of the steady-state problem. Finally, the Appendix contains the proof of the approximation Lemma stated in Section 3.

2. The existence and uniqueness result

In this section we define the concept of solution for the Dirichlet problem (1.1) and we state the existence and uniqueness result for this type of solutions when the initial data are in $L^2(\Omega)$.

Here we assume that Ω is an open bounded set in $\mathbb{R}^N$, $N \geq 2$, with boundary $\partial\Omega$ of class C^1, and the Lagrangian $f : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumptions, which we shall refer collectively as (H):

(H$_1$) f is continuous on $\overline{\Omega} \times \mathbb{R}^N$ and is a convex diffentiable function of ξ with continuous gradient for each fixed $x \in \Omega$. Further we require f to satisfy the linear growth condition

$$(2.1) \qquad C_0 \|\xi\| - C_1 \le f(x, \xi) \le M(\|\xi\| + C_2).$$

for some positive constants C_0, C_1, C_2. Moreover, f^0 exists and $f^0(x, -\xi) = f^0(x, \xi)$ for all $\xi \in \mathbb{R}^N$ and all $x \in \overline{\Omega}$.

(H$_2$) $\tilde{f}(x, \xi, t)$ is continuous on $\overline{\Omega} \times \mathbb{R}^N \times [0, +\infty[$ and convex in (ξ, t) for each fixed $x \in \overline{\Omega}$.

We consider the function $\mathbf{a}(x, \xi) = \nabla_\xi f(x, \xi)$ associated to the Lagrangian f. By the convexity of f

$$(2.2) \qquad \mathbf{a}(x, \xi) \cdot (\eta - \xi) \le f(x, \eta) - f(x, \xi),$$

and the following monotonicity condition is satisfied

$$(2.3) \qquad (\mathbf{a}(x, \eta) - \mathbf{a}(x, \xi)) \cdot (\eta - \xi) \ge 0.$$

Moreover, it is easy to see that

$$(2.4) \qquad |\mathbf{a}(x, \xi)| \le M \qquad \forall\, (x, \xi) \in \Omega \times \mathbb{R}^N.$$

We consider the function $h : \Omega \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$h(x, \xi) := \mathbf{a}(x, \xi) \cdot \xi.$$

From (2.2) and (2.1), it follows that

$$(2.5) \qquad C_0 \|\xi\| - D_1 \le h(x, \xi) \le M \|\xi\|$$

for some positive constant D_1.

We assume that

(H$_3$) $h(x, \xi) \ge 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$, h^0 exists and the function $\tilde{h}$ is continuous on $\overline{\Omega} \times \mathbb{R}^N \times [0, +\infty[$.

We need to consider the mapping $\mathbf{a}^\infty$ defined by

$$\mathbf{a}^\infty(x, \xi) := \lim_{t \to +\infty} \mathbf{a}(x, t\xi).$$

Observe that

$$h^0(x, \xi) = \mathbf{a}^\infty(x, \xi) \cdot \xi \quad \text{and} \quad C_0 \|\xi\| \le h^0(x, \xi) \le M \|\xi\|.$$

(H$_4$) $\mathbf{a}^\infty(x,\xi) = \nabla_\xi f^0(x,\xi)$ for all $\xi \neq 0$ and all $x \in \overline{\Omega}$.

In particular, as a consequence of Euler's Theorem, we have

$$f^0(x,\xi) = \mathbf{a}^\infty(x,\xi) \cdot \xi = h^0(x,\xi),$$

for all $\xi \in \mathbb{R}^N$ and all $x \in \overline{\Omega}$.

(H$_5$) $\mathbf{a}(x,\xi) \cdot \eta \leq h^0(x,\eta)$ for all $\xi, \eta \in \mathbb{R}^N$, and all $x \in \overline{\Omega}$.

Either from (H$_4$) or (H$_5$) it follows that $\mathbf{a}^\infty(x,\xi) \cdot \eta \leq h^0(x,\eta)$ for all $\xi, \eta \in \mathbb{R}^N$, $\xi \neq 0$, and all $x \in \overline{\Omega}$. Indeed, it suffices to replace ξ by $t\xi$ in (H$_5$) and let $t \to +\infty$.

Definition 2.1 Let $\varphi \in L^1(\partial\Omega)$ and $u_0 \in L^2(\Omega)$. A measurable function $u : (0,T) \times \Omega \to \mathbb{R}$ is a *solution* of (1.1) in $Q_T = (0,T) \times \Omega$ if $u \in C([0,T], L^2(\Omega))$, $u(0) = u_0$, $u'(t) \in L^2(\Omega)$, $u(t) \in BV(\Omega) \cap L^2(\Omega)$, $\mathbf{a}(x, \nabla u(t)) \in X(\Omega)$ a.e. $t \in [0,T]$, and for almost all $t \in [0,T]$ $u(t)$ satisfies:

(2.6) $u'(t) = div(\mathbf{a}(x, \nabla u(t))$ in $\mathcal{D}'(\Omega)$

(2.7) $\mathbf{a}(x, \nabla u(t)) \cdot D^s u(t) = f^0(x, D^s u(t))$

(2.8) $[\mathbf{a}(x, \nabla u(t)), \nu] \in \text{sign}(\varphi - u(t)) f^0(x, \nu(x))$ $H^{N-1} - a.e.$ on $\partial\Omega$.

Our main result is the following:

Theorem 2.2 *Let $\varphi \in L^1(\partial\Omega)$ and assume we are under assumptions (H). Given $u_0 \in L^2(\Omega)$, there exists a unique solution u of (1.1) in Q_T for every $T > 0$ such that $u(0) = u_0$.*

3. Strong solution for data in $L^2(\Omega)$

To prove Theorem 2.2 we shall use the nonlinear semigroup theory ([9]). For $\varphi \in L^1(\partial\Omega)$ we define the energy functional associated with the problem (1.1) $\Phi_\varphi : L^2(\Omega) \to [0, +\infty]$ by

$$\Phi_\varphi(u) := \int_\Omega f(x, Du) + \int_{\partial\Omega} f^0(x, \nu(x)[\varphi - u]) \, dH^{N-1}$$

if $u \in BV(\Omega) \cap L^2(\Omega)$ and

$$\Phi_\varphi(u) := +\infty \quad \text{if } u \in L^2(\Omega) \setminus BV(\Omega).$$

Note that, on the boundary, the integrand can be written in the form

$$f^0(x, \nu(x)[\varphi - u]) = |\varphi - u| f^0(x, \nu(x)).$$

Functional Φ_φ is clearly convex and has the form given in (1.19). Then, as a consequence of the Anzellotti's result ([6]) we have that Φ_φ is lower-semicontinuous. Therefore, the subdifferential $\partial \Phi_\varphi$ of Φ_φ, *i.e.* the operator in $L^2(\Omega)$ defined by

$$v \in \partial \Phi_\varphi(u) \iff \Phi_\varphi(w) - \Phi_\varphi(u) \geq \int_\Omega v(w - u) \, dx, \quad \forall \, w \in L^2(\Omega)$$

is a maximal monotone operator in $L^2(\Omega)$. Consequently, the existence and uniqueness of a solution of the abstract Cauchy problem

(3.1)
$$\begin{cases} u'(t) + \partial \Phi_\varphi(u(t)) \ni 0 & t \in]0, \infty[\\ u(0) = u_0 & u_0 \in L^2(\Omega) \end{cases}$$

follows immediately from the nonlinear semigroup theory (see [9]). Now, to get the full strength of the abstract result derived from semigroup theory we need to characterize $\partial \Phi_\varphi$. To get this characterization, we introduce the following operator $\mathcal{B}_\varphi$ in $L^2(\Omega)$. $(u, v) \in \mathcal{B}_\varphi \iff u \in BV(\Omega) \cap L^2(\Omega)$, $v \in L^2(\Omega)$ and $a(x, \nabla u) \in X(\Omega)$ satisfies:

(3.2)
$$-v = div \; \mathbf{a}(x, \nabla u) \qquad \text{in} \;\; \mathcal{D}'(\Omega)$$

(3.3)
$$\mathbf{a}(x, \nabla u) \cdot D^s u = f^0(x, D^s u) = f^0(x, \overrightarrow{D^s u})|D^s u|,$$

(3.4)
$$[\mathbf{a}(x, \nabla u), \nu] \in \text{sign} \; (\varphi - u) f^0(x, \nu(x)) \qquad H^{N-1} - \text{a.e.}.$$

Let $(u, v) \in \mathcal{B}_\varphi$, and $w \in BV(\Omega) \cap L^2(\Omega)$. Multiplying (3.2) by $w - u$, and using Green's formula (1.18), we obtain

$$\int_\Omega (w - u)v dx = - \int_\Omega (w - u) \; div \; \mathbf{a}(x, \nabla u) \; dx =$$

$$\int_\Omega (\mathbf{a}(x, \nabla u), Dw - Du) - \int_{\partial \Omega} [\mathbf{a}(x, \nabla u), \nu](w - u) \; dH^{N-1} =$$

$$= \int_\Omega (\mathbf{a}(x, \nabla u), Dw) - \int_{\partial \Omega} [\mathbf{a}(x, \nabla u), \nu](w - \varphi) \; dH^{N-1} -$$

$$- \int_\Omega (\mathbf{a}(x, \nabla u), Du) - \int_{\partial \Omega} [\mathbf{a}(x, \nabla u), \nu](\varphi - u) \; dH^{N-1} =$$

$$= \int_\Omega (\mathbf{a}(x, \nabla u), Dw) - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu](w - \varphi) \, dH^{N-1} -$$

$$- \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla u \, dx - \int_\Omega \mathbf{a}(x, \nabla u) \cdot D^s u - \int_{\partial\Omega} |\varphi - u| f^0(x, \nu(x)) \, dH^{N-1} =$$

$$= \int_\Omega (\mathbf{a}(x, \nabla u), Dw) - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu](w - \varphi) \, dH^{N-1} -$$

$$- \int_\Omega h(x, Du) - \int_{\partial\Omega} |\varphi - u| f^0(x, \nu(x)) \, dH^{N-1}.$$

Therefore, if $(u, v) \in \mathcal{B}_\varphi$, we have that

(3.5)
$$\int_\Omega (w - u) v \, dx = \int_\Omega (\mathbf{a}(x, \nabla u), Dw) - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu](w - \varphi) \, dH^{N-1}$$

$$- \int_\Omega h(x, Du) - \int_{\partial\Omega} |\varphi - u| f^0(x, \nu(x)) \, dH^{N-1},$$

for all $w \in BV(\Omega) \cap L^2(\Omega)$.

Theorem 3.1 *Let $\varphi \in L^1(\partial\Omega)$. Assume we are under assumptions (H), then the operator $\partial\Phi_\varphi$ has dense domain in $L^2(\Omega)$ and*

$$\partial\Phi_\varphi = \mathcal{B}_\varphi.$$

We note that, in the particular case of the nonparametric area integrand $f(x, \xi) = \sqrt{1 + \|\xi\|^2}$, the characterization of the subdifferential of Φ_φ given in Theorem 3.1 coincides with the one given by F. Demengel and R. Temam in [12], Theorem 3.1, where they used a different approach. More precisely, they characterized the subdifferential by means of the duality method of convex optimization introduced by R. T. Rockafellar in [19]. To prove Theorem 3.1 we need the following proposition.

Proposition 3.2 *Let $\varphi \in L^1(\partial\Omega)$. Assume we are under assumptions (H), then $L^\infty(\Omega) \subset R(I + \mathcal{B}_\varphi)$ and $D(\mathcal{B}_\varphi)$ is dense in $L^2(\Omega)$.*

We need to introduce the following sequence of auxiliar operators. For $\varphi \in W^{\frac{1}{2}, 2}(\Omega)$, let

$$W_\varphi^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) \; : \; u_{|\partial\Omega} = \varphi \quad H^{N-1} - \text{a.e.}\}.$$

For every $n \in \mathbb{N}$, consider $\mathbf{a}_n(x, \xi) := \mathbf{a}(x, \xi) + \xi/n$. We define the operator $A_{n,\varphi}$ in $L^2(\Omega)$:

$$(u, v) \in A_{n,\varphi} \iff u \in W_\varphi^{1,2}(\Omega) \cap L^\infty(\Omega), v \in L^2(\Omega), \text{and}$$

$$\int_\Omega (w - u)v \; dx \le \int_\Omega \mathbf{a}_n(x, \nabla u) \cdot \nabla(w - u) \; dx \qquad \forall \; w \in W_\varphi^{1,2}(\Omega).$$

A similar proof to the one given in Proposition 1 of [4] give us the following result.

Lemma 3.3 *Let $\varphi \in W^{\frac{1}{2},2}(\partial\Omega) \cap L^\infty(\partial\Omega)$. Then for every $n \in \mathbb{N}$ the operator $A_{n,\varphi}$ satisfies*

$$L^\infty(\Omega) \subset R(I + A_{n,\varphi}).$$

We also need an approximation lemma similar to the one given by Anzellotti in [7]. The proof of this lemma will be given in the Appendix.

Lemma 3.4 *Let Ω be an open bounded set in $\mathbb{R}^N$, $N \ge 2$, and assume that $\partial\Omega$ is of class C^1. If $v, u \in BV(\Omega)$ and $g \in L^1(\partial\Omega)$, then there exists a sequence of functions $v_j \in C^1(\overline{\Omega})$ such that*

$$(3.6) \qquad\qquad v_j \to g \qquad in \;\; L^1(\partial\Omega),$$

$$(3.7) \qquad\qquad v_j \to v \qquad in \;\; L^{N/(N-1)}(\Omega),$$

$$(3.8) \int_\Omega \sqrt{1 + |\nabla v_j(x)|^2}dx \to \int_\Omega \sqrt{1 + |Dv(x)|^2}dx + \int_{\partial\Omega} |g - v|dH^{N-1},$$

$$(3.9) \qquad\qquad \nabla v_j(x) \to \nabla v(x) \quad \lambda_N\text{-}a.e. \;\; in \;\Omega,$$

$$(3.10) \quad |\nabla v_j(x)| \to \infty \;\; and \;\; \frac{\nabla v_j(x)}{|\nabla v_j(x)|} \to \frac{Dv(x)}{|Dv(x)|} \quad |Dv|^s \;\; a.e. \;\; in \;\Omega,$$

$$(3.11) \quad |\nabla v_j(x)| \to \infty \;\; and \;\; \frac{\nabla v_j(x)}{|\nabla v_j(x)|} \to \frac{Du(x)}{|Du(x)|} \quad |Du|^{ss} \;\; a.e. \;\; in \;\Omega,$$

where $|Du|^{ss}$ denotes the part of the singular measure $|Du|^s$ which is singular with respect to $|Dv|^s$,

$$(3.12) \qquad |\nabla v_j(x)| \to \infty \;\; and \;\; \frac{\nabla v_j(x)}{|\nabla v_j(x)|} \to \frac{g(x) - v(x)}{|g(x) - v(x)|}\nu(x)$$

$$H^{N-1} \;\; a.e. \;\; in \; \{x \in \Omega : g(x) \ne v(x)\}$$

$$(3.13) \qquad |\nabla v_j(x)| \to \infty \;\; and \;\; \frac{\nabla v_j(x)}{|\nabla v_j(x)|} \to \frac{v(x) - u(x)}{|v(x) - u(x)|}\nu(x)$$

$$H^{N-1} \;\; a.e. \;\; in \; \{x \in \Omega : g(x) = v(x), u(x) \ne v(x)\}$$

The next Lemmas will be used to prove Theorem 3.1 and Proposition 3.2.

Lemma 3.5 *Let $\varphi, \varphi_n \in L^1(\partial\Omega)$, $\varphi_n \to \varphi$ in $L^1(\partial\Omega)$. Let $u_n, u \in BV(\Omega)$ and $z \in X(\Omega)$ with $div(z) \in L^2(\Omega)$. We assume that*

(3.14) $$\Phi_\varphi(u_n) \to \Phi_\varphi(u),$$

(3.15) $$\mathbf{a}(x, \nabla u_n) \rightharpoonup z \quad weakly^* \ in \ L^\infty(\Omega),$$

(3.16) $$|[z, \nu(x)]| \le f^0(x, \nu(x)) \quad a.e. \ in \ \partial\Omega,$$

(3.17) $$|z \cdot D^s u| \le f^0(x, D^s u) \quad as \ measures \ in \ \Omega,$$

(3.18)
$$\lim_{n\to\infty} \int_\Omega h(x, Du_n) + \int_{\partial\Omega} |u_n - \varphi_n| f^0(x, \nu(x)) \, dH^{N-1} =$$
$$= \int_\Omega h(x, Du) + \int_{\partial\Omega} |u - \varphi| \, f^0(x, \nu(x)) \, dH^{N-1}.$$

(3.19)
$$\int_\Omega h(x, Du) + \int_{\partial\Omega} |u - \varphi| \, f^0(x, \nu(x)) \, dH^{N-1}$$
$$\le \int_\Omega (z, Du) + \int_{\partial\Omega} [z, \nu](\varphi - u) \, dH^{N-1}.$$

Then

(3.20) $$\int_\Omega z \cdot \nabla u \, dx = \int_\Omega h(x, \nabla u) \, dx = \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla u \, dx \, ,$$

(3.21) $$z \cdot D^s u = f^0(x, D^s u) \, ,$$

(3.22) $$[z, \nu] \in \text{sign} \, (\varphi - u) f^0(x, \nu(x)) \qquad H^{N-1} - a.e.$$

Proof. By the convexity of f, we have

$$\int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u \, dx \le$$
$$\le \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx + \int_\Omega f(x, \nabla u) \, dx - \int_\Omega f(x, \nabla u_n) \, dx$$
$$\le \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx + \int_\Omega f^0(x, D^s u_n) + \int_{\partial\Omega} |u_n - \varphi_n| f^0(x, \nu(x)) dH^{N-1}$$
$$\quad + \int_{\partial\Omega} |\varphi_n - \varphi| f^0(x, \nu(x)) \, dH^{N-1} + \int_\Omega f(x, \nabla u) \, dx$$
$$\quad - \left(\int_\Omega f(x, \nabla u_n) \, dx + \int_\Omega f^0(x, D^s u_n) + \int_{\partial\Omega} |u_n - \varphi| f^0(x, \nu(x)) \right)$$
$$= \int_\Omega h(x, Du_n) \, dx + \int_{\partial\Omega} |u_n - \varphi_n| f^0(x, \nu(x)) \, dH^{N-1}$$
$$\quad + \int_{\partial\Omega} |\varphi_n - \varphi| f^0(x, \nu(x)) \, dH^{N-1} + \int_\Omega f(x, \nabla u) \, dx - \Phi_\varphi(u_n).$$

Letting $n \to \infty$, and using (3.14), (3.15) and (3.18), we obtain

$$\int_\Omega z \cdot \nabla u \; dx \leq$$

$$\leq \int_\Omega h(x, Du) + \int_{\partial\Omega} |u - \varphi| f^0(x, \nu(x)) dH^{N-1} + \int_\Omega f(x, \nabla u) dx - \Phi_\varphi(u)$$

$$= \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla u \; dx.$$

Now, since, using (3.16) and (3.17), we have

$$|[z, \nu](\varphi - u)| \leq |u - \varphi| f^0(x, \nu(x)),$$

and

$$|z \cdot D^s u| \leq f^0(x, D^s u).$$

Hence from (3.19), we obtain (3.20), (3.21) and (3.22). ∎

Lemma 3.6 i) Let $u_n \in BV(\Omega) \cap L^2(\Omega)$ and $z \in X(\Omega)$. Suppose that

$$(3.23) \qquad \mathbf{a}(x, \nabla u_n) \rightharpoonup z \qquad weakly^* \;\; in \;\; L^\infty(\Omega, \mathbb{R}^N)$$

and

$$(3.24) \qquad div\,(\mathbf{a}(x, \nabla u_n)) \rightharpoonup div\,z \qquad weakly \;\; in \;\; L^2(\Omega).$$

Then

$$(3.25) \qquad [\mathbf{a}(x, \nabla u_n), \nu(x)] \rightharpoonup [z, \nu(x)] \quad weakly \; in \; L^2(\partial\Omega) \;\; and$$

$$(3.26) \qquad |z(x) \cdot \nu(x)| \leq f^0(x, \nu(x)) \quad a.e. \; in \; \partial\Omega.$$

ii) Let $u_n \in W^{1,2}(\Omega)$. Let $\mathbf{a}_n(x, \xi) = \mathbf{a}(x, \xi) + \xi/n$. Suppose that

$$(3.27) \qquad \|u_n\|_2 \quad is \; bounded \; in \; L^2(\Omega),$$

$$(3.28) \qquad \frac{1}{n} |\nabla u_n| \to 0 \quad in \; L^2(\Omega),$$

$$(3.29) \qquad \mathbf{a}_n(x, \nabla u_n) \rightharpoonup z \qquad weakly \;\; in \;\; L^2(\Omega, \mathbb{R}^N)$$

and

$$(3.30) \qquad div\,(\mathbf{a}_n(x, \nabla u_n)) \rightharpoonup div\,z \qquad weakly \;\; in \;\; L^2(\Omega).$$

Then

$$(3.31) \qquad [\mathbf{a}_n(x, \nabla u_n), \nu(x)] \rightharpoonup [z, \nu(x)] \quad weakly \; in \; W^{1/2,2}(\partial\Omega)^* \;\; and$$

$$(3.32) \qquad |[z(x), \nu(x)]| \leq f^0(x, \nu(x)) \quad a.e. \; in \; \partial\Omega.$$

Proof. Since both proofs are based on similar arguments, we shall only prove ii). Observe that, if $\sigma \in L^2(\Omega, \mathbb{R}^N)$ and $div(\sigma) \in L^2(\Omega)$, we can define $[\sigma, \nu]$ using the integration by parts formula

$$(3.33) \qquad \int_{\partial\Omega} [\sigma, \nu]\psi = \int_{\Omega} div(\sigma)\psi + \int_{\Omega} \sigma \cdot \nabla\psi.$$

for all $\psi \in W^{1,2}(\Omega)$. This is consistent with the classical notion of trace at the boundary and it defines $[\sigma, \nu]$ as an element of $W^{1/2,2}(\partial\Omega)^*$. According to the assumptions (3.29), (3.30) we have that $[\mathbf{a}_n(x, \nabla u_n), \nu(x)] \to [z, \nu(x)]$ weakly in $W^{1/2,2}(\partial\Omega)^*$. In i), the analogous conclusion (3.25) follows from the results in [5] and the fact that $\mathbf{a}(x, \nabla u_n)$ is uniformly bounded in $L^\infty(\Omega)$. In this case, the traces $[\mathbf{a}(x, \nabla u_n), \nu(x)]$ are in $L^\infty(\partial\Omega)$.

To prove (3.32), again, we observe that [16] if $\sigma \in L^2(\Omega, \mathbb{R}^N)$ and $div(\sigma) \in L^2(\Omega)$, then there is a sequence $\sigma_k \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$ satisfying

$$(3.34) \qquad\qquad \sigma_k \to \sigma \quad \text{in } L^2(\Omega, \mathbb{R}^N),$$
$$(3.35) \qquad\qquad div\,\sigma_k \to div\,\sigma \quad \text{in } L^2(\Omega).$$

We recall the construction in [16]. We use a partition of unity θ_j, $j = 1, 2, ...p$, in $\overline{\Omega}$ with $0 \le \theta_j \le 1$, $\theta_j \in C_0^\infty(\mathbb{R}^N)$, such that if the support of θ_j intersects $\partial\Omega$, then for some bounded open cone K_j with vertex 0, every $x \in \partial\Omega \cap supp(\theta_j)$ satisfies $(x + K_j) \cap \overline{\Omega} = \emptyset$, and for some $r > 0$, every $x \in \partial\Omega \cap (supp(\theta_j) + B(0, r))$ satisfies $(x - K_j) \subset \Omega$. For each j, we choose $\rho_j \in C_0^\infty(\mathbb{R}^N)$, $0 \le \rho_j \le 1$, with $\int_{\mathbb{R}^N} \rho_j\,dx = 1$, and let $\rho_{j,k}(x) = k^N \rho_j(kx)$. If j is such that the support of θ_j intersects $\partial\Omega$, we choose ρ_j such that $supp(\rho_j) \subseteq K_j$. Then we define

$$\sigma_k = \sum_{j=1}^{p} \rho_{j,k} * (\theta_j\,\sigma\,\chi_\Omega).$$

As it was proved in [16], σ_k satisfies (3.34) and (3.35). As in the first part of the proof, we have that

$$\int_{\partial\Omega} [\sigma_k, \nu]\psi \to \int_{\partial\Omega} [\sigma, \nu]\psi$$

for all $\psi \in W^{1,2}(\Omega)$. We shall use this observation for $\sigma = \mathbf{a}_n(x, \nabla u_n)$. Previously, we extend u_n as a function in $W^{1,2}(\mathbb{R}^N)$ such that $\|u_n\|_{W^{1,2}(\mathbb{R}^N)} \le C\|u_n\|_{W^{1,2}(\Omega)}$ for some constant $C > 0$ depending only on Ω ([1]). Then we define

$$\mathbf{a}_{n,k}(x, \nabla u_n) = \sum_{j=1}^{p} \rho_{j,k} * \left(\theta_j \mathbf{a}(x, \nabla u_n)\chi_\Omega + \theta_j \frac{\nabla u_n}{n}\right).$$

Now, since $\mathbf{a}_{n,k}(x, \nabla u_n) \in C^\infty(\overline{\Omega})$, $[\mathbf{a}_{n,k}(x, \nabla u_n), \nu(x)]$ can be understood in a classical sense. Let $\psi \in W^{1/2,2}(\partial\Omega)$. We may write

$$\int_{\partial\Omega} [\mathbf{a}_{n,k}(x, \nabla u_n), \nu(x)]\psi = \sum_{j=1}^{p} \int_{\partial\Omega} [\rho_{j,k} * (\theta_j \mathbf{a}(x, \nabla u_n)\chi_\Omega), \nu(x)]\psi(x) +$$

$$+ \sum_{j=1}^{p} \frac{1}{n} \int_{\partial\Omega} [\rho_{j,k} * (\theta_j \nabla u_n), \nu(x)]\psi(x).$$

By taking k sufficiently large, we may assume that all θ_j used in the above expression are such that $supp(\theta_j)$ intersects $\partial\Omega$. We observe that

$$\int_{\partial\Omega} |[\rho_{j,k} * (\theta_j \mathbf{a}(x, \nabla u_n)\chi_\Omega), \nu(x)]||\psi(x)| \leq$$

$$\leq \int_{\partial\Omega} \int_\Omega \rho_{j,k}(x-y)\,\theta_j(y)\left|\mathbf{a}(y, \nabla u_n(y)) \cdot \nu(x)\right||\psi(x)|\,dy\,dx$$

$$\leq \int_{\partial\Omega} \int_\Omega \rho_{j,k}(x-y)\,\theta_j(y)\,f^0(y, \nu(x))\,|\psi(x)|\,dy\,dx.$$

Since

$$\nabla u_n \theta_j = \nabla(u_n \theta_j) - u_n \nabla \theta_j,$$

we may write

$$\int_{\partial\Omega} [\rho_{j,k} * (\theta_j \nabla u_n), \nu(x)]\psi(x) =$$

$$= \int_{\partial\Omega} [\nabla \rho_{j,k} * (\theta_j u_n), \nu(x)]\psi(x) - \int_{\partial\Omega} [\rho_{j,k} * (\nabla \theta_j u_n), \nu(x)]\psi(x).$$

We estimate both integrals in the right hand side of the above expression. First,

$$\left|\int_{\partial\Omega} [\nabla \rho_{j,k} * (\theta_j u_n), \nu(x)]\psi(x)\right| \leq$$

$$\leq \left\|\frac{\partial}{\partial \nu}\big(\rho_{j,k} * (u_n \theta_j)\big)\right\|_{W^{1/2,2}(\partial\Omega)^*} \|\psi\|_{W^{1/2,2}(\partial\Omega)}$$

$$\leq C\|\rho_{j,k} * (u_n \theta_j)\|_{W^{1,2}(\Omega)}\,\|\psi\|_{W^{1/2,2}(\partial\Omega)}$$

$$\leq C\|\rho_{j,k} * (u_n \theta_j)\|_{W^{1,2}(\mathbb{R}^N)}\|\psi\|_{W^{1/2,2}(\partial\Omega)}$$

$$\leq C\|u_n \theta_j\|_{W^{1,2}(\mathbb{R}^N)}\|\psi\|_{W^{1/2,2}(\partial\Omega)} \leq C\|u_n\|_{W^{1,2}(\mathbb{R}^N)}\|\psi\|_{W^{1/2,2}(\partial\Omega)}$$

$$\leq C\|u_n\|_{W^{1,2}(\Omega)}\|\psi\|_{W^{1,2}(\Omega)}$$

for some constant $C > 0$ (which may change from line to line). A similar analysis proves that

$$\left| \int_{\partial\Omega} [\rho_{j,k} * (\nabla\theta_j u_n), \nu(x)] \psi(x) \right| \leq C \|u_n\|_{W^{1,2}(\Omega)} \|\psi\|_{W^{1,2}(\Omega)}$$

for some constant $C > 0$. Taking all the above into account, we obtain

$$\left| \int_{\partial\Omega} [\mathbf{a}_{n,k}(x, \nabla u_n), \nu(x)] \, \psi \right| \leq$$
$$\leq \sum_{j=1}^{p} \int_{\partial\Omega} \int_{\Omega} \rho_{j,k}(x-y) \, \theta_j(y) \, f^0(y, \nu(x)) \, |\psi(x)| \, dy \, dx$$
$$+ \frac{C}{n} \|u_n\|_{W^{1,2}(\Omega)} \|\psi\|_{W^{1,2}(\Omega)}.$$

Letting $k \to \infty$, and taking into account the fact that θ_j is a partition of unity in $\overline{\Omega}$ and our assumptions on θ_j and K_j, we obtain

$$\left| \int_{\partial\Omega} [\mathbf{a}_n(x, \nabla u_n), \nu(x)] \, \psi \right| \leq \int_{\partial\Omega} f^0(x, \nu(x)) \, |\psi(x)| \, dx + \frac{C}{n} \|u_n\|_{W^{1,2}(\Omega)} \|\psi\|_{W^{1,2}(\Omega)}$$

Now, letting $n \to \infty$, and using (3.27), (3.28), we obtain

$$(3.36) \qquad \left| \int_{\partial\Omega} [z, \nu(x)] \, \psi \right| \leq \int_{\partial\Omega} f^0(x, \nu(x)) \, |\psi(x)| \, dx,$$

for all $\psi \in W^{1,2}(\Omega)$. Now, since $z \in L^{\infty}(\Omega)$ and $div(z) \in L^2(\Omega)$, $[z, \nu]$ coincides with the trace given in the sense of Anzellotti ([5]), and, therefore, $[z, \nu] \in L^{\infty}(\partial\Omega)$. Hence, from (3.36), we conclude that $|[z(x), \nu(x)]| \leq f^0(x, \nu(x))$. ∎

Lemma 3.7 *Suppose that any of the assumptions of Lemma 3.6 hold. Moreover we assume that*

$$(3.37) \qquad \begin{aligned} u_n \to u \quad &in \ L^2(\Omega), \quad \|u_n\|_{BV} \quad is \ bounded \\ and \quad &\mathbf{a}(x, \nabla u_n) \cdot D^s u_n = f^0(x, D^s u_n) \end{aligned}$$

Then

$$(3.38) \qquad z(x) = \mathbf{a}(x, \nabla u(x)) \qquad a.e. \ \ x \in \Omega.$$

Proof. Again, since both proofs are based on similar arguments, we shall only prove (3.38) under the assumptions given in $i)$ of Lemma 3.6. Let $0 \leq \phi \in C_0^1(\Omega)$ and $g \in C^1(\overline{\Omega})$. We observe that

$$\int_\Omega \phi[(\mathbf{a}(x, \nabla u_n), D(u_n - g)) - \mathbf{a}(x, \nabla g)D(u_n - g)] =$$

$$= \int_\Omega \phi[\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla g)) \cdot \nabla(u_n - g)] \ dx$$

$$+ \int_\Omega \phi[\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla g)] \cdot D^s(u_n - g)).$$

Since both terms at the right hand side of the above expression are positive, we have

$$\int_\Omega \phi[(\mathbf{a}(x, \nabla u_n), D(u_n - g)) - \mathbf{a}(x, \nabla g)D(u_n - g)] \geq 0.$$

Since

$$\int_\Omega \phi(\mathbf{a}(x, \nabla u_n), D(u_n - g)) =$$

$$= - \int_\Omega div(\mathbf{a}(x, \nabla u_n))\phi(u_n - g) \ dx - \int_\Omega (u_n - g)\mathbf{a}(x, \nabla u_n) \cdot \nabla\phi \ dx,$$

we get

$$\lim_{n\to\infty} \int_\Omega \phi(\mathbf{a}(x, \nabla u_n), D(u_n - g)) =$$

$$= - \int_\Omega div(z)\phi(u - g) \ dx - \int_\Omega (u - g)z \cdot \nabla\phi \ dx = \int_\Omega \phi(z, D(u - g)).$$

On the other hand,

$$\lim_{n\to\infty} \int_\Omega \phi \, \mathbf{a}(x, \nabla g)D(u_n - g) = \int_\Omega \phi \, \mathbf{a}(x, \nabla g)D(u - g).$$

Consequently, we obtain

$$\int_\Omega \phi[(z, D(u - g)) - \mathbf{a}(x, \nabla g)D(u - g)] \geq 0, \quad \forall \ 0 \leq \phi \in C_0^1(\Omega).$$

Thus the measure $(z, D(u-g)) - \mathbf{a}(x, \nabla g)D(u-g) \geq 0$. Then its absolutely continuous part

$$(z - \mathbf{a}(x, \nabla g)) \cdot \nabla(u - g) \geq 0 \quad \text{a.e. in } \Omega.$$

Since we may take a countable set dense in $C^1(\overline{\Omega})$ we have that the above inequality holds for all $x \in \tilde{\Omega}$, where $\tilde{\Omega} \subset \Omega$ is such that $\lambda_N(\Omega \setminus \tilde{\Omega}) = 0$, and all $g \in C^1(\overline{\Omega})$. Now, fixed $x \in \tilde{\Omega}$ and given $\xi \in \mathbb{R}^N$, there is $g \in C^1(\overline{\Omega})$ such that $\nabla g(x) = \xi$. Then

$$(z(x) - \mathbf{a}(x, \xi)) \cdot (\nabla u(x) - \xi) \geq 0, \quad \forall\, \xi \in R^N.$$

These inequalities imply (3.38) by an application of Minty-Browder's method in $\mathbb{R}^N$. ∎

Proof of Proposition 3.2 We divide the proof in three steps.

Step 1. Suppose first that $\varphi \in C^1(\overline{\Omega})$. Let $v \in L^\infty(\Omega)$. We shall find $u \in BV(\Omega) \cap L^2(\Omega)$ such that $(u, v - u) \in \mathcal{B}_\varphi$. That is, there is $\mathbf{a}(x, \nabla u) \in X(\Omega)$ satisfying

$$(3.39) \qquad (v - u) = -div\,\mathbf{a}(x, \nabla u), \quad \text{in } \mathcal{D}'(\Omega),$$

$$(3.40) \qquad \mathbf{a}(x, \nabla u) \cdot D^s u = f^0(x, D^s u), \quad \text{and}$$

$$(3.41) \qquad [\mathbf{a}(x, \nabla u), \nu] \in \text{sign}\,(\varphi - u) f^0(x, \nu(x)) \quad H^{N-1} - \text{a.e.}$$

By Lemma 3.3, we know that for any $n \in \mathbb{N}$ there exists $u_n \in W^{1,2}_\varphi(\Omega) \cap L^\infty(\Omega)$ such that $(u_n, v - u_n) \in A_{n,\varphi}$. Hence

$$(3.42) \qquad \int_\Omega (w - u_n)(v - u_n)\,dx \leq \int_\Omega \mathbf{a}_n(x, \nabla u_n) \cdot \nabla(w - u_n)\,dx$$

for all $w \in W^{1,2}_\varphi(\Omega)$.

Let $M_1 := \sup\{\|\varphi\|_\infty, \|v\|_\infty\}$. Then, taking $w = u_n - (u_n - M_1)^+$ as test function in (3.42), we obtain

$$\int_\Omega (u_n - M_1)^+ (u_n - v)\,dx \leq 0.$$

Hence,

$$
\begin{aligned}
\int_{\{u_n > M_1\}} (u_n - M_1)^2\,dx &\leq \int_{\{u_n > M_1\}} (u_n - M_1)(u_n - v)\,dx \\
&= \int_\Omega (u_n - M_1)^+ (u_n - v)\,dx \leq 0.
\end{aligned}
$$

Consequently, $u_n \leq M_1$ a.e. in Ω. Analogously, taking $w = u_n + (u_n + M_1)^-$ as test function, we get $-M_1 \leq u_n$ a.e. in Ω. Therefore,

$$(3.43) \qquad \|u_n\|_\infty \leq M_1 \qquad \text{for all } n \in \mathbb{N}.$$

Taking $w = w_0 \in W^{1,2}_\varphi(\Omega) \cap L^\infty(\Omega)$ in (3.42), applying Young's inequality and using (3.43) we get

$$\int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \; dx + \frac{1}{n} \int_\Omega |\nabla u_n|^2 \; dx \leq$$

$$\leq \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w_0 \; dx + \frac{1}{n} \int_\Omega \nabla u_n \cdot \nabla w_0 \; dx + \int_\Omega (w_0 - u_n)(u_n - v) \; dx$$

$$\leq M_2 \left(\int_\Omega |\nabla w_0|^2 \; dx \right)^{\frac{1}{2}} + \frac{1}{2n} \int_\Omega |\nabla u_n|^2 \; dx + \frac{1}{2n} \int_\Omega |\nabla w_0|^2 \; dx + M_3$$

$$\leq M_4 + \frac{1}{2n} \int_\Omega |\nabla u_n|^2 \; dx.$$

Hence, by (2.5), we obtain

$$(3.44) \qquad \qquad \int_\Omega |\nabla u_n| \; dx \leq M_5 \quad \forall \, n \in \mathbb{N}$$

and

$$(3.45) \qquad \qquad \frac{1}{n} \int_\Omega |\nabla u_n|^2 \; dx \leq M_6 \quad \forall n \in \mathbb{N}.$$

Thus, $\{u_n \, : \, n \in \mathbb{N}\}$ is bounded in $W^{1,1}(\Omega)$ and, by extracting a subsequence if is necessary, we may assume that u_n converges in $L^1(\Omega)$ and converges almost everywhere to some $u \in L^1(\Omega)$ as $n \to +\infty$. Now, by (3.43) and (3.44), we have that $u_n \to u$ in $L^2(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$.

Observe that by (2.4) and (3.45), $\{\mathbf{a}_n(x, \nabla u_n) \, : \, n \in \mathbb{N}\}$ is bounded in $L^2(\Omega, \mathbb{R}^N)$. Consequently we may assume that

$$(3.46) \qquad \mathbf{a}_n(x, \nabla u_n) \rightharpoonup z \quad \text{as } n \to \infty, \text{ weakly in } L^2(\Omega, \mathbb{R}^N).$$

Given $\psi \in C_0^\infty(\Omega)$, taking $w = u_n \pm \psi$ in (3.42) we obtain

$$\int_\Omega \psi(v - u_n) \; dx = \int_\Omega \mathbf{a}_n(x, \nabla u_n) \cdot \nabla \psi \; dx.$$

Letting $n \to +\infty$, we obtain

$$\int_\Omega (v - u)\psi \; dx = \int_\Omega z \cdot \nabla \psi \; dx,$$

that is,

$$(3.47) \qquad \qquad v - u = -div(z), \qquad \text{in } \mathcal{D}'(\Omega)$$

and

(3.48) $\qquad div\ \mathbf{a}_n(x, \nabla u_n) \rightharpoonup div(z) \quad$ weakly in $L^2(\Omega)$.

Since, by (3.45),

(3.49) $$\frac{1}{n}|\nabla u_n| \to 0 \quad \text{in } L^2(\Omega),$$

as a consequence of (3.46), it follows that

(3.50) $\qquad \mathbf{a}(x, \nabla u_n) \rightharpoonup z \quad$ as $n \to \infty$, weakly in $L^2(\Omega, \mathbb{R}^N)$.

Moreover, by (2.4) we may assume that

(3.51) $\qquad \mathbf{a}(x, \nabla u_n) \rightharpoonup z \quad$ as $n \to \infty$, weakly* in $L^\infty(\Omega, \mathbb{R}^N)$.

Let us prove that

(3.52) $\displaystyle \lim_{n \to \infty} \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n\ dx = \int_\Omega (z, Du) - \int_{\partial\Omega} [z, \nu](u - \varphi)\ dH^{N-1}$.

By (3.42), we have

(3.53)
$$\int_\Omega (w - u_n)(v - u_n)\ dx + \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n\ dx \leq$$
$$\leq \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w\ dx + \frac{1}{n} \int_\Omega \nabla u_n \cdot \nabla w\ dx$$

for all $w \in W^{1,2}_\varphi(\Omega)$. By Lemma 3.4, there exists $v_j \in C^1(\overline{\Omega})$ such that $v_{j|\partial\Omega} = \varphi$, $v_j \to u$ in $L^1(\Omega)$. If we set $w = v_j$ in (3.53), taking the upper limit when $n \to \infty$, we get

(3.54) $\displaystyle \int_\Omega (v_j - u)(v - u)\ dx + \limsup_{n \to \infty} \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n\ dx \leq \int_\Omega z \cdot \nabla v_j\ dx$.

Now, by Green's formula we have

$$
\begin{aligned}
\int_\Omega z \cdot \nabla v_j\ dx &= -\int_\Omega div(z) v_j\ dx + \int_{\partial\Omega} [z, \nu] \varphi\ dH^{N-1} \\
&= \int_\Omega (v - u) v_j\ dx + \int_{\partial\Omega} [z, \nu] \varphi\ dH^{N-1}.
\end{aligned}
$$

Hence, taking limit as $j \to \infty$ and applying again the Green's formula we obtain that

(3.55) $\displaystyle \lim_{j \to \infty} \int_\Omega z \cdot \nabla v_j\ dx = \int_\Omega (z, Du) - \int_{\partial\Omega} [z, \nu](u - \varphi)\ dH^{N-1}$.

Letting $j \to \infty$ in (3.54) , we have

$$(3.56) \quad \limsup_{n\to\infty} \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx \leq \int_\Omega (z, Du) - \int_{\partial\Omega} [z, \nu](u - \varphi) \, dH^{N-1}.$$

On the other hand,

$$\int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx = \int_\Omega \big(\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla v_j)\big) \cdot \nabla(u_n - v_j) \, dx +$$

$$+ \int_\Omega \big(\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla v_j)\big) \cdot \nabla v_j \, dx + \int_\Omega \mathbf{a}(x, \nabla v_j) \cdot \nabla u_n \, dx \geq$$

$$\geq \int_\Omega \big(\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla v_j)\big) \cdot \nabla v_j \, dx + \int_\Omega \mathbf{a}(x, \nabla v_j) \cdot \nabla u_n \, dx.$$

Hence

$$\liminf_{n\to\infty} \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx \geq \lim_{n\to\infty} \bigg(\int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla v_j \, dx$$

$$- \int_\Omega \mathbf{a}(x, \nabla v_j) \cdot \nabla v_j \, dx + \int_\Omega \mathbf{a}(x, \nabla v_j) \cdot \nabla u_n \, dx \bigg).$$

If we consider the $\mathbb{R}^N$-valued measures μ_n, μ on $\overline{\Omega}$ which are defined as

$$\mu_n(B) \ := \ \int_{B\cap\Omega} \nabla u_n \, dx$$

$$\mu(B) \ := \ \int_{B\cap\Omega} Du + \int_{B\cap\partial\Omega} (\varphi - u)\nu \, dH^{N-1}$$

for all Borel sets $B \subset \overline{\Omega}$, we have

$$\mu_n \rightharpoonup \mu \qquad \text{weakly as measures in} \quad \overline{\Omega}.$$

Then, since $\mathbf{a}(x, \nabla v_j(x)) \in C(\overline{\Omega}, \mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \int_\Omega \mathbf{a}(x, \nabla v_j) \cdot \nabla u_n \, dx =$$

$$= \int_\Omega \mathbf{a}(x, \nabla v_j) \, dDu + \int_{\partial\Omega} \mathbf{a}(x, \nabla v_j) \cdot \nu(\varphi - u) \, dH^{N-1}.$$

Therefore, we have

$$\liminf_{n\to\infty} \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx \geq \int_\Omega z \cdot \nabla v_j \, dx - \int_\Omega \mathbf{a}(x, \nabla v_j) \cdot \nabla v_j \, dx$$

$$+ \int_\Omega \mathbf{a}(x, \nabla v_j) \, dDu + \int_{\partial\Omega} \mathbf{a}(x, \nabla v_j) \cdot \nu(\varphi - u) \, dH^{N-1}.$$

Now, by Theorem 7.4 of [7], we have

$$\lim_{j\to\infty} \int_\Omega \mathbf{a}(x,\nabla v_j)\cdot\nabla v_j \ dx =$$

$$= \int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla u \ dx + \int_\Omega \mathbf{a}^\infty(x,\overrightarrow{D^s u})\cdot D^s u$$

$$+ \int_{\partial\Omega} \mathbf{a}^\infty(x,(\varphi-u)\nu)\cdot\nu(\varphi-u) \ dH^{N-1}.$$

On the other hand, as a consequence of Lemma 3.4, we have

$$\lim_{j\to\infty} \int_\Omega \mathbf{a}(x,\nabla v_j) \ dDu =$$

$$= \lim_{j\to\infty} \left(\int_\Omega \mathbf{a}(x,\nabla v_j)\cdot\nabla u \ dx + \int_\Omega \mathbf{a}(x,\nabla v_j) dD^s u \right)$$

$$= \int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla u \ dx + \int_\Omega \mathbf{a}^\infty(x,\overrightarrow{D^s u})\cdot D^s u$$

and

$$\lim_{j\to\infty} \int_{\partial\Omega} \mathbf{a}(x,\nabla v_j)\cdot\nu(\varphi-u) \ dH^{N-1}$$

$$= \int_{\partial\Omega} \mathbf{a}^\infty\left(x,\frac{\varphi-u}{|\varphi-u|}\nu\right)\cdot\nu(\varphi-u) \ dH^{N-1} =$$

$$= \int_{\partial\Omega} \mathbf{a}^\infty\big(x,(\varphi-u)\nu\big)\cdot\nu(\varphi-u) \ dH^{N-1}.$$

Collecting all these facts, we obtain

$$\liminf_{n\to\infty} \int_\Omega \mathbf{a}(x,\nabla u_n)\cdot\nabla u_n \ dx \ \geq \ \lim_{j\to\infty} \int_\Omega z\cdot\nabla v_j \ dx$$

$$= \int_\Omega (z,Du) - \int_{\partial\Omega} [z,\nu](u-\varphi) \ dH^{N-1}.$$

Combining this inequality with (3.56), we obtain (3.52).

Our next purpose will be to show that

$$(3.57) \qquad \int_\Omega h(x,Du) + \int_{\partial\Omega} |\varphi-u| f^0(x,\nu(x)) \ dH^{N-1}$$

$$= \int_\Omega (z,Du) - \int_{\partial\Omega} [z,\nu](u-\varphi) \ dH^{N-1}.$$

According to [6], there exists a sequence $\{w_j\} \subset C^1(\Omega) \cap BV(\Omega)$ such that $w_{j|\partial\Omega} = \varphi$,

$$w_j \to u \ \text{ in } \ L^1(\Omega), \quad \text{ and } \quad \Phi_\varphi(w_j) \to \Phi_\varphi(u).$$

Now, by the convexity of f, we have

$$\int_\Omega f(x, \nabla u_n) \, dx \leq \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx$$
$$- \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w_j \, dx + \int_\Omega f(x, \nabla w_j) \, dx.$$

Thus,

$$\Phi_\varphi(u_n) \leq \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx - \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w_j \, dx + \Phi_\varphi(w_j).$$

Using (3.52), it follows that

$$\limsup_{n \to \infty} \Phi_\varphi(u_n) \leq$$

$$\leq \int_\Omega (z, Du) - \int_{\partial\Omega} [z, \nu](u - \varphi) \, dH^{N-1} - \lim_{n \to \infty} \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w_j \, dx + \Phi_\varphi(w_j)$$

$$= \int_\Omega (z, Du) - \int_{\partial\Omega} [z, \nu](u - \varphi) \, dH^{N-1} - \int_\Omega z \cdot \nabla w_j \, dx + \Phi_\varphi(w_j).$$

Since

$$\lim_{j \to \infty} \int_\Omega z \cdot \nabla w_j \, dx = \lim_{j \to \infty} \left(- \int_\Omega div(z)w_j \, dx + \int_{\partial\Omega} [z, \nu]\varphi \, dH^{N-1} \right) =$$

$$= - \int_\Omega div(z)u \, dx + \int_{\partial\Omega} [z, \nu]\varphi \, dH^{N-1} = \int_\Omega (z, Du) - \int_{\partial\Omega} [z, \nu](u - \varphi) \, dH^{N-1},$$

letting $j \to \infty$ in the above inequality, we obtain

$$\limsup_{n \to \infty} \Phi_\varphi(u_n) \leq \lim_{j \to \infty} \Phi_\varphi(w_j) = \Phi_\varphi(u).$$

Thus, by the lower-semicontinuity of Φ_φ, we get

(3.58) $$\Phi_\varphi(u) = \lim_{n \to \infty} \Phi_\varphi(u_n).$$

Now,

$$\Phi_\varphi(u) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde{\mu}) \qquad \text{and} \qquad \Phi_\varphi(u_n) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde{\mu}_n).$$

Hence, (3.58) yields

$$\lim_{n \to \infty} \int_{\overline{\Omega}} \tilde{f}(x, \tilde{\mu}_n) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde{\mu}).$$

Then, applying Theorem 3 of [18], it follows that

$$(3.59) \qquad \int_{\overline{\Omega}} \tilde{h}(x, \tilde{\mu}) = \lim_{n \to \infty} \int_{\overline{\Omega}} \tilde{h}(x, \tilde{\mu}_n) = \lim_{n \to \infty} \int_{\Omega} \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n.$$

Since

$$
\begin{aligned}
\int_{\overline{\Omega}} \tilde{h}(x, \tilde{\mu}) &= \int_{\overline{\Omega}} \tilde{h}(x, \tilde{\mu}^a(x)) \, dx + \int_{\overline{\Omega}} \tilde{h}\Big(x, \frac{d\tilde{\mu}^s}{d|\tilde{\mu}^s|}(x)\Big) \, d|\tilde{\mu}^s| \\
&= \int_{\overline{\Omega}} \tilde{h}(x, \mu^a(x), 1) \, dx + \int_{\overline{\Omega}} \tilde{h}\Big(x, \frac{d\mu^s}{d|\mu^s|}(x), 0\Big) \, d|\mu^s| \\
&= \int_{\overline{\Omega}} h(x, \mu^a(x)) \, dx + \int_{\overline{\Omega}} h^0\Big(x, \frac{d\mu^s}{d|\mu^s|}(x)\Big) \, d|\mu^s| \\
&= \int_{\Omega} h(x, \nabla u(x)) \, dx + \int_{\Omega} h^0\big(x, \overrightarrow{D^s u}(x)\big) \, d|D^s u| \\
&\quad + \int_{\partial\Omega} h^0\Big(x, \frac{(\varphi - u) \cdot \nu}{|(\varphi - u) \cdot \nu|}\Big) \, dH^{N-1} \\
&= \int_{\Omega} h(x, Du) + \int_{\partial\Omega} |\varphi - u| f^0(x, \nu(x)) \, dH^{N-1},
\end{aligned}
$$

(3.57) follows from (3.52) and (3.59).

By (3.49), (3.50) and (3.48), applying Lemma 3.6 (ii), we get

$$(3.60) \qquad |[z(x), \nu(x)]| \leq f^0(x, \nu(x)) \quad \text{a.e. in } \partial\Omega.$$

Let $v_j \in C^1(\overline{\Omega})$ be a sequence such that $v_j \to u$ in $L^2(\Omega)$ and $\int_\Omega |\nabla v_j| \to \|Du\|$. According to (H_5), we have

$$|\mathbf{a}(x, \nabla u_n) \cdot \nabla v_j| \leq f^0(x, \nabla v_j).$$

Then, if $\psi, \phi \in C^1(\Omega)$, with $0 \leq \psi \leq \phi$, we have

$$\left| \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla v_j \, \psi \, dx \right| \leq \int_\Omega f^0(x, \nabla v_j) \psi \, dx,$$

and, letting $n \to \infty$, we get

$$\left| \int_\Omega z \cdot \nabla v_j \, \psi \, dx \right| \leq \int_\Omega f^0(x, \nabla v_j) \psi \, dx.$$

Now, since

$$\left| \int_\Omega z \cdot \nabla v_j \, \psi \, dx \right| = \left| - \int_\Omega div(z) v_j \psi \, dx - \int_\Omega v_j z \cdot \nabla \psi \, dx \right|,$$

letting $j \to \infty$ we obtain that

$$|\langle (z, Du), \psi \rangle| = \left| - \int_\Omega div(z) u \psi \, dx - \int_\Omega u z \cdot \nabla \psi \, dx \right|$$

$$\leq \int_\Omega \psi f^0(x, Du) \leq \int_\Omega \phi f^0(x, Du).$$

Hence

$$\langle |(z, Du)|, \phi \rangle \leq \int_\Omega \phi f^0(x, Du).$$

Thus, we have

$$|(z, Du)| \leq f^0(x, Du) \qquad \text{as measures in } \Omega.$$

Then, the singular parts also satisfy a similar inequality,

$$(3.61) \qquad |z \cdot D^s u| \leq f^0(x, D^s u) \qquad \text{as measures in } \Omega.$$

Now, by (3.58), (3.51), (3.60) and (3.61), the assumptions of Lemma 3.5 are satisfied, and we have

$$(3.62) \qquad \int_\Omega z \cdot \nabla u \, dx = \int_\Omega h(x, \nabla u) \, dx = \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla u \, dx,$$

$$(3.63) \qquad \qquad z \cdot D^s u = f^0(x, D^s u),$$

$$(3.64) \qquad [z, \nu] \in \text{sign} \ (\varphi - u) f^0(x, \nu(x)) \qquad H^{N-1} - \text{a.e.}$$

Moreover, since the assumptions of Lemma 3.7 hold, we have that

$$(3.65) \qquad \qquad z(x) = \mathbf{a}(x, \nabla u(x)) \qquad \text{a.e. } x \in \Omega.$$

Observe that (3.39) follows from (3.47) and (3.65); (3.40) is a consequence of (3.62), (3.63) and (3.65); and (3.41) follows from (3.64) and (3.65). This concludes the proof in the case $\varphi \in C^1(\overline{\Omega})$.

Step 2. Suppose now we are in the general case, that is, $\varphi \in L^1(\partial\Omega)$. Take $\varphi_j \in C^1(\overline{\Omega})$ such that $\varphi_j \to \varphi$ in $L^1(\partial\Omega)$. Given $v \in L^\infty(\Omega)$, from the Step 1, there exists $u_j \in D(\mathcal{B}_{\varphi_j})$ such that $(u_j, v - u_j) \in \mathcal{B}_{\varphi_j}$. Hence, we have

$$(3.66) \qquad -div(\mathbf{a}(x, \nabla u_j)) = v - u_j, \qquad \text{in } \ \mathcal{D}'(\Omega),$$

$$(3.67) \qquad \qquad \mathbf{a}(x, \nabla u_j) \cdot D^s u_j = f^0(x, D^s u_j),$$

$$(3.68) \quad [\mathbf{a}(x, \nabla u_j), \nu] \in \text{sign}(\varphi_j - u_j) f^0(x, \nu(x)) \qquad H^{N-1} - \text{a.e.}$$

By (3.66), (3.67) and (3.68), we get

$$
\int_\Omega \mathbf{a}(x, \nabla u_j) \cdot \nabla u_j \, dx + \int_\Omega f^0(x, D^s u_j)
$$

(3.69)
$$
+ \int_{\partial\Omega} |\varphi_j - u_j| f^0(x, \nu(x)) \, dH^{N-1} + \int_\Omega u_j^2 \, dx
$$
$$
= \int_\Omega u_j v \, dx + \int_{\partial\Omega} \big(\mathbf{a}(x, \nabla u_j) \cdot \nu\big) \varphi_j \, dH^{N-1}.
$$

From (3.69), using Young's inequality and (2.5), we obtain that

$$
C_0 \|D u_j\| + C_0 \int_{\partial\Omega} |\varphi_j - u_j| f^0(x, \nu(x)) \, dH^{N-1} + \frac{1}{2} \int_\Omega u_j^2 \, dx \le C \quad \forall \, j \in \mathbb{N},
$$

for some constant $C > 0$. It follows that there exists $u \in BV(\Omega) \cap L^2(\Omega)$, such that

(3.70)
$$
u_j \rightharpoonup u \quad \text{weakly in } L^2(\Omega),
$$
$$
u_j \to u \quad \text{in } L^q(\Omega) \text{ for all } 1 \le q < \frac{N}{N-1}.
$$

Hence,

(3.71)
$$
\int_\Omega u^2 \, dx \le \limsup_{j \to \infty} \int_\Omega u_j^2 \, dx.
$$

After passing to a subsequence, if necessary, we may assume that

(3.72) $\mathbf{a}(x, \nabla u_j) \rightharpoonup z \quad \text{as } j \to \infty, \text{ weakly* in } L^\infty(\Omega, \mathbb{R}^N)$

and

(3.73) $-div(z) = v - u \qquad \text{in} \quad \mathcal{D}'(\Omega).$

According to [6], Fact 3.3, there exists a sequence $\{w_k\} \subset C^1(\Omega) \cap BV(\Omega)$ such that $w_{k|\partial\Omega} = \varphi$,

(3.74) $w_k \to u \quad \text{in } L^2(\Omega) \quad \text{and} \quad \Phi_\varphi(w_k) \to \Phi_\varphi(u).$

Now, by the convexity of f we have

$$
\int_\Omega f(x, \nabla u_j) \, dx \le \int_\Omega \mathbf{a}(x, \nabla u_j) \cdot \nabla u_j \, dx - \int_\Omega \mathbf{a}(x, \nabla u_j) \cdot \nabla w_k \, dx
$$
$$
+ \int_\Omega f(x, \nabla w_k) \, dx.
$$

Thus, having in mind (3.66), (3.67) and (3.68), we get

$$
\begin{aligned}
\Phi_{\varphi_j}(u_j) &= \int_\Omega f(x, \nabla u_j)\, dx + \int_\Omega f^0(x, D^s u_j) + \int_{\partial\Omega} |u_j - \varphi_j| f^0(x, \nu(x))\, dH^{N-1} \\
&\le \int_\Omega f(x, \nabla w_k)\, dx + \int_\Omega \mathbf{a}(x, \nabla u_j) \cdot \nabla u_j\, dx + \int_\Omega f^0(x, D^s u_j) + \\
&\quad + \int_{\partial\Omega} |u_j - \varphi_j| f^0(x, \nu(x))\, dH^{N-1} - \int_\Omega \mathbf{a}(x, \nabla u_j) \cdot \nabla w_k\, dx \\
&\le \int_\Omega f(x, \nabla w_k)\, dx + \int_\Omega (v - u_j) u_j\, dx \\
&\quad + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_j), \nu]\varphi_j\, dH^{N-1} - \int_\Omega \mathbf{a}(x, \nabla u_j) \cdot \nabla w_k\, dx.
\end{aligned}
$$

Using (3.71) and (3.72), it follows that

$$
\limsup_{j\to\infty} \Phi_\varphi(u_j) = \limsup_{j\to\infty} \Phi_{\varphi_j}(u_j) \le
$$

$$
\le \int_\Omega f(x, \nabla w_k)\, dx + \int_\Omega uv\, dx - \int_\Omega u^2\, dx + \int_{\partial\Omega} [z, \nu]\varphi\, dH^{N-1} - \int_\Omega z \cdot \nabla w_k\, dx
$$

$$
\le \int_\Omega f(x, \nabla w_k)\, dx + \int_\Omega (v - u)u\, dx + \int_\Omega div(z) w_k\, dx.
$$

Hence, by (3.73), letting $k \to \infty$, we arrive to

$$
\limsup_{j\to\infty} \Phi_\varphi(u_j) \le \lim_{k\to\infty} \Phi_\varphi(w_k) = \Phi_\varphi(u).
$$

Thus, by the lower-semicontinuity of Φ_φ, we get

(3.75) $$ \Phi_\varphi(u) = \lim_{j\to\infty} \Phi_\varphi(u_j). $$

Applying Theorem 3 of [18] as in the Step 1, it follows that

(3.76)
$$
\begin{aligned}
\lim_{j\to\infty} \int_\Omega h(x, Du_j) &+ \int_{\partial\Omega} |u_j - \varphi_j| f^0(x, \nu(x))\, dH^{N-1} \\
&= \int_\Omega h(x, Du) + \int_{\partial\Omega} |u - \varphi| f^0(x, \nu(x))\, dH^{N-1}.
\end{aligned}
$$

On the other hand, by Green's formula, (3.66), (3.67) and (3.68), we have

$$
\begin{aligned}
\int_\Omega h(x, Du_j) &+ \int_{\partial\Omega} |u_j - \varphi_j| f^0(x, \nu(x))\, dH^{N-1} \\
&= \int_\Omega \big(\mathbf{a}(x, \nabla u_j), Du_j\big) + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_j), \nu](\varphi_j - u_j)\, dH^{N-1} \\
&= \int_\Omega u_j(v - u_j)\, dx + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_j), \nu]\varphi_j\, dH^{N-1}.
\end{aligned}
$$

Since $[\mathbf{a}(x, \nabla u_j), \nu] \rightharpoonup [z, \nu]$ weakly* in $L^\infty(\partial\Omega)$, letting $j \to +\infty$, and using (3.76), it follows that

$$\int_\Omega h(x, Du) + \int_{\partial\Omega} |u - \varphi| f^0(x, \nu(x))\, dH^{N-1}$$

$$\leq \int_\Omega u(v - u)\, dx + \int_{\partial\Omega} [z, \nu]\varphi\, dH^{N-1}$$

$$= \int_\Omega (z, Du) + \int_{\partial\Omega} [z, \nu](\varphi - u)\, dH^{N-1}.$$

Now, by Lemma 3.6 (i), we have

$$|[z(x), \nu(x)]| \leq f^0(x, \nu(x)) \qquad H^{N-1} - a.e \ \text{in} \ \partial\Omega.$$

Moreover, as in the Step 1, we get

$$|z \cdot D^s u| \leq f^0(x, D^s u) \quad \text{as measures in} \ \Omega.$$

With this and using Lemma 3.5, we obtain

$$(3.77) \qquad \int_\Omega z \cdot \nabla u\, dx = \int_\Omega h(x, \nabla u)\, dx = \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla u\, dx,$$

$$(3.78) \qquad\qquad z \cdot D^s u = f^0(x, D^s u),$$

$$(3.79) \qquad [z, \nu] \in \text{sign}\ (\varphi - u) f^0(x, \nu(x)) \qquad H^{N-1} - \text{a.e.}$$

As in the Step 1, to get that $(u, v - u) \in \mathcal{B}_\varphi$, we only need to prove that

$$(3.80) \qquad\qquad div\ z = div\ \mathbf{a}(x, \nabla u), \qquad \text{in} \ \ \mathcal{D}'(\Omega),$$

and

$$(3.81) \qquad [z, \nu] = [\mathbf{a}(x, \nabla u), \nu] \qquad H^{N-1} - \text{a.e. on} \ \partial\Omega.$$

Now, by (3.66), (3.70) and using Fatou's Lemma, we are able to adapt the proof of Lemma 3.7 obtaining that $z(x) = \mathbf{a}(x, \nabla u(x))$ a.e. in Ω and this implies both (3.80) and (3.81).

Step 3. To prove the density of $D(\mathcal{B}_\varphi)$ in $L^2(\Omega)$, we prove that $C_0^\infty(\Omega) \subseteq \overline{D(\mathcal{B}_\varphi)}^{L^2(\Omega)}$. Let $v \in C_0^\infty(\Omega)$. By the above, $v \in R(I + \frac{1}{n}\mathcal{B}_\varphi)$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $u_n \in D(\mathcal{B}_\varphi)$ such that $(u_n, n(v - u_n)) \in \mathcal{B}_\varphi$. Consequently, we have $\mathbf{a}(x, \nabla u_n) \in X(\Omega)$, $n(v - u_n) = -div(\mathbf{a}(x, \nabla u_n))$ in

$\mathcal{D}'(\Omega)$ and

$$\int_\Omega (w - u_n) n(v - u_n) \, dx =$$
$$= \int_\Omega (\mathbf{a}(x, \nabla u_n), Dw) - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu](w - \varphi) \, dH^{N-1}$$
$$- \int_\Omega h(x, Du_n) - \int_{\partial\Omega} |\varphi - u_n| f^0(x, \nu(x)) \, dH^{N-1}.$$

for every $w \in BV(\Omega) \cap L^2(\Omega)$. Taking $w = v$, we get

$$\int_\Omega (v - u_n)^2 \, dx = \frac{1}{n} \left(\int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla v \, dx - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu](v - \varphi) dH^{N-1} \right.$$
$$\left. - \int_\Omega h(x, Du_n) - \int_{\partial\Omega} |\varphi - u_n| f^0(x, \nu(x)) \, dH^{N-1} \right)$$
$$\leq \frac{1}{n} \left(\int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla v \, dx - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu](v - \varphi) \, dH^{N-1} \right)$$
$$\leq \frac{M}{n} \left(\int_\Omega |\nabla v| \, dx + \int_{\partial\Omega} |v - \varphi| \, dH^{N-1} \right).$$

Letting $n \to \infty$, it follows that $u_n \to v$ in $L^2(\Omega)$. Therefore $v \in \overline{D(\mathcal{B}_\varphi)}^{L^2(\Omega)}$ and the proof is complete. ∎

Proof of Theorem 3.1 First, we prove that $\mathcal{B}_\varphi \subset \partial\Phi_\varphi$. Let $(u, v) \in \mathcal{B}_\varphi$ and $w \in W_\varphi^{1,2}(\Omega)$. Then, by (2.2), and applying Green's formula we get

$$\int_\Omega (w - u)v \, dx = - \int_\Omega (w - u) \, div \, \mathbf{a}(x, \nabla u) \, dx$$
$$= \int_\Omega (\mathbf{a}(x, \nabla u), Dw - Du) - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu](\varphi - u) \, dH^{N-1}$$
$$= \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla w \, dx - \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla u \, dx - \int_\Omega \mathbf{a}(x, \nabla u) \cdot D^s u$$
$$- \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu](\varphi - u) \, dH^{N-1}$$
$$\leq \int_\Omega f(x, \nabla w) \, dx - \int_\Omega f(x, Du) \, dx - \int_{\partial\Omega} |\varphi - u| f^0(x, \nu(x)) \, dH^{N-1}$$
$$= \Phi_\varphi(w) - \Phi_\varphi(u).$$

Suppose that $w \in BV(\Omega) \cap L^2(\Omega)$. According to [6], Fact 3.3, there exists a sequence $w_n \in W_\varphi^{1,2}(\Omega)$, with $w_n \to w$ in $L^2(\Omega)$, and $\Phi_\varphi(w_n) \to \Phi_\varphi(w)$.

Then, by the above inequality, we have

$$\int_\Omega (w_n - u)v \ dx \le \Phi_\varphi(w_n) - \Phi_\varphi(u).$$

Now, letting $n \to \infty$, we get

$$\int_\Omega (w - u)v \ dx \le \Phi_\varphi(w) - \Phi_\varphi(u),$$

and therefore, $(u, v) \in \partial \Phi_\varphi$.

Since $\mathcal{B}_\varphi \subset \partial \Phi_\varphi$, and, by Proposition 3.2, $L^\infty(\Omega) \subset R(I + \mathcal{B}_\varphi)$, we have $\partial \Phi_\varphi = \overline{\mathcal{B}_\varphi}^{L^2(\Omega)}$. To finish the proof we only need to prove that the operator $\mathcal{B}_\varphi$ is closed. Let $(u_n, v_n) \in \mathcal{B}_\varphi$, and assume that $(u_n, v_n) \to (u, v)$ in $L^2(\Omega) \times L^2(\Omega)$. Let us prove that $(u, v) \in \mathcal{B}_\varphi$. Since $(u_n, v_n) \in \mathcal{B}_\varphi$, we know that $\mathbf{a}(x, \nabla u_n) \in X(\Omega)$ is such that

$$(3.82) \qquad\qquad -v_n = div \ \mathbf{a}(x, \nabla u_n), \qquad \text{in } \mathcal{D}'(\Omega),$$

$$(3.83) \qquad\qquad \mathbf{a}(x, \nabla u_n) \cdot D^s u_n = f^0(x, D^s u_n),$$

$$(3.84) \quad [\mathbf{a}(x, \nabla u_n), \nu] \in \text{sign}\,(\varphi - u_n)f^0(x, \nu(x)) \qquad H^{N-1} - \text{a.e.}$$

Multiplying (3.82) by u_n and applying Green's formula we obtain

$$-\int_\Omega u_n v_n \ dx = \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu]\varphi \ dH^{N-1} - \int_\Omega h(x, Du_n)$$
$$-\int_{\partial\Omega} |\varphi - u_n|f^0(x, \nu(x)) \ dH^{N-1}.$$

Hence,

$$(3.85) \qquad \int_\Omega h(x, Du_n) \le \int_\Omega u_n v_n \ dx + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu]\varphi \ dH^{N-1}.$$

From (2.5) and (3.85), we have

$$C_0 \int_\Omega |Du_n| \ dx - D_1 \lambda_N(\Omega) \ \le \ \int_\Omega h(x, Du_n) \ dx$$
$$\le \ \int_\Omega u_n v_n \ dx + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu]\varphi \ dH^{N-1}.$$

Hence,

$$(3.86) \qquad\qquad \int_\Omega |Du_n| \ dx \le M_1 \qquad \forall \ n \in \mathbb{N}.$$

Therefore, $u \in BV(\Omega) \cap L^2(\Omega)$. On the other hand, since $\|\mathbf{a}(x, \nabla u_n)\|_\infty \leq M$, we may assume that

$$(3.87) \qquad \mathbf{a}(x, \nabla u_n) \rightharpoonup z \quad \text{in the weak}^* \text{ topology of } L^\infty(\Omega, \mathbb{R}^N),$$

with $\|z\|_\infty \leq M$. Moreover, since $v_n \to v$ in $L^2(\Omega)$, we have that $v = -div(z)$ in $\mathcal{D}'(\Omega)$. By the definition of the weak trace on $\partial\Omega$ of the normal component of z, it is easy to see that

$$(3.88) \qquad [\mathbf{a}(x, \nabla u_n), \nu] \rightharpoonup [z, \nu] \qquad \text{weakly}^* \text{ in } L^\infty(\partial\Omega).$$

Now, we prove the convergence of the energies. According to [6], Fact 3.3, there exists a sequence $w_j \in C^1(\Omega) \cap BV(\Omega)$, with $w_{j|\partial\Omega} = \varphi$, $w_j \to u$ in $L^1(\Omega)$, and $\Phi_\varphi(w_j) \to \Phi_\varphi(u)$. Moreover, looking at the proof of Fact 3.3 in [6], we have that, $w_j = w_j^1 + w_j^2$ with $w_{j|\partial\Omega}^1 = u_{|\partial\Omega}$ and $w_j^1 \to u$ in $L^1(\Omega)$, $w_{j|\partial\Omega}^2 = \varphi - u_{|\partial\Omega}$, $w_j^2 \to 0$ in $L^1(\Omega)$, and, using [5], Lemma 1.8, we have that

$$\int_\Omega (z, Dw_j^1) \to \int_\Omega (z, Du).$$

By the convexity of f and taking (3.83) and (3.84) into account we have

$$\Phi_\varphi(u_n) = \int_\Omega f(x, \nabla u_n)\, dx + \int_\Omega f^0(x, D^s u_n) + \int_{\partial\Omega} |u_n - \varphi| f^0(x, \nu(x)) dH^{N-1}$$

$$\leq \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n\, dx - \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w_j\, dx + \int_\Omega f(x, \nabla w_j)\, dx$$

$$+ \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot D^s u_n + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu](\varphi - u_n)\, dH^{N-1}$$

$$= \int_\Omega (\mathbf{a}(x, \nabla u_n), Du_n) - \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w_j\, dx + \Phi_\varphi(w_j)$$

$$+ \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu](\varphi - u_n)\, dH^{N-1} = \Phi_\varphi(w_j) - \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w_j\, dx$$

$$- \int_\Omega div(\mathbf{a}(x, \nabla u_n)) u_n\, dx + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu]\varphi\, dH^{N-1}$$

$$= \Phi_\varphi(w_j) - \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla w_j\, dx + \int_\Omega v_n u_n\, dx + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu]\varphi\, dH^{N-1}.$$

Hence, by (3.87) and (3.88), it follows that

$$\limsup_{n\to\infty} \Phi_\varphi(u_n) \leq \Phi_\varphi(w_j) - \int_\Omega (z, Dw_j) + \int_\Omega uv\, dx + \int_{\partial\Omega} [z, \nu]\varphi\, dH^{N-1}$$

$$= \Phi_\varphi(w_j) - \int_\Omega (z, Dw_j^1) - \int_\Omega (z, Dw_j^2) - \int_\Omega div(z)u\, dx + \int_{\partial\Omega} [z, \nu]\varphi\, dH^{N-1}$$

$$= \Phi_\varphi(w_j) - \int_\Omega (z, Dw_j^1) + \int_\Omega div(z)w_j^2 \, dx + \int_{\partial\Omega} [z, \nu]u \, dH^{N-1} - \int_\Omega div(z)u \, dx.$$

Letting $j \to \infty$, we have that

$$\limsup_{n\to\infty} \Phi_\varphi(u_n) \le \Phi_\varphi(u) - \int_\Omega (z, Du) + \int_{\partial\Omega} [z, \nu]u \, dH^{N-1} - \int_\Omega div(z)u \, dx$$
$$= \Phi_\varphi(u).$$

Finally, by the lower-semicontinuity of Φ_φ, we obtain

$$(3.89) \qquad\qquad \Phi_\varphi(u) = \lim_{n\to\infty} \Phi_\varphi(u_n).$$

If we consider the $\mathbb{R}^N$-valued measures μ_n, μ on $\overline{\Omega}$ which are defined as

$$\mu_n(B) \;\; := \;\; \int_{B\cap\Omega} Du_n + \int_{B\cap\partial\Omega} (\varphi - u_n)\nu \, dH^{N-1},$$
$$\mu(B) \;\; := \;\; \int_{B\cap\Omega} Du + \int_{B\cap\partial\Omega} (\varphi - u)\nu \, dH^{N-1}$$

for all Borel sets $B \subset \overline{\Omega}$, we have

$$\mu_j \rightharpoonup \mu \qquad \text{weakly as measures in } \overline{\Omega}.$$

Moreover,

$$\Phi_\varphi(u) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde\mu) \qquad \text{and} \qquad \Phi_\varphi(u_n) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde\mu_n).$$

Hence, (3.89) yields

$$\lim_{n\to\infty} \int_{\overline{\Omega}} \tilde{f}(x, \tilde\mu_n) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde\mu).$$

Then, applying [18], Theorem 3, it follows that

$$\int_{\overline{\Omega}} \tilde{h}(x, \tilde\mu) \;\; = \;\; \lim_{n\to\infty} \int_{\overline{\Omega}} \tilde{h}(x, \tilde\mu_n)$$
$$= \;\; \lim_{n\to\infty} \int_\Omega h(x, Du_n) + \int_{\partial\Omega} |u_n - \varphi| f^0(x, \nu(x)) \, dH^{N-1}.$$

Since

$$\int_{\overline{\Omega}} \tilde{h}(x, \tilde\mu) = \int_\Omega h(x, Du) + \int_{\partial\Omega} |\varphi - u| f^0(x, \nu(x)) \, dH^{N-1},$$

we have

$$\int_\Omega h(x, Du) + \int_{\partial\Omega} |\varphi - u| f^0(x, \nu(x)) \, dH^{N-1} =$$

(3.90)

$$= \lim_{n\to\infty} \int_\Omega h(x, Du_n) + \int_{\partial\Omega} |u_n - \varphi| f^0(x, \nu(x)) \, dH^{N-1}.$$

Now, since

$$\lim_{n\to\infty} \int_\Omega h(x, Du_n) + \int_{\partial\Omega} |u_n - \varphi| f^0(x, \nu(x)) \, dH^{N-1}$$

$$= \lim_{n\to\infty} \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla u_n \, dx + \int_\Omega f^0(x, D^s u_n)$$

$$+ \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu](\varphi - u_n) \, dH^{N-1}$$

$$= \lim_{n\to\infty} \int_\Omega (\mathbf{a}(x, \nabla u_n), Du_n) + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu](\varphi - u_n) \, dH^{N-1}$$

$$= \lim_{n\to\infty} \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu]\varphi \, dH^{N-1} - \int_\Omega div(\mathbf{a}(x, \nabla u_n)) u_n \, dx$$

$$= \int_{\partial\Omega} [z, \nu]\varphi \, dH^{N-1} - \int_\Omega div(z) u \, dx$$

$$= \int_\Omega (z, Du) + \int_{\partial\Omega} [z, \nu](\varphi - u) \, dH^{N-1},$$

we finally obtain

(3.91)
$$\int_\Omega h(x, Du) + \int_{\partial\Omega} |\varphi - u| f^0(x, \nu(x)) \, dH^{N-1}$$

$$= \int_\Omega (z, Du) + \int_{\partial\Omega} [z, \nu](\varphi - u) \, dH^{N-1}.$$

Again, by (3.87) and (3.88) we can apply Lemma 3.6 obtaining that

(3.92) $$|[z(x), \nu(x)]| \leq f^0(x, \nu(x)) \quad \text{a.e. in } \partial\Omega.$$

Moreover, acting as in the proof of Proposition 3.2, we get that

(3.93) $$|z \cdot D^s u| \leq f^0(x, D^s u) \quad \text{as measures in } \Omega.$$

Hence by (3.89), (3.90), (3.91), (3.87) (3.93) and (3.92), we can apply Lemma 3.5, to obtain

(3.94) $$\int_\Omega z \cdot \nabla u \, dx = \int_\Omega h(x, \nabla u) \, dx = \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla u \, dx,$$

(3.95) $$z \cdot D^s u = f^0(x, D^s u),$$

(3.96) $$[z, \nu] \in \text{sign}\, (\varphi - u) f^0(x, \nu(x)) \qquad H^{N-1} - \text{a.e.}$$

Now, using Lemma 3.7, we have

(3.97) $$div \ z = div \ \mathbf{a}(x, \nabla u), \qquad in \ \mathcal{D}'(\Omega),$$

and

(3.98) $$[z, \nu] = [\mathbf{a}(x, \nabla u), \nu] \qquad H^{N-1} - a.e. \ on \ \partial\Omega.$$

Since $v = -div(z)$ in $\mathcal{D}'(\Omega)$, taking (3.97) into account, we get

$$v = -div(\mathbf{a}(x, \nabla u)), \qquad in \ \mathcal{D}'(\Omega),$$

and, using (3.94), (3.95) and (3.97), we get

$$\mathbf{a}(x, \nabla u) \cdot D^s u = f^0(x, D^s u).$$

Finally, by (3.96) and (3.98) we get

$$[\mathbf{a}(x, \nabla u), \nu] \in sign \ (\varphi - u) f^0(x, \nu(x)) \qquad H^{N-1} - a.e.$$

Therefore, $(u, v) \in \mathcal{B}_\varphi$. ∎

Proof of Theorem 2.2 Let $(S(t))_{t \geq 0}$ be the semigroup in $L^2(\Omega)$ generated by the subdifferential of Φ_φ. Then by the nonlinear semigroup theory ([9]), given $u_0 \in L^2(\Omega) = \overline{D(\partial\Phi_\varphi)}$, $u(t) = S(t)u_0$ is the only strong solution of problem (3.1). Thus, by Theorem 3.1, we have that for almost all $t \in [0, +\infty[$, $u(t) \in D(\mathcal{B}_\varphi)$ and $-u'(t) \in \mathcal{B}_\varphi(u(t))$. This concludes the proof. ∎

4. Behaviour of the solution

We have the following weak form of the maximum principle.

Theorem 4.1 *Suppose u_1 and u_2 are two solutions of (1.1) corresponding to initial data $u_{1,0}$ and $u_{2,0}$ in $L^2(\Omega)$ and boundary data φ_1 and φ_2 in $L^1(\partial\Omega)$, respectively. If*

$$u_{1,0} \geq u_{2,0} \quad and \quad \varphi_1 \geq \varphi_2,$$

then $u_1 \geq u_2$.

Proof. For almost all $t \in [0, +\infty[$, we have $u_i'(t) \in L^2(\Omega)$, $u_i(t) \in BV(\Omega) \cap L^2(\Omega)$, $\mathbf{a}(x, \nabla u_i(t)) \in X(\Omega)$, and

(4.1) $\quad u_2'(t) - u_1'(t) = div[\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t))], \quad in \ \mathcal{D}'(\Omega),$

(4.2) $\qquad \mathbf{a}(x, \nabla u_i(t)) \cdot D^s u_i(t) = f^0(x, D^s u_i(t)),$

(4.3) $[\mathbf{a}(x, \nabla u_i(t)), \nu] \in sign(\varphi_i - u_i(t)) f^0(x, \nu(x)) \quad H^{N-1} - a.e. \ on \ \partial\Omega.$

Multiplying in (4.1) by $\left(u_2(t) - u_1(t)\right)^+$, integrating in Ω, and using Green's formula, we get

$$\frac{1}{2} \int_\Omega \frac{d}{dt} \left[\left(u_2(t) - u_1(t)\right)^+\right]^2 \, dx =$$

(4.4)

$$= \int_\Omega div\left[\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t))\right]\left(u_2(t) - u_1(t)\right)^+ \, dx$$

$$= -\int_\Omega \left(\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t)), D\left(\left(u_2(t) - u_1(t)\right)^+\right)\right) +$$

$$+ \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t)), \nu]\left(u_2(t) - u_1(t)\right)^+ \, dH^{N-1}.$$

Now, by the chain rule for BV-functions ([2], [14], Lemma 1.2), there exists a scalar function $\eta(t)$, with $0 \leq \eta(t) \leq 1$, such that

$$\int_\Omega \left(\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t)), D\left(\left(u_2(t) - u_1(t)\right)^+\right)\right) =$$

$$= \int_{\{u_2 \geq u_1\}} \left(\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t)) \cdot \left(\nabla u_2(t) - \nabla u_1(t)\right) \, dx +$$

$$+ \int_\Omega \eta(t)\left(\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t)) \cdot D^s\left(u_2(t) - u_1(t)\right).$$

Observe that, by the monotonicity of $\mathbf{a}$, (H_5) and (4.2), we have that

(4.5) $$\int_\Omega \left(\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t)), D\left(\left(u_2(t) - u_1(t)\right)^+\right)\right) \geq 0.$$

On the other hand, since $\varphi_1 \geq \varphi_2$, from (4.3), it is easy to see that

(4.6) $$\int_{\partial\Omega} [\mathbf{a}(x, \nabla u_2(t)) - \mathbf{a}(x, \nabla u_1(t)), \nu]\left(u_2(t) - u_1(t)\right)^+ \, dH^{N-1} \leq 0.$$

From (4.4), (4.5) and (4.6), it follows that

$$\frac{1}{2} \int_\Omega \frac{d}{dt} \left[\left(u_2(t) - u_1(t)\right)^+\right]^2 \, dx \leq 0.$$

Since $u_{1,0} \geq u_{2,0}$, we have $u_1 \geq u_2$, and the proof is concluded. ∎

We shall now prove that the solution $u(t)$ stabilizes as $t \to +\infty$ by converging to a solution of the steady-state problem. To do that, we follow the proof of Theorem 4.2 in [17].

Theorem 4.2 *Suppose $u_0 \in L^2(\Omega) \cap BV(\Omega)$ and $\varphi \in L^\infty(\partial\Omega)$. Then the solution $u(t)$ of (1.1) converges as $t \to +\infty$ to some limit $w \in \mathcal{B}_\varphi^{-1}(0)$ in the following sense:*

$$u(t) \to w \quad \text{strongly in} \quad L^1(\Omega) \quad \text{and weakly in} \quad L^2(\Omega).$$

Proof. Since $\mathcal{B}_\varphi$ is the subdifferential of Φ_φ, by a classical result of Bruck ([10], Theorem 4), to prove the weak convergence in $L^2(\Omega)$, it is sufficient to prove that Φ_φ attains its minimum in $L^2(\Omega)$. In fact, let $\{u_n\}$ be a minimizing sequence for Φ_φ. Without loss of generality, we may assume that $u_n \in BV(\Omega) \cap L^2(\Omega)$. Now, by approximation we may assume that $u_n \in W^{1,1}(\Omega) \cap L^2(\Omega)$. Denote by $J : \mathbb{R} \to \mathbb{R}$ the truncature function

$$J(r) := \begin{cases} -\|\varphi\|_\infty & \text{if} \quad r < -\|\varphi\|_\infty \\ r & \text{if} \quad |r| \leq \|\varphi\|_\infty \\ \|\varphi\|_\infty & \text{if} \quad r > \|\varphi\|_\infty. \end{cases}$$

If we take $w_n := J \circ u_n$, $w_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, and using that $|J'| \leq 1$, we have

$$\Phi_\varphi(w_n) = \int_\Omega f(x, \nabla w_n) \, dx + \int_{\partial\Omega} |w_n - \varphi| f^0(x, \nu(x)) \, dH^{N-1} =$$

$$= \int_{\{|u_n| \leq \|\varphi\|_\infty\}} f(x, \nabla u_n) \, dx + \int_{\partial\Omega} |J \circ u_n - J \circ \varphi| f^0(x, \nu(x)) \, dH^{N-1} \leq$$

$$\leq \int_\Omega f(x, \nabla u_n) \, dx + \int_{\partial\Omega} |u_n - \varphi| f^0(x, \nu(x)) \, dH^{N-1}.$$

Thus, $\{w_n\}$ is still a minimizing sequence for Φ_φ. Moreover, this sequence is bounded in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$, hence, relatively compact in $L^1(\Omega)$. We may extract a subsequence converging in $L^1(\Omega)$ to some $\overline{u} \in L^1(\Omega) \cap BV(\Omega)$. Therefore,

$$\Phi_\varphi(\overline{u}) = \inf_{u \in L^2(\Omega)} \Phi_\varphi(u).$$

Then, by Bruck's result ([10], Theorem 4), there exists $w \in \mathcal{B}_\varphi^{-1}(0)$, such that $u(t) \to w$ weakly in $L^2(\Omega)$. Finally, we prove the strong convergence in $L^1(\Omega)$. Since $(u(t), -u'(t)) \in \partial\Phi_\varphi$, using [9], Lemma 3.3, we have

$$\frac{d}{ds}\Phi_\varphi\big(u(s)\big) = -\int_\Omega u'(s)^2 \, dx \leq 0,$$

hence,

$$\Phi_\varphi\big(u(t)\big) \leq \Phi_\varphi\big(u_0\big) \quad \forall \, t > 0.$$

Thus, $\{u(t) \, : \, t \geq 0\}$ is bounded in $BV(\Omega)$, and therefore relatively compact in $L^1(\Omega)$. The result follows. ∎

5. Appendix

In this appendix we prove the approximation Lemma. Before giving the proof, let us construct a substitute for the distance function to the boundary $d(., \partial\Omega)$. That construction would be unnecessary if $\partial\Omega$ would be of class $W^{2,\infty}$ ([7]). We follow the proof of Lemma 5.1 in [7] for C^2 domains.

If $\partial\Omega$ is a manifold of class C^1, then there is some $\varepsilon > 0$ such that for all points $y \in \Omega$ such that $d(y, \Omega) < \varepsilon$ there is $z \in \partial\Omega$ and $t \in (0, \varepsilon)$ such that $y = z - t\nu(z)$, $\nu(z)$ being the outer unit normal to $\partial\Omega$ at z ([11]). In other words, $\Omega^\varepsilon := \{x \in \Omega : x = y - t\nu(y),\ y \in \partial\Omega,\ t \in (0, \varepsilon)\}$ is open. Then there is a function $D \in C^1(\overline{\Omega})$ such that $D = 0$ on $\partial\Omega$, $D > 0$ on Ω and $\nabla D(x) = -\nu(x)$ for all $x \in \partial\Omega$. This is a consequence of Withney's extension Theorem ([15], p.48, [13], p.245). Indeed, since

$$\nu(y) \frac{x - y}{|x - y|} \to 0 \ \text{ as } x, y \to p,\ x \neq y,\ x, y \in \partial\Omega,$$

by Withney's Theorem , we know that there exists a function $\tilde{D} \in C^1(\overline{\Omega})$ such that $\tilde{D} = 0$ on $\partial\Omega$ and $\nabla\tilde{D}(x) = -\nu(x)$ for all $x \in \partial\Omega$. Now, let $y \in \partial\Omega$ and $t \in (0, \varepsilon)$. Using the mean value theorem, we know that

$$\tilde{D}(y - t\nu(y)) = \tilde{D}(y) - t\nabla\tilde{D}(y - s\nu(y)) \cdot \nu(y) = -t\nabla\tilde{D}(y - s\nu(y)) \cdot \nu(y).$$

Since $\tilde{D} \in C^1(\overline{\Omega})$, we have

$$\tilde{D}(y - t\nu(y)) = t(1 + \omega(t))$$

where $\omega(t) = o(1)$ as $t \to 0+$ and is the modulus of continuity of $\nabla\tilde{D}$. Without loss of generality we may assume that $\varepsilon > 0$ is such that $\omega(t) < \frac{1}{2}$ for all $t \in (0, \varepsilon)$. In particular, we have that

(5.1) $$\tilde{D}(x) > 0 \qquad \text{for all } x \in \Omega^\varepsilon.$$

We shall modify $\tilde{D}$ so that the modified function is > 0 in Ω. Let $\eta \in C([0, \infty))$, $\eta(t) > 0$, for all $t \in (0, \infty)$, $\eta(t) = o(t)$ as $t \to 0+$. Let Ω_1 be an open set, $\overline{\Omega_1} \subset \Omega$, with smooth boundary $\partial\Omega_1 \subset \Omega^\varepsilon$ such that $0 < \delta - \eta(\delta) < \tilde{D}(x) < \delta + \eta(\delta)$ for all $x \in \partial\Omega_1$ for some $\delta > 0$. Let Ω_2' be an open set with smooth boundary such that $\overline{\Omega_2'} \subset \Omega_1$ and $\eta(\delta) < d(\partial\Omega_1, \partial\Omega_2') < 2\eta(\delta)$, where $d(\partial\Omega_1, \partial\Omega_2') = \inf\{|x - y| : x \in \partial\Omega_1,\ y \in \partial\Omega_2'\}$. Let $d_{\partial\Omega_2'}$ be the distance function to $\partial\Omega_2'$, > 0 in Ω_2', negative outside. Let $d_{\partial\Omega_2', n} = \rho_n * d_{\partial\Omega_2'}$, ρ_n being a positive regularizing kernel. Observe that $\|\nabla d_{\partial\Omega_2', n}\|_\infty \leq 1$. We may choose n large enough, and Ω_2 such that $\overline{\Omega_2} \subset \Omega_2'$, $\eta(\delta) < d(\partial\Omega_1, \partial\Omega_2) <$

$2\eta(\delta)$, $0 < d_{\partial\Omega'_2,n} < \eta(\delta)$ in $\partial\Omega_2$, and $d_{\partial\Omega'_2,n} > 0$ in Ω_2. Let $B_{1,2} = \Omega_1 \setminus \overline{\Omega_2}$. Then, using again Withney's extension Theorem, there is a function $R \in C^1(\overline{B_{1,2}})$ such that $R = \tilde{D} - \delta$ and $\nabla R = \nabla\tilde{D}$ on $\partial\Omega_1$, and $R = d_{\partial\Omega'_2,n}$, $\nabla R = \nabla d_{\partial\Omega'_2,n}$ on $\partial\Omega_2$. Moreover, $\|\nabla R\|_\infty$ is bounded by a constant depending on $\|\tilde{D}\|_{\infty,\partial\Omega_1}$, $\|d_{\partial\Omega'_2,n}\|_{\infty,\partial\Omega_2}$, $\|\nabla\tilde{D}\|_{\infty,\partial\Omega_1}$, $\|\nabla d_{\partial\Omega'_2,n}\|_{\infty,\partial\Omega_2}$ and

$$\sup_{x\in\partial\Omega_1, y\in\partial\Omega_2} \frac{|\tilde{D}(x) - \delta - d_{\partial\Omega'_2,n}(y)|}{|x - y|} \le \frac{2\eta(\delta)}{\eta(\delta)} = 2.$$

We define $D : \overline{\Omega} \to \mathbb{R}$ by

$$\begin{aligned} D &= \tilde{D} \quad \text{in } \overline{\Omega_1}, \\ D &= R + \delta \quad \text{in } B_{1,2}, \\ D &= d_{\partial\Omega'_2,n} + \delta \quad \text{in } \overline{\Omega_2}. \end{aligned}$$

Then $D \in C^1(\overline{\Omega})$, $D = 0$ on $\partial\Omega$, $D > 0$ on Ω and $\nabla D(x) = -\nu(x)$ for all $x \in \partial\Omega$.

Proof of Lemma 3.4 We may think that u and v are extended as BV functions in $\mathbb{R}^N$ in such a way that

$$(5.2) \qquad \int_{\partial\Omega} |Du| = \int_{\partial\Omega} |Dv| = 0.$$

We consider a family of radially symmetric positive mollifiers

$$\eta_j = \frac{1}{\tau_j^N} \eta\left(\frac{x}{\tau_j}\right), \quad \eta \ge 0, \quad \int_{\mathbb{R}^N} \eta(x)dx = 1, \quad \tau_j \downarrow 0+,$$

and we set

$$(5.3) \qquad z_j = \eta_j * \left(v + \frac{u}{j}\right)$$

Clearly, we have $z_j \in C^1(\overline{\Omega})$ and obviously we have

$$(5.4) \qquad z_j \to v \quad \text{in } L^{N/(N-1)}(\Omega).$$

Also, from (5.2) it follows that

$$(5.5) \qquad \int_\Omega \sqrt{1 + |Dz_j(x)|^2}dx \to \int_\Omega \sqrt{1 + |Dv(x)|^2}dx.$$

This implies, by the Theorem of convergence of traces for BV functions that

$$(5.6) \qquad z_j|_{\partial\Omega} \to v|_{\partial\Omega} \quad \text{in } L^1(\partial\Omega).$$

By the Theorem of differentiation of measures ([7]), we obtain

$$(5.7) \qquad Dz_j(x) \to \nabla v(x) \quad \lambda_N \text{ a.e. in } \Omega.$$

Indeed, since $Dz_j = \eta_j * Dv + \frac{1}{j}\eta_j * Du$, this is a consequence of the four following limits

$$(5.8) \qquad \lim_j [\eta_j * \nabla v](x) = \nabla v(x) \quad \lambda_N \text{ a.e. in } \Omega,$$

$$(5.9) \qquad \lim_j [\eta_j * (Dv)^s](x) = (Dv)^s(x) = 0 \quad \lambda_N \text{ a.e. in } \Omega,$$

$$(5.10) \qquad \lim_j \frac{1}{j}[\eta_j * \nabla u](x) = \nabla u(x)\lim_j \frac{1}{j} = 0 \quad \lambda_N \text{ a.e. in } \Omega,$$

$$(5.11) \qquad \lim_j \frac{1}{j}[\eta_j * (Du)^s](x) = 0 \quad \lambda_N \text{ a.e. in } \Omega,$$

since $(Du)^s$, $(Dv)^s$ are singular with respect to λ_N and $|\nabla u(x)| < \infty$ λ_N a.e. in Ω. In the same way, using the Theorem of differentiation of measures, we have

$$(5.12) \qquad \lim_j [\eta_j * \nabla v](x) = 0 \quad |Dv|^s \text{ a.e. in } \Omega,$$

$$(5.13) \qquad \lim_j [\eta_j * \nabla u](x) = 0 \quad |Dv|^s \text{ a.e. in } \Omega,$$

$$(5.14) \qquad \lim_j [\eta_j * (Du)^{ss}](x) = 0 \quad |Dv|^s \text{ a.e. in } \Omega,$$

$$(5.15) \qquad \lim_j \frac{1}{j}[\eta_j * (Du)^{sa}](x) = (Du)^{sa}(x)\lim \frac{1}{j} = 0 \quad |Dv|^s \text{ a.e. in } \Omega,$$

where $(Du)^{sa}$, $(Du)^{ss}$ denote the absolutely continuous and singular part of $(Du)^s$ with respect to $(Dv)^s$, and we obtain

$$(5.16) \qquad \lim_j \frac{Dz_j(x)}{|Dz_j(x)|} = \lim_j \frac{Dz_j(x)}{|[\eta_j * (Dv)^s]|(x)} = \frac{Dv}{|Dv|}(x) \quad |Dv|^s \text{ a.e. in } \Omega.$$

Similarly

$$(5.17) \qquad \lim_j |Dz_j(x)| = \lim_j |[\eta_j * |Dv|^s](x)| = \infty \quad |Dv|^s \text{ a.e. in } \Omega.$$

Next, we prove that for a suitable choice of the numbers τ_j one has

$$(5.18) \qquad \lim_j \frac{Dz_j(x)}{(1/j)[\eta_j * |Du|^{ss}](x)} = \frac{Du}{|Du|}(x) \quad |Du|^{ss} \text{ a.e..}$$

Assuming this, it is easy to prove that

$$(5.19) \qquad \lim_j \frac{Dz_j(x)}{|Dz_j(x)|} = \frac{Du}{|Du|}(x) \quad |Du|^{ss} \text{ a.e..}$$

Indeed,

$$
\begin{aligned}
Dz_j &= \eta_j * Dv(x) + \frac{1}{j}\eta_j * Du(x) \\
&= \eta_j * \nabla v(x) + \eta_j * (Dv)^s(x) + \frac{1}{j}\eta_j * \nabla u(x) \\
&\quad + \frac{1}{j}\eta_j * (Du)^{sa}(x) + \frac{1}{j}\eta_j * (Du)^{ss}(x).
\end{aligned}
$$

Since $\eta_j*\nabla v(x) \to 0$, $\eta_j*(Dv)^s(x) \to 0$, $\frac{1}{j}\eta_j*\nabla u(x) \to 0$, $\frac{1}{j}\eta_j*(Du)^{sa}(x) \to 0$ $|Du|^{ss}$-a.e., we see that (5.19) follows from (5.18). To prove (5.18) we observe that

$$
(5.20) \qquad \frac{Dz_j(x)}{(1/j)[\eta_j * |Du|^{ss}](x)} = \frac{[\eta_j * Dv](x)}{(1/j)[\eta_j * |Du|^{ss}](x)} + \frac{[\eta_j * Du](x)}{[\eta_j * |Du|^{ss}](x)}.
$$

Since

$$
(5.21) \qquad \frac{[\eta_j * Du](x)}{[\eta_j * |Du|^{ss}](x)} \to \frac{Du}{|Du|}(x) \quad |Du|^{ss} \text{ a.e.},
$$

it is sufficient to prove that

$$
(5.22) \qquad \frac{[\eta_j * Dv](x)}{\frac{1}{j}[\eta_j * |Du|^{ss}](x)} \to 0 \quad |Du|^{ss} \text{ a.e..}
$$

To prove (5.22), we define

$$
(5.23) \qquad a_\tau(x) = \frac{[\eta_\tau * Dv](x)}{[\eta_\tau * |Du|^{ss}](x)}.
$$

Since Dv and $|Du|^{ss}$ are mutually singular, then

$$
a_\tau(x) \to 0 \quad |Du|^{ss} \text{ a.e..}
$$

Thus, if we consider the sets

$$
E(\tau, j) = \left\{ x \in \Omega : |a_\tau(x)| > \frac{1}{j^2} \right\},
$$

for any fixed $j \in \mathbb{N}$ we have

$$
\lim_{\tau \to 0} |Du|^{ss}(E(\tau, j)) = 0.
$$

For each $j \in \mathbb{N}$, there is some τ_j such that

$$
|Du|^{ss}(E(\tau_j, j)) < \frac{1}{2^j},
$$

that is

$$|Du|^{ss}(\{x \in \Omega : j|a_\tau(x)| > \frac{1}{j}\}) < \frac{1}{2^j}.$$

This easily implies that

$$\lim_{j \to \infty} ja_{\tau_j}(x) = 0, \quad |Du|^{ss} \text{ a.e..}$$

which is exactly (5.22). Moreover, we may choose τ_j such that $\frac{1}{j}[\eta_j * |Du|^{ss}](x) \to \infty$ $|Du|^{ss}$ a.e.. From this, and (5.18), it follows that

$$(5.24) \qquad\qquad |Dz_j|(x) \to \infty \quad |Du|^{ss} \text{ a.e..}$$

We observe that up to know we have not used neither the hypothesis on the regularity of $\partial\Omega$ nor the regularity of g.

We observe that the functions z_j that we have constructed satisfy some of the requirements of the Lemma but not all of them, in particular, (3.6), (3.8), (3.12), (3.13) have yet to be satisfied. For that, we construct suitable correction functions σ_j and ρ_j around the boundary and we shall define

$$v_j = z_j + \sigma_j + \rho_j.$$

Let $g_j \in C^1(\partial\Omega)$ be such that $g_j \to g$ in $L^1(\partial\Omega)$. We shall construct the sequence of functions $\sigma_j \in C^1(\overline{\Omega})$ such that

$$(5.25) \qquad\qquad \sigma_j = g_j - z_j \quad \text{on } \partial\Omega,$$

$$(5.26) \qquad\qquad \int_\Omega |\sigma_j|^{\frac{N}{N-1}} \to 0,$$

$$(5.27) \qquad\qquad \sigma_j(x) = 0 \quad \text{if } D(x) > \varepsilon_j + \varepsilon_j^2,$$

$$(5.28) \qquad\qquad \int_\Omega \psi \cdot D\sigma_j \to \int_{\partial\Omega} \psi \cdot \nu(g - v)\, dH^{N-1}$$

for all $\psi \in C(\overline{\Omega}, \mathbb{R}^N)$,

$$(5.29) \qquad \int_\Omega |D\sigma_j| \to \int_{\partial\Omega} |v - g|\, dH^{N-1},$$

$$(5.30) \quad |D(\sigma_j + z_j)(x)| \to \infty \quad H^{N-1} \text{ a.e. in } T = \{x \in \partial\Omega : g(x) \neq v(x)\},$$

and

$$(5.31) \qquad\qquad \frac{D(\sigma_j + z_j)(x)}{|D(\sigma_j + z_j)(x)|} \to \frac{g(x) - v(x)}{|g(x) - v(x)|}\nu(x)$$

H^{N-1} a.e. in $T = \{x \in \partial\Omega : g(x) \neq v(x)\}$.

Construction of σ_j.

For each number $\varepsilon \in (0, \varepsilon_0)$ we consider a function $h_\varepsilon(t) : [0, \infty) \to [0, \infty)$ such that

$$h_\varepsilon \in C^1([0, \infty)),$$

$$h'_\varepsilon(t) \le 0, \quad h'_\varepsilon(0) = -\frac{1}{\varepsilon},$$

$$h'_\varepsilon(t) \text{ is not decreasing,}$$

$$h_\varepsilon(0) = 1, \ h_\varepsilon(t) = 0 \text{ for } t \ge \varepsilon + \varepsilon^2.$$

Let $\{\varepsilon_n\}_{n=1}^\infty$ be a decreasing sequence of numbers such that

$$2\varepsilon_1 < \varepsilon_0 < 1, \quad \lim_j \varepsilon_j = 0.$$

Now, let $G \in W^{1,1}(\Omega)$ such that $G_{|\partial\Omega} = g$. Since $g_j \in C^1(\partial\Omega)$, we may consider a function $G_j \in C^1(\overline{\Omega})$ which is an extension of g_j. We may assume that $G_j \to G$ in $L^1(\Omega)$ and $\int_\Omega |\nabla G_j| \to \int_\Omega |\nabla G|$. We define

(5.32) $$\sigma_j = [G_j(x) - z_j(x)] h_{\varepsilon_j}(D(x)).$$

Clearly, $\sigma_j \in C^1(\overline{\Omega})$,

$$\sigma_j = g_j - z_j \quad \text{on } \partial\Omega,$$

and, if $D(x) > \varepsilon_j + \varepsilon_j^2$, then $h_{\varepsilon_j}(D(x)) = 0$, and, therefore

$$\sigma_j(x) = 0.$$

Now,

$$\int_\Omega |\sigma_j|^{N/(N-1)} = \int_{\Omega_{2\varepsilon_j}} |\sigma_j|^{N/(N-1)} \le \int_{\Omega_{2\varepsilon_j}} |G_j(x) - z_j(x)|^{N/(N-1)}$$

where, for any $\varepsilon > 0$, we denote

$$\Omega_\varepsilon = \{x \in \Omega : D(x) < \varepsilon\}.$$

The functions G_j, z_j being independent of ε_j, we may choose $\varepsilon_j > 0$ small enough such that

$$\int_{\Omega_{2\varepsilon_j}} |G_j(x) - z_j(x)|^{N/(N-1)} < \frac{1}{j}.$$

Hence

$$\int_\Omega |\sigma_j|^{N/(N-1)} \to 0 \quad \text{as } j \to \infty.$$

Let $\psi \in C(\overline{\Omega}, \mathbb{R}^N)$. Since

$$\nabla\sigma_j(x) = \nabla(G_j - z_j)(x)h_{\varepsilon_j}(D(x)) + (G_j - z_j)(x)h'_{\varepsilon_j}(D(x))\nabla D(x),$$

we have

$$
\begin{aligned}
\int_\Omega \psi(x) \cdot \nabla\sigma_j(x)dx &= \int_\Omega \psi(x) \cdot (\nabla G_j(x) - \nabla z_j(x))h_{\varepsilon_j}(D(x))dx \\
&\quad + \int_\Omega (G_j(x) - z_j(x))\psi(x) \cdot \nabla(h_{\varepsilon_j}(D(x)))dx \\
&= \int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} \psi(x) \cdot (\nabla G_j(x) - \nabla z_j(x))h_{\varepsilon_j}(D(x))dx \\
&\quad + \int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} (G_j(x) - z_j(x))\psi(x) \cdot h'_{\varepsilon_j}(D(x))\nabla D(x)dx.
\end{aligned}
$$

Again, since $|h_\varepsilon| \leq 1$ for all $\varepsilon > 0$, a proper choice of ε_j guarantees that

$$\int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} \psi(x) \cdot (\nabla G_j(x) - \nabla z_j(x))h_{\varepsilon_j}(D(x))dx \to 0$$

as $j \to \infty$. Now, by our choice of G_j, (5.5) and a proper choice of ε_j, we have that
(5.33)
$$\int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} (G_j(x) - z_j(x))\psi(x) \cdot h'_{\varepsilon_j}(D(x))\nabla D(x)dx \to \int_{\partial\Omega} \psi \cdot \nu(g - v)dH^{N-1}.$$

Indeed, using the change of variable's formula ([13], p. 118, [20], p. 96),

$$
\begin{aligned}
&\int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} (G_j(x) - z_j(x))\psi(x) \cdot h'_{\varepsilon_j}(D(x))\nabla D(x)dx \\
&= \int_0^{\varepsilon_j + \varepsilon_j^2} \int_{[D=\lambda]} (G_j(y) - z_j(y))\psi(y) \cdot h'_{\varepsilon_j}(D(y))\frac{\nabla D(y)}{|\nabla D(y)|}dH^{N-1}(y)d\lambda \\
&= (\varepsilon_j + \varepsilon_j^2) \int_{[D=\lambda_j]} (G_j(y) - z_j(y))\psi(y) \cdot h'_{\varepsilon_j}(D(y))\frac{\nabla D(y)}{|\nabla D(y)|}dH^{N-1}(y)
\end{aligned}
$$

for some $\lambda_j \in (0, \varepsilon_j + \varepsilon_j^2)$ by the intermediate value Theorem. Now, since G_j, z_j do not depend on our choice of ε_j, by choosing $\varepsilon_j \to 0+$ sufficiently fast, we obtain(5.33). Hence

$$\int_\Omega \psi(x) \cdot \nabla\sigma_j(x)dx \to \int_{\partial\Omega} \psi \cdot \nu(g - v)dH^{N-1}$$

as $j \to \infty$. In particular, we have

(5.34)
$$\liminf_j \int_\Omega |\nabla \sigma_j(x)| dx \geq \int_{\partial\Omega} |g - v| dH^{N-1}.$$

On the other hand, we have

$$\int_\Omega |\nabla \sigma_j(x)| dx \leq \int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} |\nabla G_j(x) - \nabla z_j(x)| dx$$
$$+ \int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} |G_j(x) - z_j(x)| |h'_{\varepsilon_j}(D(x))| |\nabla D(x)| dx.$$

Again, a suitable choice of ε_j guarantees that

$$\int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} |\nabla G_j(x) - \nabla z_j(x)| dx \to 0$$

as $j \to \infty$. Similarly, the properties of G_j, z_j and a choice of ε_j imply that

$$\int_{\Omega_{\varepsilon_j + \varepsilon_j^2}} |G_j(x) - z_j(x)| |h'_{\varepsilon_j}(D(x))| |\nabla D(x)| dx \to \int_{\partial\Omega} |g(x) - v(x)| dH^{N-1}.$$

Hence
$$\limsup_j \int_\Omega |\nabla \sigma_j(x)| dx \leq \int_{\partial\Omega} |g(x) - v(x)| dH^{N-1}.$$

This, together with (5.34) proves that

(5.35)
$$\lim_j \int_\Omega |\nabla \sigma_j(x)| dx = \int_{\partial\Omega} |g - v| dH^{N-1}.$$

Finally, since

$$D\sigma_j + Dz_j = \nabla G_j h_{\varepsilon_j}(D) + \nabla z_j (1 - h_{\varepsilon_j}(D)) + (G_j - z_j) h'_{\varepsilon_j}(D) \nabla D,$$

we may write on $\partial\Omega$

(5.36)
$$D\sigma_j + Dz_j = \nabla G_j - (G_j - z_j) h'_{\varepsilon_j}(0) \nu(x).$$

Hence, on $\partial\Omega$, we have

$$\frac{D\sigma_j + Dz_j}{|D\sigma_j + Dz_j|} = \frac{\nabla G_j - (G_j - z_j) h'_{\varepsilon_j}(0)\nu}{|\nabla G_j - (G_j - z_j) h'_{\varepsilon_j}(0)\nu|} = \frac{\varepsilon_j \nabla G_j + (G_j - z_j)\nu}{|\varepsilon_j \nabla G_j + (G_j - z_j)\nu|}.$$

Now, choosing ε_j such that $\varepsilon_j \nabla G_j \to 0$ as $j \to \infty$, we obtain that

$$(5.37) \qquad \frac{D\sigma_j + Dz_j}{|D\sigma_j + Dz_j|} \to \frac{g(x) - v(x)}{|g(x) - v(x)|}\nu(x)$$

H^{N-1} a.e. in $T = \{x \in \partial\Omega : g(x) \neq v(x)\}$. Next, a proper choice of ε_j in (5.36) guarantees that

$$(5.38) \qquad |D\sigma_j(x) + Dz_j(x)| \to \infty$$

H^{N-1} a.e. in T.

Next, we construct a sequence of functions $\rho_j \in C^1(\overline{\Omega})$ such that

$$(5.39) \qquad \rho_j \ = \ 0 \qquad \text{on } \partial\Omega,$$

$$(5.40) \qquad \int_\Omega |\rho_j|^{\frac{N}{N-1}} \ \to \ 0,$$

$$(5.41) \qquad \rho_j \ = \ 0 \qquad \text{if } D(x) > \delta_j^2$$

for some $\delta_j > 0$,

$$(5.42) \qquad \int_\Omega |D\rho_j| \to 0\,;$$

for H^{N-1} -a.e. $x \in T$, there is $j_0(x)$ such that

$$(5.43) \qquad D\rho_j(x) = 0 \quad \text{for all } j \geq j_0(x);$$

$$(5.44) \qquad \text{if we set } v_j = z_j + \sigma_j + \rho_j \text{ then (3.13) holds.}$$

Construction of ρ_j.

For all $\delta > 0$ consider a function $\psi_\delta : [0, \infty) \to [0, \infty)$ such that

$$\begin{cases} \psi_\delta \in C^1([0, \infty)), \\ \psi_\delta(0) = 0, \quad \psi_\delta(t) = 0 \text{ for } t \geq \delta^2, \\ |\psi_\delta'(t)| \leq \dfrac{4}{\delta}, \text{ for } t \in (0, \delta^2), \\ \psi_\delta'(0) \geq \dfrac{1}{\delta}, \\ \displaystyle\int_0^\infty |\psi_\delta'(t)|dt \leq 2\delta. \end{cases}$$

We define $\zeta : \partial\Omega \to \mathbb{R}$

(5.45) $\qquad \zeta(x) = \begin{cases} \dfrac{u(x) - g(x)}{|u(x) - g(x)|} & \text{if } g(x) = v(x) \text{ and } g(x) \neq u(x) \\[2mm] 0 & \text{elsewhere} \end{cases}$

Let ζ_j be a sequence of functions in $C^1(\partial\Omega)$ converging to ζ in $L^1(\partial\Omega)$. Now, we may assume that ζ is the trace of a function $\Theta \in W^{1,1}(\Omega)$ and ζ_j are traces of functions $\Theta_j \in C^1(\overline{\Omega})$ such that $\Theta_j \to \Theta$ in $L^1(\Omega)$ and $\int_\Omega |D\Theta_j| \to \int_\Omega |D\Theta|$. Let δ_j be a decreasing sequence of positive numbers that converges to 0 and consider the functions

(5.46) $\qquad\qquad\qquad \rho_j(x) = \Theta_j(x)\psi_{\delta_j}(D(x)).$

Clearly, $\rho_j \in C^1(\overline{\Omega})$, $\rho_j(x) = 0$ if $x \in \partial\Omega$. Also (5.41) holds. Since, by our choice of the functions ψ_δ, we have

(5.47) $\qquad\qquad\qquad\qquad |\psi_\delta(t)| \leq 2\delta.$

Now,

$$\int_\Omega |\rho_j|^{N/(N-1)} \leq 2\delta_j \int_\Omega |\Theta_j|^{N/(N-1)}$$

which tends to 0 as $j \to \infty$, which proves (5.40).

Our purpose now is to choose the functions ζ_j such that (5.43) holds. For that, we consider the sets

$$N^+ = \{x \in \partial\Omega : \zeta(x) = 1\},$$
$$N^- = \{x \in \partial\Omega : \zeta(x) = -1\},$$
$$N = N^+ \cup N^-.$$

We consider increasing sequences of compact sets $K_j^+ \subseteq N^+$, $K_j^- \subseteq N^-$ such that

$$\lim_j H^{N-1}(N^+ \setminus K_j^+) = \lim_j H^{N-1}(N^- \setminus K_j^-) = 0.$$

We consider also decreasing sequences of open sets $G_j^+ \supseteq N^+$, $G_j^- \supseteq N^-$ such that

$$\lim_j H^{N-1}(G_j^+ \setminus N^+) = \lim_j H^{N-1}(G_j^- \setminus N^-) = 0.$$

Now, we take functions $\zeta_j^+, \zeta_j^- \in C^1(\partial\Omega)$ with values in $[0,1]$ such that

(5.48) $\qquad\qquad\qquad \zeta_j^+(x) = \begin{cases} 1 & \text{in } K_j^+ \\ 0 & \text{in } \partial\Omega \setminus G_j^+ \end{cases}$

$$(5.49) \qquad \zeta_j^-(x) = \begin{cases} 1 & \text{in } K_j^- \\ 0 & \text{in } \partial\Omega \setminus G_j^-, \end{cases}$$

and we set

$$\zeta_j = \zeta_j^+ - \zeta_j^-.$$

The functions ζ_j satisfy

$$(5.50) \qquad \zeta_j(x) = \begin{cases} 1 & \text{in } K_j^+ \setminus G_j^- \\ 0 & \text{in } \partial\Omega \setminus (G_j^+ \cup G_j^-) \\ -1 & \text{in } K_j^- \setminus G_j^+. \end{cases}$$

Moreover

$$(5.51) \qquad \lim_j H^{N-1}(T \cap (G_j^+ \cup G_j^-)) = 0.$$

Recall that the functions Θ_j are extensions of ζ_j to Ω. Now,

$$\nabla \rho_j(x) = \nabla \Theta_j(x) \psi_{\delta_j}(D(x)) + \Theta_j(x) \psi_{\delta_j}'(D(x)) \nabla D(x).$$

If $x \in \partial\Omega$, then

$$\nabla \rho_j(x) = -\Theta_j(x) \psi_{\delta_j}'(0) \nu(x).$$

Now, using (5.51), for almost all $x \in T = \{x \in \partial\Omega : g(x) \neq v(x)\}$, there exists $j_0(x) \in \mathbb{N}$ such that $\zeta_j(x) = 0$ for all $j \geq j_0(x)$. Hence, also

$$\nabla \rho_j(x) = 0 \quad \text{for all } j \geq j_0(x).$$

Next

$$\int_\Omega |\nabla \rho_j| \;\leq\; 2\delta_j \int_\Omega |\nabla \Theta_j| + \|\Theta_j\|_\infty \|\nabla D\|_\infty \int_{\Omega_{\delta_j^2}} |\psi_{\delta_j}'(D(x))| dx$$

$$\leq\; 2\delta_j \int_\Omega |\nabla \Theta_j| + \|\Theta_j\|_\infty \|\nabla D\|_\infty \frac{4}{\delta_j} \int_{\Omega_{\delta_j^2}} dx.$$

Now, for j large enough

$$\int_{\Omega_{\delta_j^2}} dx = \int_0^{\delta_j^2} \int_{[D=\lambda]} \frac{dH^{N-1}(z)}{|\nabla D(z)|} d\lambda \leq 2 \int_0^{\delta_j^2} \int_{[D=\lambda]} dH^{N-1}(z) d\lambda \leq C\delta_j^2$$

where C depends on $Per(\partial\Omega)$. Hence

$$\int_\Omega |\nabla \rho_j| \leq 2\delta_j \int_\Omega |\nabla \Theta_j| + \|\Theta_j\|_\infty \|\nabla D\|_\infty 4 C \delta_j.$$

Now, choosing δ_j we may guarantee that

$$\int_\Omega |\nabla \rho_j| \to 0 \quad \text{as} \quad j \to \infty.$$

Since

$$\nabla v_j = \nabla G_j h_{\varepsilon_j} + \nabla z_j (1 - h_{\varepsilon_j}) + (G_j - z_j) h'_{\varepsilon_j}(D) \nabla D + \nabla \Theta_j \psi_{\delta_j} + \Theta_j \psi'_{\delta_j} \nabla D,$$

on $\partial \Omega$, we have

$$(5.52) \quad \nabla v_j(x) = \nabla G_j(x) - (G_j(x) - z_j(x)) h'_{\varepsilon_j}(0) \nu(x) - \Theta_j(x) \psi'_{\delta_j}(0) \nu(x)$$

and we may write on $\partial \Omega$

$$
\begin{aligned}
\frac{\nabla v_j}{|\nabla v_j|} &= \frac{\nabla G_j - (G_j - z_j) h'_{\varepsilon_j}(0)\nu - \Theta_j \psi'_{\delta_j}(0)\nu}{|\nabla G_j - (G_j - z_j) h'_{\varepsilon_j}(0)\nu - \Theta_j \psi'_{\delta_j}(0)\nu|} = \\
&= \frac{\delta_j \nabla G_j - \delta_j (G_j - z_j) h'_{\varepsilon_j}(0)\nu - \delta_j \Theta_j \psi'_{\delta_j}(0)\nu}{|\delta_j \nabla G_j - \delta_j (G_j - z_j) h'_{\varepsilon_j}(0)\nu - \delta_j \Theta_j \psi'_{\delta_j}(0)\nu|}.
\end{aligned}
$$

Now, we choose δ_j such that $\delta_j \nabla G_j \to 0$, $\delta_j (G_j - z_j) h'_{\varepsilon_j}(0) \to 0$, as $j \to \infty$, we obtain that

$$\frac{\nabla v_j}{|\nabla v_j|} \to \frac{g(x) - u(x)}{|g(x) - u(x)|} \nu(x)$$

H^{N-1} a.e. on $\{x \in \partial \Omega : g(x) = v(x), u(x) \neq v(x)\}$. By choosing δ_j to converge sufficiently fast to 0, from (5.52), we obtain that

$$|\nabla v_j(x)| \to \infty$$

H^{N-1} a.e. on $\{x \in \partial \Omega : g(x) = v(x), u(x) \neq v(x)\}$.

Let us now check that $v_j = z_j + \sigma_j + \rho_j$ satisfies the required properties. Since $v_j = g_j$ on $\partial \Omega$, (3.6) follows immediately. The property (3.7) follows from (5.4), (5.26) and (5.40). To check (3.8), let $\psi \in C^1(\overline{\Omega}, \mathbb{R}^N)$, $\psi = (\psi_1, ..., \psi_n)$ and $\psi_{N+1} \in C^1(\overline{\Omega}, \mathbb{R})$. Using (3.6) and (3.7), we obtain

$$
\begin{aligned}
\lim_j &\int_\Omega \Big[\sum_{i=1}^N \psi_i(x) D_i v_j(x) + \psi_{N+1}(x) \Big] dx \\
&= -\lim_j \int_\Omega div\, \psi(x) v_j(x) dx + \int_{\partial \Omega} g_j \psi \cdot \nu \, dH^{N-1} + \int_\Omega \psi_{N+1}(x) dx \\
&= -\int_\Omega div\, \psi(x) v(x) dx + \int_{\partial \Omega} g\psi \cdot \nu \, dH^{N-1} + \int_\Omega \psi_{N+1}(x) dx \\
&= \int_\Omega [\psi(x) \cdot Dv(x) + \psi_{N+1}(x)] dx + \int_{\partial \Omega} (g - v)\psi \cdot \nu \, dH^{N-1}.
\end{aligned}
$$

Now, because of the lower semicontinuity of the total variation with respect to weak convergence, we have

$$\int_\Omega \sqrt{1 + |Dv|^2} + \int_{\partial\Omega} |g - v| dH^{N-1} \leq \liminf_j \int_\Omega \sqrt{1 + |Dv_j|^2} dx.$$

On the other hand, since

$$\int_\Omega \sqrt{1 + |\nabla v_j|^2} dx \leq \int_\Omega \sqrt{1 + |\nabla z_j|^2} dx + \int_\Omega |\nabla \sigma_j| dx + \int_\Omega |\nabla \rho_j| dx,$$

using (5.5), (5.29) and (5.42) we obtain that

$$\limsup_j \int_\Omega \sqrt{1 + |Dv_j|^2} dx \leq \int_\Omega \sqrt{1 + |Dv|^2} + \int_{\partial\Omega} |g - v| dH^{N-1}.$$

This proves (3.8). Now, using (5.7), (5.27) and (5.41) we obtain (3.9). Next, we observe that (3.10) is a consequence of (5.16), (5.27) and (5.41). In the same way, (3.11) is a consequence of (5.19), (5.27) and (5.41). We observe that (3.12) follows from (5.31) and (5.43). Fina lly, (3.13) has already been proved. ∎

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