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ON FILTER CONVERGENCE OF SERIES

Abstract

A series $\sum x_k$ is $\mathcal{F}$-convergent to s if the sequence $(\sum_{k=1}^{n} x_k)$ of its partial sums is $\mathcal{F}$-convergent to s. We describe filters $\mathcal{F}$ for which $\mathcal{F}$-convergence of a series $\sum x_k$ implies $\mathcal{F}$-convergence to 0 of the series terms x_k. If (x_k) is small enough with respect to a given filter $\mathcal{F}$, then there is an $\mathcal{F}$-subseries $\sum_{k\in I} x_k$ which is absolutely convergent in the usual sense. Filters corresponding to summable ideals, Erdős-Ulam ideals, matrix summability ideals, lacunary ideals and Louveau-Veličković ideals are considered.

1 Introduction and preliminaries

The basic property of the classical series theory says that the terms of a convergent series of reals form a null sequence; i.e., they must tend to zero. This property, in turn, implies the existence of an absolutely convergent subseries of a given convergent series. The question we pose in this paper is: how can these properties be carried over to the notion of filter convergence?

We generalize some results of [9] and [7] and answer the above question as follows. First, in Section 2, we characterize those filters $\mathcal{F}$ for which $\mathcal{F}$-convergence of a series $\sum x_k$ implies $\mathcal{F}$-convergence to 0 of the sequence of the series terms (x_k). We also consider those filters $\mathcal{F}$ for which $\mathcal{F}$-convergence of a series $\sum x_k$ implies the usual convergence to 0 of an $\mathcal{F}$-subsequence $(x_k)_{k\in I}$ of (x_k). Then, in Section 3, we describe for some filters $\mathcal{F}$ the relation between

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the speed of a sequence converging to 0 and the existence of an $\mathcal{F}$-subseries of this sequence which is absolutely convergent in the usual way. This also yields some generalization of the following (T) property [2] of a filter $\mathcal{F}$: *if* $\sum x_k$ *is an* $\mathcal{F}$*-convergent series, then there is a subseries* $\sum_{k\in A} x_k$, $A \in \mathcal{F}$, *which is convergent in the usual sense.* In [2], J. Cervenansky, T. Salat and V. Toma proved that filters $\mathcal{F}_{st}, \mathcal{F}_{\mathcal{E}\mathcal{U}_{1/n}}, \mathcal{F}^{1/n}$ (see the definitions below) do not have the (T) property. In Section 3, we establish some weaker version of the (T) property for classes of filters which contain $\mathcal{F}_{st}, \mathcal{F}_{\mathcal{E}\mathcal{U}_{1/n}}, \mathcal{F}^{1/n}$.

Recall that a *filter* $\mathcal{F}$ on $\mathbb{N}$ is a non-empty collection of subsets of $\mathbb{N}$ satisfying the following axioms: $\emptyset \notin \mathcal{F}$; if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$; and for every $A \in \mathcal{F}$, if $B \supset A$, then $B \in \mathcal{F}$.

The dual to the notion of filter is the notion of ideal. An *ideal* $\mathcal{I}$ on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under taking finite unions and subsets of its elements. Given a filter $\mathcal{F}$ on $\mathbb{N}$, we have the corresponding ideal of complements $\mathcal{I}_{\mathcal{F}} = \{\mathbb{N} \setminus A : \ A \in \mathcal{F}\}$ on $\mathbb{N}$. And vice versa, a given ideal $\mathcal{I}$ on $\mathbb{N}$ gives rise to the filter $\mathcal{F}_{\mathcal{I}} = \{\mathbb{N} \setminus A : \ A \in \mathcal{I}\}$. Due to this correspondence, sometimes we will say "ideal" while studying the corresponding "filter" and the other way around.

One more family corresponding to a filter is the family of all its stationary sets. A subset of $\mathbb{N}$ is called *stationary* with respect to $\mathcal{F}$, or just $\mathcal{F}$-stationary, if it has a nonempty intersection with each member of the filter. In other words, an $A \subset \mathbb{N}$ is $\mathcal{F}$-stationary if and only if A does not belong to $\mathcal{I}_{\mathcal{F}}$. Denote the collection of all $\mathcal{F}$-stationary sets by $\mathcal{F}^*$. It is easy to check that the collection $\mathcal{F}^{**} = \{A \subset \mathbb{N} : \ A \cap J \neq \emptyset, \ J \in \mathcal{F}^*\}$ is equal to $\mathcal{F}$.

For an $I \in \mathcal{F}^*$, we call the collection of sets $\mathcal{F} \restriction I = \{A \cap I : \ A \in \mathcal{F}\}$ *the trace of* $\mathcal{F}$ *on* I, which is evidently a filter on I. By $\mathcal{F}(I)$, we denote the filter $\{A \subset \mathbb{N} : A \supset B, \ B \in \mathcal{F} \restriction I\}$ on $\mathbb{N}$ generated by the trace of $\mathcal{F}$ on I.

Any subset of $\mathbb{N}$ is either a member of $\mathcal{F}$ or a member of $\mathcal{I}_{\mathcal{F}}$, or the set and its complement are both $\mathcal{F}$-stationary sets.

A sequence (x_n), $n \in \mathbb{N}$, in a topological space X is said to be $\mathcal{F}$-*convergent* to x, written $x = \mathcal{F}\text{-}\lim x_n$ or $x_n \to_{\mathcal{F}} x$, if for every neighborhood U of x, the set $\{n \in \mathbb{N} : x_n \in U\}$ belongs to $\mathcal{F}$; equivalently $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}_{\mathcal{F}}$. In particular, if one takes $\mathcal{F}$ as the filter $\mathcal{F}r$ whose ideal Fin consists of finite sets (the *Fréchet filter*), then $\mathcal{F}$-convergence coincides with the ordinary one.

It is natural to define $\mathcal{F}$-convergence of a series as $\mathcal{F}$-convergence of the sequence of its partial sums. However, this definition is not the only possible one. For example, S. Głąb and M. Olczyk in [6] introduced another definition based on the Cauchy condition. With their definition, the filter convergence of a series implies the filter convergence of its terms to zero—as opposed to the situation that we consider here.

In this paper, we define a series $\sum x_k$ of reals to be $\mathcal{F}$*-convergent* to s if the sequence $(s_n) = (\sum_{k=1}^n x_k)$ is $\mathcal{F}$-convergent to s, and we write $s = \mathcal{F}\text{-}\sum_k x_k$, or simply $s = \sum_{\mathcal{F}} x_k$ when there is only one possible summing index.

Theorem 1.1. *[1] Let X be a topological space, $x_n, x \in X$, and let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then the following conditions are equivalent:*

(i) (x_n) is $\mathcal{F}$-convergent to x;

(ii) (x_n) is $\mathcal{F}(I)$-convergent to x for every $I \in \mathcal{F}^*$;

(iii) x is a cluster point of $(x_n)_{n \in I}$ for every $I \in \mathcal{F}^*$.

According to this theorem, it is natural to introduce the following notions. A sequence $(x_n)_{n \in I}$ with $I \in \mathcal{F}^*$ is called an $\mathcal{F}$*-subsequence* of the sequence (x_n). A series $\sum_{n \in I} x_n$ with $I \in \mathcal{F}^*$ is called an $\mathcal{F}$*-subseries* of the series $\sum_{n \in \mathbb{N}} x_n$.

The natural ordering on the set of filters on $\mathbb{N}$ is defined as follows: $\mathcal{F}_1 \succ \mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$. A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be *free* if it dominates the Fréchet filter. When we say "filter" below, we mean a free filter on $\mathbb{N}$. In particular, all ordinary convergent sequences and series will be automatically $\mathcal{F}$-convergent.

Let us consider some examples of filters.

Recall that the *statistical convergence filter* is defined as the set

$$\mathcal{F}_{st} = \Big\{ A \subset \mathbb{N} : \lim_{n \to \infty} \frac{|A \cap \{1, ..., n\}|}{n} = 1 \Big\}.$$

A *summable ideal* is the set

$$\mathcal{I}^s = \Big\{ A \subset \mathbb{N} : \sum_{k \in A} s_k < \infty \Big\},$$

where $s = (s_k)$ is a sequence of non-negative real numbers such that $\sum_{k=1}^\infty s_k = \infty$. A *summable filter* is the filter $\mathcal{F}^s = \mathcal{F}_{\mathcal{I}^s}$ corresponding to $\mathcal{I}^s$.

An example of a filter dominating $\mathcal{F}^s$ is the *Erdős-Ulam* filter $\mathcal{F}_{\mathcal{EU}_s}$. It is determined by its ideal

$$\mathcal{EU}_s = \Big\{ A \subset \mathbb{N} : \lim_{n \to \infty} \frac{\sum_{k \in A \cap [1,n]} s_k}{\sum_{k=1}^n s_k} = 0 \Big\}.$$

All the filters above are the examples of filters with the following important property. A filter $\mathcal{F}$ is a P*-filter* if, for every sequence of filter elements A_n, there is $A_\infty \in \mathcal{F}$ such that $|A_\infty \setminus A_n| < \infty$ for every $n \in \mathbb{N}$. An ideal is a P*-ideal* if the corresponding filter of complements is a P-filter.

Identifying the set $2^{\mathbb{N}}$ with the Cantor space $\{0,1\}^{\mathbb{N}}$ we can talk about F_σ, Borel, analytic filters and ideals.

Definition 1.2. A map $\phi : 2^{\mathbb{N}} \to [0,\infty]$ is a *submeasure on* $\mathbb{N}$ if

- $\phi(\emptyset) = 0$ and $\phi(\{n\}) < \infty$ for every $n \in \mathbb{N}$;
- ϕ is monotone: $A \subset B \subset \mathbb{N}$ implies $\phi(A) \le \phi(B)$;
- ϕ subadditive: $A, B \subset \mathbb{N}$ implies $\phi(A \cup B) \le \phi(A) + \phi(B)$.

It is *lower semicontinuous* if $\phi(A) = \lim_{n\to\infty} \phi(A \cap [1,n])$ for every $A \subset \mathbb{N}$.

There are two ideals associated with a lower semicontinuous submeasure ϕ on $\mathbb{N}$:

$$\mathrm{Fin}(\phi) = \{A \subset \mathbb{N} : \phi(A) < \infty\}$$

and

$$\mathrm{Exh}(\phi) = \{A \subset \mathbb{N} : \lim_{n\to\infty} \phi(A \setminus [1,n]) = 0\}.$$

Theorem 1.3. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then*

(i) (K. Mazur, [11]) $\mathcal{I}$ is an F_σ ideal if and only if $\mathcal{I} = \mathrm{Fin}(\phi)$ for some lower semicontinuous submeasure ϕ on $\mathbb{N}$.

(ii) (S. Solecki, [12]) $\mathcal{I}$ is an analytic P-ideal if and only if $\mathcal{I} = \mathrm{Exh}(\phi)$ for some lower semicontinuous submeasure ϕ on $\mathbb{N}$. Every analytic P-ideal is an $F_{\sigma\delta}$.

Every summable ideal $\mathcal{I}^s$ is an F_σ P-ideal; $\mathcal{I}^s$ is determined by a lower semicontinuous submeasure $\phi(A) = \sum_{n\in A} s_n$, and $\mathcal{I}^s = \mathrm{Fin}(\phi) = \mathrm{Exh}(\phi)$.

Every Erdős-Ulam ideal $\mathcal{EU}_s$ equals $\mathrm{Exh}(\phi)$, and the corresponding lower semicontinuous submeasure ϕ is given by

$$\phi(A) = \sup_{n\in\mathbb{N}} \frac{\sum_{i\in A\cap[1,n]} s_i}{\sum_{i=1}^{n} s_i}.$$

A wide class of analytical P-ideals containing all the Erdős-Ulam ideals are ideals $\mathcal{I}_\tau$ determined by summability matrices τ: $\tau_{n,i} \ge 0$, $\sum_{j=1}^{\infty} \tau_{n,j} \le 1$ for every $n \in \mathbb{N}$; $\limsup_{n\to\infty} \sum_{j=1}^{\infty} \tau_{n,j} > 0$; $\lim_{n\to\infty} \tau_{n,j} = 0$ for every $n \in \mathbb{N}$. We define $\mathcal{I}_\tau$ to be a *matrix summability ideal* if $\mathcal{I}_\tau$ is equal to $\mathrm{Exh}(\phi)$, where $\phi(A) = \sup_{n\in\mathbb{N}} \sum_{i\in A} \tau_{n,i}$. For a connection of $\mathcal{I}_\tau$ with summable ideals, see [8, Lemma 4].

Two more classes of ideals will appear in Section 3.

2 $\mathcal{F}$-convergence to 0 of terms of $\mathcal{F}$-convergent series

In [9], the coincidence of the limit point range and the sum range of an $\mathcal{F}$-convergent series along a filter $\mathcal{F}$ was studied. It was shown that an essential property for this coincidence is the existence of a null subsequence of an $\mathcal{F}$-convergent series. The characterization of this property [9, Proposition 2] is that for every $A \in \mathcal{F}$, there exists $s \in A$ such that $s+1 \in A$; the 1-shift property. In this section, we generalize this result from [9].

The property that the terms of convergent series converge to 0 is characteristic only for the usual convergence of series. Indeed, one can easily check the following statement from [9]. We present it with a proof for the reader's convenience.

Proposition 2.1. *Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then $\mathcal{F}$ is the Fréchet filter if, for every $\mathcal{F}$-convergent series $\sum x_k$, the terms $x_k \to_{k\to\infty} 0$.*

Proof. Suppose $\mathcal{F}$ is not the Fréchet filter. This means that there is some infinite $C \subset \mathbb{N}$ such that $A = \mathbb{N} \setminus C \in \mathcal{F}$. Denote by E the infinite set $A \cap (C+1)$. Now let (x_k) be the following sequence: x_k is 1 for $k \in E$, -1 for $k \in (E-1)$ and 0 for $k \in \mathbb{N} \setminus (E \cup (E-1))$. Then $\sum x_k$ $\mathcal{F}$-converges to 0, but $x_k \not\to_{k\to\infty} 0$. $\square$

Let us describe two classes of filters with similar properties: $\mathcal{F}$-convergence to 0 of terms of a series and existence of a subsequence converging to 0.

Definition 2.2. A filter $\mathcal{F}$ is said to have *the shift property,* or is called *shift invariant*, if $A \in \mathcal{F}$ implies $A+1 \in \mathcal{F}$.

Proposition 2.3. *The following conditions are equivalent:*

1. *$\mathcal{F}$ is shift invariant;*
2. *For every $I \in \mathcal{F}^*$ and $A \in \mathcal{F}$, $(A+1) \cap A \cap I \neq \emptyset$;*
3. *$\mathcal{I}_{\mathcal{F}}$ is a* shift invariant ideal*; i.e., $A \in \mathcal{I}_{\mathcal{F}}$ implies $A+1 \in \mathcal{I}_{\mathcal{F}}$.*

Proof. The equivalence of (1) and (2) follows from the definition of a stationary set and the equality $\mathcal{F}^{**} = \mathcal{F}$. Let us check the equivalence of (1) and (3). Implication (1) $\Rightarrow$ (3): for every $A \in \mathcal{I}_{\mathcal{F}}$, the set $B := \mathbb{N} \setminus A \in \mathcal{F}$ and thus $B \cap (B+1) \in \mathcal{F}$. Since $A+1 \subset \mathbb{N} \setminus (B \cap (B+1))$, we deduce that $A+1 \in \mathcal{I}_{\mathcal{F}}$. Implication (3) $\Rightarrow$ (1): if $B \in \mathcal{F}$, then $A := \mathbb{N} \setminus B \in \mathcal{I}_{\mathcal{F}}$ and $A \cup (A+1) \in \mathcal{I}_{\mathcal{F}}$. Since $B+1$ contains $\mathbb{N} \setminus (A \cup (A+1))$, we deduce that $B+1 \in \mathcal{F}$. $\square$

Theorem 2.4. *Let $\mathcal{F}$ be a filter on $\mathbb{N}$. The following conditions are equivalent:*

1. *For every sequence* (x_k) *such that* $\sum x_k$ $\mathcal{F}$*-converges, the sequence* (x_k) *is* $\mathcal{F}$*-convergent to* 0*;*
2. $\mathcal{F}$ *has the shift property.*

PROOF. (2)⇒(1). Let $\sum_{\mathcal{F}} x_k = x$ and suppose that (x_k) does not $\mathcal{F}$-converge to 0. Then by Theorem 1.1, there is $\delta > 0$ and $J \in \mathcal{F}^*$ such that $|x_k| > \delta$ for $k \in J$. For $\varepsilon = \delta/4$, let us find $A \in \mathcal{F}$ such that $|x - \sum_{k=1}^{s} x_k| < \varepsilon$ for every $s \in A$. Applying condition (2) of Proposition 2.3, we take $s \in A$ such that $s+1 \in A \cap J$. Then we come to a contradiction:

$$\varepsilon > \left|x - \sum_{k=1}^{s+1} x_k\right| \geq |x_{s+1}| - \left|x - \sum_{k=1}^{s} x_k\right| > \delta - \varepsilon > 2\varepsilon.$$

(1)⇒(2). Suppose that condition (1) of Proposition 2.3 does not hold. Then there is $A \in \mathcal{F}$ such that $(A+1) \notin \mathcal{F}$, and so there is $I \in \mathcal{F}^*$ such that $(A+1) \cap I = \emptyset$. This means that $A \cap (I-1) = \emptyset$. Consider $J := A \cap I$. We have $J \in \mathcal{F}^*$, $J - 1 \in \mathcal{I}_{\mathcal{F}}$ and $J \cap (J-1) = \emptyset$. Let (x_k) be the following sequence: $x_k = -1$ for $k \in J-1$, $x_k = 1$ for $k \in J$ and $x_k = 0$ for all other k. Then $\sum x_k$ is $\mathcal{F}$-convergent, since $\sum_{k \in B} x_k = 0$, where $B = \mathbb{N} \setminus (J-1) \in \mathcal{F}$; but (x_k) does not $\mathcal{F}$-converge to 0. □

Let us also consider the following weaker property.

Definition 2.5. A filter $\mathcal{F}$ is said to have *the 1-shift property,* or is called *1-shift invariant*, if $A \in \mathcal{F}$ implies $A + 1 \in \mathcal{F}^*$.

Proposition 2.6. *A filter* $\mathcal{F}$ *is 1-shift invariant if and only if for every* $A \in \mathcal{F}$*,* $A \cap (A+1) \neq \emptyset$*.*

PROOF. One direction is obvious. Thus, it is sufficient to show that $A \cap (A+1) \in \mathcal{F}^*$. Let us take an arbitrary $B \in \mathcal{F}$ and check that $B \cap (A \cap (A+1)) \neq \emptyset$. We know that $B \cap A \in \mathcal{F}$. Hence, by the propositions assumption, we get $B \cap A \cap (B \cap A + 1) \neq \emptyset$. Since $B \cap A + 1 \subset A + 1$, we have $B \cap A \cap (A+1) \neq \emptyset$. That is what we needed to check. □

This proposition means that the 1-shift property is the property of the same name from [9]. So, the following theorem can be found in [9] or can be proved the same way as Theorem 2.4 above.

Theorem 2.7. *Let* $\mathcal{F}$ *be a filter on* $\mathbb{N}$*. The following conditions are equivalent:*

1. *For every sequence* (x_k) *such that* $\sum x_k$ $\mathcal{F}$*-converges, there is a null subsequence* (x_{k_n})*; i.e.,* $(x_{k_n}) \to 0$*;*

2. $\mathcal{F}$ has the 1-shift property.

Theorem 2.8. *Let $\mathcal{F}$ be a 1-shift invariant P-filter. Then every sequence (x_k) with $\mathcal{F}$-convergent series $\sum x_k$ has an $\mathcal{F}$-subsequence $(x_k)_{k\in I}$, $I \in \mathcal{F}^*$, which is $\mathcal{F}$-convergent to* 0.

Proof. Let $\mathcal{F}$-$\sum x_k = x$. Since $\mathcal{F}$ is a P-filter, there is $A \in \mathcal{F}$ such that $\lim_{n\to\infty} \sum_{k=1}^{a_n} x_k = x$, where (a_n) is an increasing numeration of A. Suppose $(x_k) \not\to_{\mathcal{F}(I)} 0$ for any $I \in \mathcal{F}^*$. According to the 1-shift property, we can take $I = (A+1) \cap A$. By Theorem 1.1, there are $\delta > 0$ and $J \in \mathcal{F}^*$, with $J \subset I$, such that $|x_k| > \delta$ for $k \in J$. Let us take $\varepsilon = \delta/4$ and N big enough to satisfy $|x - \sum_{k=1}^{a_n} x_k| < \varepsilon$ for all $a_n \geq N$. For every $s \in J$, we have $s-1 \in A$, and taking $s > N$, we come to a contradiction:

$$\varepsilon > \left|x - \sum_{k=1}^{s} x_k\right| \geq |x_s| - \left|x - \sum_{k=1}^{s-1} x_k\right| > \delta - \varepsilon > 2\varepsilon.$$

□

If $\mathcal{F}$ is a P-filter, then for every $I \in \mathcal{F}^*$, the filter $\mathcal{F}(I)$ is also a P-filter. Thus, we can strengthen the statement of Theorem 2.8.

Corollary 2.9. *If $\mathcal{F}$ is a 1-shift invariant P-filter, then every sequence (x_k) with $\mathcal{F}$-convergent series $\sum x_k$ has an $\mathcal{F}$-subsequence $(x_k)_{k\in I}$, $I \in \mathcal{F}^*$, which converges to* 0.

The shift property is a strictly stronger condition than the 1-shift property even for P-filters. One can construct a filter which is 1-shift invariant, but not shift invariant; for example, a filter on $\mathbb{N}$ isomorphic to the Fubini product $\{\mathbb{N}\} \times \mathcal{F}r = \{(n, A) : \ n \in \mathbb{N}, A \in \mathcal{F}r\}$.

Let us find examples of filters with the shift property.

Proposition 2.10. *Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytical P-filter such that $\phi(A) = \sup_{\mu\leq\phi} \mu(A)$, where μ is a measure and $\mu(\{n\}) \geq \mu(\{n+1\})$ for every $n \in \mathbb{N}$. Then $\mathcal{I}$ is a shift invariant ideal.*

Proof. Let us take an arbitrary $I \in \mathcal{I}$. We have $\lim_{k\to\infty} \phi(I \setminus [1,k]) = 0$. For $I+1$, by the propositions condition and semicontinuity of ϕ, we get

$$\phi\left((I+1) \setminus [1,k]\right) = \lim_{n\to\infty} \phi\left((I+1) \cap (k,n]\right)$$
$$= \lim_{n\to\infty} \sup_{\mu\leq\phi} \sum_{i\in I\cap(k,n]} \mu(\{i+1\}) \leq \lim_{n\to\infty} \phi\left(I \cap (k,n]\right) = \phi(I \setminus [1,k]),$$

and so the sequence converges to 0 as $k \to \infty$. Thus, we obtain $I+1 \in \mathcal{I}$, and by Proposition 2.3, the statement is proved. □

Corollary 2.11. *The following ideals $\mathcal{I}$ are shift invariant:*

- *$\mathcal{I}$ determined by a summability matrix τ such that $\tau_{i,j} \geq \tau_{i,j+1}$ for all $i, j \in \mathbb{N}$;*
- *$\mathcal{I} = \mathcal{EU}_s$ and $s_i \geq s_{i+1}$ for all $i \in \mathbb{N}$;*
- *$\mathcal{I} = \mathcal{I}^s$ and $s_i \geq s_{i+1}$ for all $i \in \mathbb{N}$.*

3 Null filter subsequences and filter subseries

Let us consider another approach to the study of terms of a series which converge to 0 with respect to a given filter. In [7], we studied sequences of a Hilbert space which have 0 weak filter limit, but can tend to infinity in norm. A connection to the question that we study here was established. In particular, it was shown that there is a convergent $\mathcal{F}_{st}$-subseries $\sum_{n\in I} x_n < \infty$ of positive reals x_n if and only if there is an $\mathcal{F}_{st}$-subsequence of $(nx_n)_{n\in J}$ converging to 0. In this section, we extend the results obtained in [7] to lacunary statistical convergence filters and Louveau-Veličković filters. For filters corresponding to general matrix summability ideals, the problem is open.

Definition 3.1. Let H be an infinite-dimensional separable Hilbert space. For a given filter $\mathcal{F}$ on $\mathbb{N}$, let us say that a sequence (a_n) of positive reals is *$\mathcal{F}$-admissible* if there is a sequence $(x_n) \subset H$ with $\|x_n\| = a_n$ such that $\mathcal{F}\text{-}\lim x_n = 0$ in weak topology.

Let us state the main theorem of the paper [7].

Theorem 3.2. *Let $\mathcal{F}$ be a filter on $\mathbb{N}$. The following properties of a sequence $a_n > 0$, $n \in \mathbb{N}$, are equivalent:*

1. *(a_n) is $\mathcal{F}$-admissible;*
2. *For every $I \in \mathcal{F}^*$,*
$$\sum_{n\in I} a_n^{-2} = \infty;$$
3. *$\mathcal{F}$ dominates the summable filter $\mathcal{F}^{a^{-2}}$; or dually $\mathcal{I}^{a_n^{-2}} \subset \mathcal{I}_{\mathcal{F}}$.*

Summarizing results obtained in [7], namely Theorem 5.9, Remark 5.10 and Theorem 5.14, we have the following theorem.

Theorem 3.3. *Let ϕ be a lower semicontinuous submeasure on $\mathbb{N}$ and $\mathcal{F}$ the filter corresponding either to the ideal $\mathcal{I} = \mathrm{Exh}(\phi)$ or to the ideal $\mathcal{I} = \mathrm{Fin}(\phi)$. A sequence (a_n) of positive numbers is $\mathcal{F}$-admissible if, for every $J \in \mathcal{F}^*$, $a_n\sqrt{\phi(n)} \not\to_J \infty$.*

Moreover, for the Erdős-Ulam ideals and summable ideals, the opposite implication is also true.

Let us apply now the equivalence of negation of the statement of Theorem 3.3 and the negation of (2) in Theorem 3.2. If we write $\frac{1}{a_n^2\varphi_n} \to_J 0$ instead of $a_n\sqrt{\varphi_n} \to_J \infty$ and put $a_n = 1/\sqrt{x_n}$, then we obtain the following corollary on $\mathcal{F}$-subseries.

Corollary 3.4. *Let $\mathcal{F}$ be a filter which corresponds to either an Erdős-Ulam ideal or a summable ideal, and let φ_n be $\frac{s_n}{\sum_{k=1}^n s_k}$ or s_n, accordingly. For a sequence (x_n) of positive reals, the following statements are equivalent:*

(1) there is an $I \in \mathcal{F}^*$ such that $\sum_{n\in I} x_n < \infty$;

(2) there is a $J \in \mathcal{F}^*$ such that $x_n/\varphi_n \to_{n\in J} 0$.

These statements can be also reformulated as follows:

(1) there is a convergent $\mathcal{F}$-subseries of $\sum x_n$;

(2) there is an $\mathcal{F}$-subsequence of (x_n/φ_n) converging to 0.

Let us unify the results of Corollary 2.9 and Corollary 2.11 with the above Corollary 3.4. We have the following weak analogue of the (T) property for the Erdős-Ulam and summable ideals with the 1-shift property.

Corollary 3.5. *Let $\mathcal{F}$ be a filter which corresponds to either an Erdős-Ulam ideal or a summable ideal with $s_i \geq s_{i+1}$, and let φ_n be equal to $\frac{s_n}{\sum_{k=1}^n s_k}$ or s_n, accordingly. If $\sum x_k$ is $\mathcal{F}$-convergent, then there is an $I \in \mathcal{F}^*$ such that $\sum_{k\in I} \varphi_k |x_k| < \infty$.*

Now let us check the "moreover" part of Theorem 3.3 and get results similar to Corollaries 3.5 and 3.4 for the class of lacunary statistical filters.

Recall the definition of lacunary sequence [5] and introduce the corresponding ideal.

Let (θ_r) be a rapidly increasing sequence of naturals such that $\Delta\theta_r := \theta_r - \theta_{r-1} \to_{r\to\infty} \infty$. A *lacunary ideal* $\mathcal{I}_\theta$ is defined as follows: $I \in \mathcal{I}_\theta$ if and only if

$$\lim_{r\to\infty} \frac{|(\theta_{r-1}, \theta_r] \cap I|}{\Delta\theta_r} = 0.$$

It is easy to see that the corresponding matrix summability elements are

$$\tau_{r,i} = \begin{cases} 0, & \text{when } i \notin (\theta_{r-1}, \theta_r] \\ \frac{1}{\Delta\theta_r}, & \text{when } i \in (\theta_{r-1}, \theta_r] \end{cases},$$

and $\phi(\{i\}) = \sup_{n\in\mathbb{N}} \tau_{n,i} = \frac{1}{\Delta\theta_r}$, where θ_r is the number for which $i \in (\theta_{r-1}, \theta_r]$. Some results on lacunary statistical convergence may be found in [5].

Defining for every $I \notin \mathcal{I}_\theta$ the sets $D_n(I) = (\theta_{n-1}, \theta_n] \cap I$, we obtain the following lemma.

Lemma 3.6. *Let $\mathcal{I}_\theta$ be a lacunary ideal. Then $I \notin \mathcal{I}_\theta$ if and only if there are $\varepsilon > 0$ and a subsequence $n_1 < n_2 < \dots$ such that*

$$1 > \frac{|D_{n_j}(I)|}{\Delta\theta_{n_j}} > \varepsilon \quad \textit{for all} \quad j \in \mathbb{N}. \tag{3.1}$$

Theorem 3.7. *Let $\mathcal{F}_\theta$ correspond to a lacunary ideal $\mathcal{I}_\theta$. A sequence (a_n) is not $\mathcal{F}_\theta$-admissible if and only if there is $J \in \mathcal{F}_\theta^*$ such that*

$$a_n \sqrt{\frac{1}{\Delta\theta_{r(n)}}} \to_{n\in J} \infty, \quad \textit{where} \ \ n \in (\theta_{r(n)-1}, \theta_{r(n)}].$$

Proof. Theorem 3.3 gives us the "only if" part . So, it is the "if" part that we need to prove. Let us have $J \in \mathcal{F}_\theta^*$ such that $a_n\sqrt{\frac{1}{\Delta\theta_{r(n)}}} \to \infty$ along J. Lemma 3.6 gives us (n_j) and $\varepsilon > 0$ such that $D_{n_j}(J)$ satisfies (3.1). Obviously, for every subsequence $(m_j) \subset (n_j)$, the inequality (3.1) also holds. Thus, $I = \bigsqcup_{j=1}^\infty D_{m_j}(J) \in \mathcal{F}_\theta^*$. To prove our statement, it is sufficient to find (m_j) such that $\sum_{n\in I} a_n^{-2} < \infty$.

Let us write $f(n) = a_n\sqrt{\frac{1}{\Delta\theta_{r(n)}}}$ and choose a subsequence (m_j) such that $\min_{k\in D_{m_j}} f^2(k) > 2^j$. We have

$$\begin{aligned}\sum_{n\in I} a_n^{-2} &= \sum_{j=1}^\infty \sum_{k\in D_{m_j}(J)} a_k^{-2} \le \sum_{j=1}^\infty \sum_{k\in D_{m_j}(J)} \frac{1}{f^2(k)} \frac{1}{\Delta\theta_{r(k)}} \\ &< \sum_{j=1}^\infty \frac{1}{2^j} \frac{|D_{m_j}(J)|}{\Delta\theta_{r(m_j)}} < \infty.\end{aligned}$$

$\square$

Corollary 3.8. *Let $\mathcal{F}_\theta$ be a lacunary statistical convergence filter and (x_n) a sequence of positive reals. The following statements are equivalent:*

(1) there is an $\mathcal{F}_\theta$-subseries of $\sum x_n$ which is convergent;

(2) there is an $\mathcal{F}_\theta$-subsequence of $(\Delta\theta_{r(n)}x_n)$ converging to 0.

Every lacunary statistical convergence ideal is obviously a shift invariant ideal. So, the above corollary, together with Corollary 2.9, gives us the following result.

Corollary 3.9. *Let $\mathcal{F}_\theta$ be a lacunary statistical convergence filter. If $\sum x_k$ is $\mathcal{F}_\theta$-convergent, then there is an $I \in {\mathcal{F}_\theta}^*$ such that $\sum_{k\in I} \frac{|x_k|}{\Delta\theta_{r(k)}} < \infty$.*

Finally, let us have the description of $\mathcal{F}$-admissibility and its corollaries for Louveau-Veličković ideals.

Let us mention that Louveau-Veličković ideals are examples of non-density ideals (c.f. [4]). Note that lacunary statistical convergence ideals are density ideals, as are Erdős-Ulam ideals. Density ideals are defined in [3] as follows. Assume that D_k are pairwise disjoint intervals on $\mathbb{N}$ and μ_n a measure that is concentrated on D_n. Then $\phi = \sup_n \mu_n$ is a lower semicontinuous submeasure, and $\mathcal{Z}_\mu = \mathrm{Exh}(\phi)$ is called a *density ideal.*

Recall the definition of Louveau-Veličković ideal [10]. Let (θ_r) be an increasing sequence of naturals. Let D_r be pairwise disjoint intervals on $\mathbb{N}$ such that $|D_r| = 2^{\theta_r}$. Let ϕ_r be a submeasure on D_r given by

$$\phi_r(I) = \frac{\log_2(|I \cap D_r| + 1)}{\theta_r}.$$

Then $\phi = \sup_r \phi_r$ is a lower semicontinuous submeasure, and $\mathcal{LV}_\theta = \mathrm{Exh}(\phi)$ is called the *Louveau-Veličković ideal.* It is easy to see that the following lemma is true.

Lemma 3.10. *Let $\mathcal{LV}_\theta$ be a Louveau-Veličković ideal. Then $I \notin \mathcal{LV}_\theta$ if and only if there are $\varepsilon > 0$ and a subsequence $n_1 < n_2 < \dots$ such that*

$$2 > \frac{\log_2(|I \cap D_{n_j}| + 1)}{\theta_{n_j}} > \varepsilon \quad \text{for all} \quad j \in \mathbb{N}. \tag{3.2}$$

Theorem 3.11. *Let $\mathcal{F}$ correspond to $\mathcal{I} = \mathcal{LV}_\theta$. A sequence (a_n) is not $\mathcal{F}$-admissible if and only if there is $J \in \mathcal{F}^*$ such that*

$$a_n\sqrt{\frac{1}{|J \cap D_{r(n)}|}} \to_{n\in J} \infty, \quad \text{where} \quad n \in D_{r(n)}.$$

PROOF. Let us first suppose that a sequence (a_n) is not $\mathcal{F}$-admissible. For a given I, we write $f(k,I) = a_k\sqrt{\frac{1}{|I\cap D_{r(k)}|}}$, and for a natural n, we set $I^{<n} = \{k \in I : f(k,I) < n\}$ and $I^{\geq n} = \{k \in I : f(k,I) \geq n\}$. Then there is $I \in \mathcal{F}^*$ such that for every n, we have

$$\begin{aligned}
\infty > \sum_{k\in I} a_k^{-2} &= \sum_{r=1}^{\infty}\sum_{k\in D_r\cap I}\frac{1}{|D_r\cap I|f^2(k,I)}\\
&\geq \sum_{r=1}^{\infty}\left(\sum_{k\in D_r\cap I^{<n}}\frac{1}{|D_r\cap I|n^2} + \sum_{k\in D_r\cap I^{\geq n}}\frac{1}{|D_r\cap I|\max_{k\in D_r} f^2(k,I)}\right)\\
&= \sum_{r=1}^{\infty}\frac{|D_r\cap I^{<n}|}{|D_r\cap I|n^2} + \sum_{r=1}^{\infty}\frac{|D_r\cap I^{\geq n}|}{|D_r\cap I|\max_{k\in D_r} f^2(k,I)}. \qquad (3.3)
\end{aligned}$$

For this $I \in \mathcal{F}^*$ and the sets $I^{\geq n}$ and $I^{<n}$ from (3.3), there are two possible cases: (1) There are infinitely many n such that $I^{<n} \in \mathcal{I}$; (2) $I^{<n} \in \mathcal{F}^*(I)$ for all n starting from some natural number.

Since $(I^{<n})$ is an increasing sequence, the case (1) gives us that $I^{<n} \in \mathcal{I}$ for all n. For P-filters, this means that we can find $A \in \mathcal{F}$ such that $|I\cap A\cap I^{<n}| < \infty$ for every n. Thus, for every n, there is a natural N such that for all $k > N$, $k \in I \cap A$, we have $f(k,I) \geq n$. We conclude that $f(k, I\cap A) \geq f(k,I) \to \infty$ along $I \cap A$.

Now let us show that the case (2) gives us that all sets $I^{\geq n}$ are also in $\mathcal{F}^*(I)$. First note that

$$|I^{\geq n}\cap D_i| + |I^{<n}\cap D_i| = |I\cap D_i| \geq x_i|I^{<n}\cap D_i|$$

for some $x_i \to +\infty$, since the first series from (3.3) converges. Thus,

$$|I^{\geq n}\cap D_i| \geq (x_i - 1)|I^{<n}\cap D_i|,$$

and for those i for which $|I^{<n}\cap D_i| \geq 1$, we have

$$|I^{\geq n}\cap D_i| + 1 \geq \frac{(x_i-1)}{2}\left(|I^{<n}\cap D_i| + 1\right).$$

So, we get

$$\frac{\log_2(|I^{\geq n}\cap D_i| + 1)}{\theta_i} \geq \frac{\log_2(|I^{<n}\cap D_i| + 1)}{\theta_i} + \frac{\log_2\frac{(x_i-1)}{2}}{\theta_i}. \qquad (3.4)$$

Since the last summand is nonnegative when $i \to \infty$, we obtain that $I^{\geq n}$ is in $\mathcal{F}^*(I)$.

Our last step is to apply Lemma 3.10 and choose one D_{m_n} for each $I^{\geq n}$ such that the condition (3.2) holds and (m_n) is increasing. Then by the same lemma, the set $J = \bigsqcup_{n=1}^{\infty} D_{m_n} \cap I^{\geq n}$ is in $\mathcal{F}^*$, actually in $\mathcal{F}^*(I)$, and for every n, there is a natural N such that for all $k > N$, $k \in J$, we have $f(k, I) \geq n$. So, this case also gives us that $f(k, J) \geq f(k, I) \to \infty$ along J.

Now we consider the opposite direction. Let us have $J \in \mathcal{F}^*$ such that $f(k, J) = a_k \sqrt{\frac{1}{|J \cap D_{r(k)}|}} \to \infty$ along J. Lemma 3.10 gives us (n_j) and $\varepsilon > 0$ such that $D_{n_j} \cap J$ satisfies (3.2). Obviously, for every subsequence $(m_j) \subset (n_j)$, the inequality (3.2) also holds, and thus, $I = \bigsqcup_{j=1}^{\infty} D_{m_j} \cap J \in \mathcal{F}^*$. We choose (m_j) such that $\min_{k \in D_{m_j}} f^2(k, J) \geq 2^j$. Hence, we have

$$\sum_{n \in I} a_n^{-2} = \sum_{j=1}^{\infty} \sum_{k \in J \cap D_{m_j}} a_k^{-2} = \sum_{j=1}^{\infty} \sum_{k \in J \cap D_{m_j}} \frac{1}{f^2(k)|J \cap D_{m_j}|}$$
$$\leq \sum_{j=1}^{\infty} \frac{1}{\min_{k \in D_{m_j}} f^2(k)} < \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty.$$

The absence of $\mathcal{F}$-admissibility is proved. □

Corollary 3.12. *Let $\mathcal{F}$ be a filter which corresponds to the Louveau-Veličković ideal. For a sequence (x_n) of positive reals, the following statements are equivalent:*

(1) there is an $I \in \mathcal{F}^*$ such that $\sum_{n \in I} x_n < \infty$;

(2) there is a $J \in \mathcal{F}^*$ such that

$$|J \cap D_{r(n)}| x_n \to_{n \in J} 0, \quad \text{where} \quad n \in D_{r(n)}.$$

Finally, to obtain the analogue of Corollaries 3.5 and 3.9 for the Louveau-Veličković ideals, let us check that they have the shift property.

Proposition 3.13. *The Louveau-Veličković ideals are shift invariant.*

Proof. Suppose $I \in \mathcal{LV} = \mathrm{Exh}(\phi)$; that is, $\lim_{k \to \infty} \phi(I \setminus [1, k]) = 0$. Let us check the same limit for $I + 1$. We have

$$\begin{aligned}
&\phi\left((I+1) \setminus [1,k]\right) = \lim_{n \to \infty} \phi\left((I+1) \cap (k, n]\right) \\
&\quad = \lim_{n \to \infty} \sup_r \frac{\log_2(|(I+1) \cap (k,n] \cap D_r| + 1)}{\theta_r} \\
&\quad \leq \phi\left(I \setminus [1,k]\right) + \lim_{n \to \infty} \sup_r \frac{1}{\theta_r} \log_2 \left(\frac{|(I+1) \cap (k,n] \cap D_r| + 1}{|I \cap (k,n] \cap D_r| + 1} \right). \qquad (3.5)
\end{aligned}$$

The supremum in the last summand of (3.5) is reached at $r = r(k)$, which tends to ∞ when $k \to \infty$. This supremum is

$$\leq \frac{1}{\theta_{r(k)}} \log_2 \left(1 + \frac{1}{|I \cap (k, n] \cap D_{r(k)}| + 1}\right) \leq \frac{1}{\theta_{r(k)}} \log_2 2 \to_{k \to \infty} 0.$$

Thus, $\phi\left((I+1) \setminus [1, k]\right)$ is also convergent to 0 as $k \to \infty$. $\square$

Corollary 3.14. *Let $\mathcal{F}$ be a Louveau-Veličković filter. If $\sum x_k$ is $\mathcal{F}$-convergent, then there is an $I \in \mathcal{F}^*$ such that $\sum_{k \in I} \frac{|x_k|}{|I \cap D_{r(k)}|} < \infty$, where $k \in D_{r(k)}$.*

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