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A STIELTJES TYPE EXTENSION OF THE L^r-PERRON INTEGRAL

Abstract

We explore properties of L^r-derivates with respect to a monotone increasing Lipschitz function. We then define L^r-ex-major and L^r-ex-minor functions with respect to a monotone increasing Lipschitz function and use these to define a Perron-Stieltjes type integral which extends the integral of L. Gordon.

1 Introduction

In 1914, O. Perron [3] developed an extension of the Lebesgue integral based on major and minor functions and upper and lower Dini derivates. The classical derivative of a function F is Perron integrable, and F is the indefinite integral of its derivative. Calderon and Zygmund then introduced the L^r-derivative, which has applications in harmonic analysis [1]. Later, L. Gordon developed a Perron-type integral that recovers a function from its L^r-derivative [2].

In [7], Tikare and Chaudhary defined L^r-derivates with respect to a Lipschitz function of order 1. They then defined a Perron-type integral which recovers a function from its L^r-derivative with respect to a Lipschitz function. In the present paper, we modify the integration process given in [7] so that it extends the integral of L. Gordon [2].

Throughout this paper, a Lipschitz function will mean a Lipschitz function of order 1, and $r \in [1, \infty)$.

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2 Definitions and elementary properties of the $L^{r,\phi}$-derivates

For completeness, here we restate the definitions of the L^r-derivates with respect to a Lipschitz function found in [7].

Definition 1. *[7] Let $f \in L^r[a,b]$, let ϕ be a monotone increasing Lipschitz function defined on $[a,b]$, and let $h \to 0^+$.*

We define the upper right $L^{r,\phi}$-derivate, denoted $D_r^+ f(x;\phi)$, to be the greatest lower bound of all α such that

$$\left(\frac{1}{h}\int_0^h \left[f(x+t) - f(x) - \alpha(\phi(x+t) - \phi(x))\right]_+^r dt\right)^{\frac{1}{r}} = o(h). \tag{1}$$

If no real number α satisfies (1), then we set $D_r^+ f(x;\phi) = +\infty$. If (1) holds for every real number α, then we set $D_r^+ f(x;\phi) = -\infty$.

We define the lower right $L^{r,\phi}$-derivate, denoted $D_{+,r} f(x;\phi)$, to be the least upper bound of all α such that

$$\left(\frac{1}{h}\int_0^h \left[f(x+t) - f(x) - \alpha(\phi(x+t) - \phi(x))\right]_-^r dt\right)^{\frac{1}{r}} = o(h). \tag{2}$$

If no real number α satisfies (2), then we set $D_{+,r} f(x;\phi) = -\infty$. If (2) holds for every real number α, then we set $D_{+,r} f(x;\phi) = +\infty$.

We define the upper left $L^{r,\phi}$-derivate, denoted $D_r^- f(x;\phi)$, to be the greatest lower bound of all α such that

$$\left(\frac{1}{h}\int_0^h \left[-f(x-t) + f(x) - \alpha(-\phi(x-t) + \phi(x))\right]_+^r dt\right)^{\frac{1}{r}} = o(h). \tag{3}$$

If no real number α satisfies (3), then we set $D_r^- f(x;\phi) = +\infty$. If (3) holds for every real number α, then we set $D_r^- f(x;\phi) = -\infty$.

Finally, we define the lower left $L^{r,\phi}$-derivate, denoted $D_{-,r} f(x;\phi)$, to be the least upper bound of all α such that

$$\left(\frac{1}{h}\int_0^h \left[-f(x-t) + f(x) - \alpha(-\phi(x-t) + \phi(x))\right]_-^r dt\right)^{\frac{1}{r}} = o(h). \tag{4}$$

If no real number α satisfies (4), then we set $D_{-,r} f(x;\phi) = -\infty$. If (4) holds for every real number α, then we set $D_{-,r} f(x;\phi) = +\infty$.

Definition 2. *[7] We define the upper (two-sided) $L^{r,\phi}$-derivate as follows:*

$$\overline{D}_r f(x;\phi) = \max\left\{D_r^+ f(x;\phi), D_r^- f(x;\phi)\right\}.$$

Similarly we define the lower (two-sided) $L^{r,\phi}$-derivate as follows:

$$\underline{D}_r f(x;\phi) = \min\left\{D_{+,r} f(x;\phi), D_{-,r} f(x;\phi)\right\}.$$

Definition 3. *Let f and ϕ satisfy the hypotheses of Definition 1 and let $h \to 0^+$. If $\overline{D}_r f(x;\phi)$ and $\underline{D}_r f(x;\phi)$ are the same real number, then we say that f is $L^{r,\phi}$-differentiable at x and denote the common value by $D_r f(x,\phi)$.*

If the ϕ is omitted from the notation for an $L^{r,\phi}$-derivate or $L^{r,\phi}$-derivative, then it is assumed that ϕ is the identity function, and we have the L^r-derivates and L^r-derivatives from [2].

It is clear that if ϕ is strictly decreasing in a neighborhood of x, then none of the $L^{r,\phi}$-derivates at x can be finite; therefore, unless otherwise indicated, in this paper we will assume that ϕ is monotone increasing.

We will make use of the following.

Theorem 4. *[7] Let f and ϕ satisfy the hypotheses of Definition 1. Then either $D_r^+ f(x;\phi) = \pm\infty$ or $D_r^+ f(x;\phi)$ is the minimum of all real numbers α such that*

$$\left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - \alpha(\phi(x+t) - \phi(x))]_+^r dt\right)^{\frac{1}{r}} = o(h),$$

where ϕ is a monotone increasing Lipschitz function.

Similar conditions hold for each of the other $L^{r,\phi}$-derivates.

Indeed, we now show that in order for ϕ to have finite $L^{r,\phi}$-derivates at x, ϕ must be strictly increasing in a neighborhood of x and must not increase too slowly.

Theorem 5. *Let f and ϕ satisfy the hypotheses of Definition 1, and let $x \in [a,b]$. If $D_r^+\phi(x) = 0$, that is, if*

$$\left(\frac{1}{h}\int_0^h (\phi(x+t) - \phi(x))^r dt\right)^{\frac{1}{r}} = o(h) \text{ as } h \to 0^+, \tag{5}$$

then both $D_r^+ f(x;\phi)$ and $D_{+,r} f(x;\phi)$ are infinite.

Similarly if $D_r^- \phi(x) = 0$, that is, if

$$\left(\frac{1}{h}\int_0^h (\phi(x) - \phi(x-t))^r\, dt\right)^{\frac{1}{r}} = o(h) \text{ as } h \to 0^+,$$

then both $D_r^- f(x;\phi)$ and $D_{-,r} f(x;\phi)$ are infinite.

Proof. We will prove that $D_r^+ \phi(x) = 0$ implies that $D_r^+ f(x;\phi)$ is infinite; the other cases have similar proofs.

Suppose

$$\left(\frac{1}{h}\int_0^h [f(x+t) - f(x)]_+^r dt\right)^{\frac{1}{r}} = o(h) \text{ as } h \to 0^+ \tag{6}$$

and let $\alpha \in R$. We then have by Minkowski's inequality

$$\begin{aligned}&\left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - \alpha(\phi(x+t) - \phi(x))]_+^r\, dt\right)^{\frac{1}{r}}\\ &\leq \left(\frac{1}{h}\int_0^h [f(x+t) - f(x)]_+^r\, dt\right)^{\frac{1}{r}} + |\alpha|\left(\frac{1}{h}\int_0^h (\phi(x+t) - \phi(x))^r\, dt\right)^{\frac{1}{r}}.\end{aligned}$$

Both of the terms on the right hand side are $o(h)$, so that $D_r^+ f(x;\phi) = -\infty$.

Also by Minkowski's inequality, we have

$$\begin{aligned}&\left(\frac{1}{h}\int_0^h [f(x+t) - f(x)]_+^r\, dt\right)^{\frac{1}{r}}\\ &\leq \left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - \alpha(\phi(x+t) - \phi(x))]_+^r\, dt\right)^{\frac{1}{r}}\\ &\qquad + |\alpha|\left(\frac{1}{h}\int_0^h (\phi(x+t) - \phi(x))^r\, dt\right)^{\frac{1}{r}},\end{aligned}$$

so that if (6) does not hold, then $D_r^+ f(x;\phi) = +\infty$, and the result is proved. □

Corollary 6. *If $D_r^+ f(x;\phi)$ or $D_{+,r} f(x;\phi)$ is finite, then $D_r^+ \phi(x) > 0$, and if $D_r^- f(x;\phi)$ or $D_{-,r} f(x;\phi)$ is finite, then $D_r^- \phi(x) > 0$.*

Theorem 7. *Let f and ϕ satisfy the hypotheses of Definition 1, and let $x \in [a,b]$. Then,*

1. $D_r^+\phi(x) > 0$ *implies* $D_r^+ f(x;\phi) \geq D_{+,r} f(x;\phi)$,
2. $D_r^-\phi(x) > 0$ *implies* $D_r^- f(x;\phi) \geq D_{-,r} f(x;\phi)$,
3. $D_r^+\phi(x) > 0$ *and* $D_r^-\phi(x) > 0$ *imply* $\overline{D}_r f(x;\phi) \geq \underline{D}_r f(x;\phi)$.

Proof. It is clear that (3) follows from (1) and (2). We will prove that $D_r^+ f(x;\phi) \geq D_{+,r} f(x;\phi)$; the proof for the left $L^{r,\phi}$-derivates is similar. If $D_r^+ f(x;\phi) = +\infty$, then there is nothing to prove. We first assume that $D_r^+ f(x;\phi)$ is finite. Suppose that β could take the place of α in (1) and γ could take the place of α in (2), and suppose by way of contradiction that $\gamma > \beta$. We then have

$$\begin{aligned} 0 &\leq (\gamma-\beta)\left(\frac{1}{h}\int_0^h (\phi(x+t)-\phi(x))^r\,dt\right)^{\frac{1}{r}} \\ &\leq \left(\frac{1}{h}\int_0^h [f(x+t)-f(x)-\beta(\phi(x+t)-\phi(x))]_+^r\,dt\right)^{\frac{1}{r}} \\ &\qquad + \left(\frac{1}{h}\int_0^h [f(x+t)-f(x)-\gamma(\phi(x+t)-\phi(x))]_-^r\,dt\right)^{\frac{1}{r}}. \end{aligned}$$

The last two terms are $o(h)$. This contradicts the fact that $D_r^+\phi(x) > 0$, so either $D_{+,r} f(x;\phi)$ is a finite number less than or equal to $D_r^+ f(x;\phi)$ or $D_{+,r} f(x;\phi) = -\infty$.

Finally we consider the case where $D_r^+ f(x;\phi) = -\infty$. Assume by way of contradiction that $D_{+,r} f(x;\phi) \neq -\infty$; i.e., there exists γ that could take the place of α in (2). The preceding inequality shows that if $\beta < \gamma$, then

$$\left(\frac{1}{h}\int_0^h [f(x+t)-f(x)-\beta(\phi(x+t)-\phi(x))]_+^r\,dt\right)^{\frac{1}{r}} \neq o(h).$$

This means that $D_r^+ f(x;\phi) > -\infty$, and the theorem is proved. □

It is clear that if f is $L^{r,\phi}$-differentiable at x, then $D_r^+\phi(x) > 0$ and $D_r^-\phi(x) > 0$. Therefore, the following is a consequence of Theorem 7.

Corollary 8. *If f is $L^{r,\phi}$-differentiable at x, then $D_r f\left(x,\phi\right)$ is the unique real number α such that*

$$\left(\frac{1}{h}\int_{-h}^{h}\left|f\left(x+t\right)-f\left(x\right)-\alpha\left(\phi\left(x+t\right)-\phi\left(x\right)\right)\right|^{r}dt\right)^{\frac{1}{r}}=o\left(h\right).$$

In addition, all four $L^{r,\phi}$-derivates are equal to $D_r f(x,\phi)$.

We now show that the upper $L^{r,\phi}$-derivate is subadditive, the lower $L^{r,\phi}$-derivate is superadditive and the $L^{r,\phi}$-derivative is additive.

Theorem 9. *Let f satisfy the hypotheses of Definition 1, and let $x \in [a,b]$. Let f_1 and f_2 be in $L^r[a,b], 1 \le r < \infty$, and let ϕ be a monotone increasing Lipschitz function defined on $[a,b]$ such that $D_r^+\phi(x) > 0$. Let $f = f_1 + f_2$. Then*

1. $D_r^+ f\left(x;\phi\right) \le D_r^+ f_1\left(x;\phi\right) + D_r^+ f_2\left(x;\phi\right)$ *and*
2. $D_{+,r} f\left(x;\phi\right) \ge D_{+,r} f_1\left(x;\phi\right) + D_{+,r} f_2\left(x;\phi\right)$

if the right side of each inequality is defined. Similar inequalities hold for the left and two-sided $L^{r,\phi}$-derivates.

If f_1 is $L^{r,\phi}$-differentiable at x and f_2 is $L^{r,\phi}$-differentiable at x, then f is $L^{r,\phi}$-differentiable at x and $D_r f\left(x;\phi\right) = D_r f_1\left(x;\phi\right) + D_r f_2\left(x;\phi\right)$.

Proof. We sketch the proof of (1). If the right hand side of the inequality is $+\infty$, then there is nothing to prove. If the right hand side is finite, then the result holds by Minkowski's inequality.

If the right hand side is $-\infty$, we may assume that $D_r^+ f_1\left(x;\phi\right) = -\infty$. Let $\beta \in \mathbb{R}$, let $\alpha_2 > D_r^+ f_2\left(x;\phi\right)$ and let $\alpha_1 = \beta - \alpha_2$. An application of Minkowski's inequality proves the result. □

3 Relation between $L^{r,\phi}$-derivates and L^r-derivates.

If ϕ is L^r-differentiable at a point x, then we have the following.

Theorem 10. *Let f satisfy the hypotheses of Definition 1, and let ϕ be a monotone increasing Lipschitz function defined on $[a,b]$ which is L^r-differentiable at x with $D_r\phi\left(x\right) > 0$. Then f is $L^{r,\phi}$-differentiable at x if and only if f is L^r-differentiable at x, and in this case we have*

$$D_r f\left(x\right) = D_r\phi\left(x\right) D_r f\left(x,\phi\right). \tag{7}$$

PROOF. Let $\beta = D_r\phi(x)$. Suppose f is $L^{r,\phi}$-differentiable at x and let $\alpha = D_r f(x,\phi)$. We then have

$$\begin{aligned}
&\left(\frac{1}{h}\int_{-h}^{h} |f(x+t) - f(x) - \alpha\beta t|^r \, dt\right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{h}\int_{-h}^{h} |f(x+t) - f(x) - \alpha(\phi(x+t) - \phi(x))|^r \, dt\right)^{\frac{1}{r}} \\
&\qquad + |\alpha| \left(\frac{1}{h}\int_{-h}^{h} |\phi(x+t) - \phi(x) - \beta t|^r \, dt\right)^{\frac{1}{r}}.
\end{aligned}$$

Both of the terms on the righthand side are $o(h)$, so f is L^r-differentiable at x and (7) holds.

Conversely, suppose f is L^r-differentiable at x and let $\xi = D_r f(x)$. Then we have that

$$\begin{aligned}
&\left(\frac{1}{h}\int_{-h}^{h} \left|f(x+t) - f(x) - \frac{\xi}{\beta}(\phi(x+t) - \phi(x))\right|^r \, dt\right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{h}\int_{-h}^{h} |f(x+t) - f(x) - \xi t|^r \, dt\right)^{\frac{1}{r}} \\
&\qquad + \left|\frac{\xi}{\beta}\right| \left(\frac{1}{h}\int_{-h}^{h} |\phi(x+t) - \phi(x) - \beta t|^r \, dt\right)^{\frac{1}{r}}.
\end{aligned}$$

Both of the terms on the righthand side are $o(h)$, so f is $L^{r,\phi}$-differentiable at x and (7) holds.

□

Theorem 11. *Let ϕ be a monotone increasing Lipschitz function defined on $[a,b]$. Then $\underline{D}_r f(x;\phi) \geq 0$ if and only if $\underline{D}_r f(x) \geq 0$.*

PROOF. Let γ be the identity function. Suppose $D_{+,r} f(x;\phi) \geq 0$. Let $P_{f,\phi}(\alpha)$ mean that

$$\left(\frac{1}{h}\int_{0}^{h} [f(x+t) - f(x) - \alpha(\phi(x+t) - \phi(x))]_-^r \, dt\right)^{\frac{1}{r}} = o(h).$$

Suppose $\alpha \leq \beta$. Then because ϕ is monotone increasing, we have that $P_{f,\phi}(\beta)$ implies $P_{f,\phi}(\alpha)$.

By Theorem 4, we have that if $D_{+,r}f\left(x;\phi\right)\geq 0$, then $P_{f,\phi}\left(0\right)$. We then have that

$$\left(\frac{1}{h}\int_0^h\left[f\left(x+t\right)-f\left(x\right)-0\left(\phi\left(x+t\right)-\phi\left(x\right)\right)\right]_-^r\,dt\right)^{\frac{1}{r}}=o\left(h\right)$$

so that

$$\left(\frac{1}{h}\int_0^h\left[f\left(x+t\right)-f\left(x\right)-0\left(\gamma\left(x+t\right)-\gamma\left(x\right)\right)\right]_-^r\,dt\right)^{\frac{1}{r}}=o\left(h\right),$$

and so $D_{+,r}f\left(x\right)\geq 0$. The converse follows similarly. Also, the result for the lower left L^r-derivate follows similarly. □

Theorem 12. *Let ϕ be a monotone increasing Lipschitz function defined on $[a,b]$. If $\overline{D}_r\phi(x)$ is finite and if $\overline{D}_rf(x;\phi)<\infty$, then $\overline{D}_rf(x)<\infty$.*

PROOF. We first work on the right side; the proof for the left side is similar. Since $D_r^+f(x;\phi)<\infty$, there exists a real number α such that (1) holds. We wish to prove that there exists β such that

$$\left(\frac{1}{h}\int_0^h[f(x+t)-f(x)-\beta t]_+^r\,dt\right)^{\frac{1}{r}}=o(h).$$

Let $D_r^+\phi(x)=\eta$, where $0\leq\eta<\infty$. By Corollary 6, we also have that $\eta>0$. We then have

$$
\begin{aligned}
&\left(\frac{1}{h}\int_0^h [f(x+t)-f(x)-\alpha\eta t]_+^r dt\right)^{\frac{1}{r}} \\
&= \left(\frac{1}{h}\int_0^h [f(x+t)-f(x)-\alpha\eta t+\alpha(\phi(x+t)-\phi(x))\right. \\
&\qquad \left. -\alpha(\phi(x+t)-\phi(x))]_+^r dt\right)^{\frac{1}{r}} \\
&\le \left(\frac{1}{h}\int_0^h [f(x+t)-f(x)-\alpha(\phi(x+t)-\phi(x))]_+^r dt\right)^{\frac{1}{r}} \\
&\qquad + \left(\frac{1}{h}\int_0^h [\alpha(\phi(x+t)-\phi(x))-\alpha\eta t]_+^r dt\right)^{\frac{1}{r}} \\
&\le o(h)+|\alpha|\left(\frac{1}{h}\int_0^h [(\phi(x+t)-\phi(x))-\eta t]_+^r dt\right)^{\frac{1}{r}} \\
&\le o(h).
\end{aligned}
$$

We may therefore conclude that $D_r^+ f(x) < \infty$, and the theorem is proved. □

4 Relation between $L^{r,\phi}$-continuity and L^r-continuity

Definition 13. *[7] Let $1 \le r < \infty$. A function $f \in L^r([a,b])$ is said to be L^r-continuous with respect to ϕ (or simply $L^{r,\phi}$-continuous) at $x_0 \in [a,b]$ if for some number k,*

$$\int_{[a,b]\cap[x_0-h,x_0+h]} |f(x)-f(x_0)-k(\phi(x)-\phi(x_0))|^r dx = o(h). \tag{8}$$

In particular, if $k=0$, we will simply say that f is L^r-continuous at x.

Theorem 14. *Given a Lipschitz function ϕ, a function $f : [a,b] \to R$ is L^r-continuous with respect to ϕ if and only if f is L^r-continuous.*

Proof. Let f be L^r-continuous. We need to show that (8) holds for any Lipschitz function ϕ and any k. Let M be a positive constant such that for any $x_1, x_2 \in [a,b]$ we have

$$|\phi(x_2) - \phi(x_1)| \leq M|x_2 - x_1|.$$

By Minkowski's inequality we have

$$\begin{aligned}
&\left(\int_{[a,b]\cap[x_0-h,x_0+h]} |f(x) - f(x_0) - k(\phi(x) - \phi(x_0))|^r dx\right)^{\frac{1}{r}} \\
&\leq \left(\int_{[a,b]\cap[x_0-h,x_0+h]} |f(x) - f(x_0)|^r dx\right)^{\frac{1}{r}} + |k| \left(\int_{[a,b]\cap[x_0-h,x_0+h]} |\phi(x) - \phi(x_0)|^r dx\right)^{\frac{1}{r}} \\
&\leq o(h) + |k|M \left(\int_{[a,b]\cap[x_0-h,x_0+h]} |x - x_0|^r dx\right)^{\frac{1}{r}} \\
&\leq o(h) + |k|M \left(\int_{[a,b]\cap[x_0-h,x_0+h]} |h|^r dx\right)^{\frac{1}{r}} \\
&\leq o(h) + (|k|M)\,(h)\,(2h)^{\frac{1}{r}} \\
&\leq o(h).
\end{aligned}$$

Conversely, supposing that (8) holds for some ϕ and some k, we also have, by Minkowski's inequality,

$$\begin{aligned}
\left(\int_{[a,b]\cap[x_0-h,x_0+h]} |f(x) - f(x_0)|^r dx\right)^{\frac{1}{r}} &\leq \left(\int_{[a,b]\cap[x_0-h,x_0+h]} |f(x) - f(x_0) - k(\phi(x) - \phi(x_0)|^r dx\right)^{\frac{1}{r}} \\
&\quad + |k| \left(\int_{[a,b]\cap[x_0-h,x_0+h]} |\phi(x) - \phi(x_0))|^r dx\right)^{\frac{1}{r}} \\
&\leq o(h).
\end{aligned}$$

□

5 Further properties of the $L^{r,\phi}$-derivates.

We will need the following as we develop the theory of $L^{r,\phi}$-ex-major functions.

Theorem 15. *Suppose that $f \in L^r([a,b])$, that ϕ is a monotone increasing Lipschitz function defined on $[a,b]$ and that $\underline{D}_r f(x;\phi) \geq 0$, except perhaps on a countable set E' where, however, f is L^r-continuous. Then f is monotone increasing on $[a,b]$.*

The proof will require several lemmas, including the following extension of [2] Lemma 2.

Definition 16. *Let* $0 \leq p \leq 1$ *and let* E *be a measurable subset of* $[a,b]$. *Let* $x \in (a,b)$. *We will say that* x *is a point of* p*-lower density of* E *if*

$$\lim_{h \to 0^+} \inf \frac{\lambda\left(E \cap (x-h, x+h)\right)}{2h} = p. \tag{9}$$

Definition 17. *Let* $0 \leq p \leq 1$ *and let* E *be a measurable subset of* $[a,b]$. *Let* $x \in [a,b)$. *We will say that* x *is a point of* p*-lower right-hand density of* E *if*

$$\lim_{h \to 0^+} \inf \frac{\lambda\left(E \cap (x, x+h)\right)}{h} = p. \tag{10}$$

For convenience we will assume that if $b \in E$, *then* b *is a point of* 1*-lower right-hand density of* E.

Definition 18. *Let* $0 \leq p \leq 1$ *and let* E *be a measurable subset of* $[a,b]$. *Let* $x \in (a,b]$. *We will say that* x *is a point of* p*-lower left-hand density of* E *if*

$$\lim_{h \to 0^+} \inf \frac{\lambda\left(E \cap (x-h, x)\right)}{h} = p. \tag{11}$$

For convenience we will assume that if $a \in E$, *then* a *is a point of* 1*-lower left-hand density of* E.

Lemma 19. *Let* R *and* L *be nonempty disjoint measurable sets such that* $[a,b] = R \cup L$, *and suppose that there exist* $p_1 > 1/2$ *so that every point of* R *is a point of* p_1*-lower right-hand density of* R, *and* $p_2 > 1/2$ *so that every point of* L *is a point of* p_2*-lower left-hand density of* L. *The every point of* R *is to the right of every point of* L.

Proof. Suppose to the contrary that there exist $x_1 \in R$ and $x_2 \in L$ such that $a \leq x_1 < x_2 \leq b$. Choose $q \in (1/2, p_1 \wedge p_2)$ as well as $m > 1/(2q-1)$. Let

$$g(x) = (x-d)^{-1} \int_a^x \left(\chi_R(t) - \chi_L(t)\right) dt,$$

where $x \in [a,b]$ and $d < a - m(b-a)$. We will show that g fails to achieve a maximum value on $[x_1, x_2]$. Let us show that if $x_0 \in [x_1, x_2) \cap R$, then $g(x)$ increases as we move slightly to the right of x_0. Let $x_3 \in (x_0, b)$ be such that if $\xi \in (x_0, x_3)$, then

$$\frac{\lambda\left(R \cap (x_0, \xi)\right)}{\xi - x_0} > q.$$

Letting $N = 1/(\xi - d)(x_0 - d)$, and noting that $N > 0$, we have

$$\begin{aligned}
&g(\xi) - g(x_0) \\
&= (\xi - d)^{-1} \int_a^{\xi} (\chi_R(t) - \chi_L(t))\, dt - (x_0 - d)^{-1} \int_a^{x_0} (\chi_R(t) - \chi_L(t))\, dt \\
&= N\left[(x_0 - d) \int_a^{\xi} (2\chi_R(t) - 1)\, dt - (\xi - d) \int_a^{x_0} (2\chi_R(t) - 1)\, dt\right] \\
&= N\left[(x_0 - d) \int_{x_0}^{\xi} (2\chi_R(t) - 1)\, dt - (\xi - x_0) \int_a^{x_0} (2\chi_R(t) - 1)\, dt\right] \\
&> N\left[m(b-a)(2q-1)(\xi - x_0) - (\xi - x_0)(b - a)\right] \\
&> 0.
\end{aligned}$$

Now suppose $x_0 \in (x_1, x_2] \cap L$. Let $x_3 \in (a, x_0)$ be such that if $\xi \in (x_3, x_0)$, then

$$\frac{\lambda(L \cap (\xi, x_0))}{x_0 - \xi} > q.$$

We then have

$$\begin{aligned}
&g(x_0) - g(\xi) \\
&= (x_0 - d)^{-1} \int_a^{x_0} (\chi_R(t) - \chi_L(t))\, dt - (\xi - d)^{-1} \int_a^{\xi} (\chi_R(t) - \chi_L(t))\, dt \\
&= (\xi - d)^{-1} \int_a^{\xi} (\chi_L(t) - \chi_R(t))\, dt - (x_0 - d)^{-1} \int_a^{x_0} (\chi_L(t) - \chi_R(t))\, dt \\
&= N\left[(x_0 - d) \int_a^{\xi} (2\chi_L(t) - 1)\, dt - (\xi - d) \int_a^{x_0} (2\chi_L(t) - 1)\, dt\right] \\
&= N\left[(x_0 - \xi) \int_a^{\xi} (2\chi_L(t) - 1)\, dt - (\xi - d) \int_{\xi}^{x_0} (2\chi_L(t) - 1)\, dt\right] \\
&< N\left[(x_0 - \xi)(b - a) - m(b-a)(2q-1)(x_0 - \xi)\right] \\
&< 0.
\end{aligned}$$

We then have that $g(x)$ increases as we move slightly to the left of x_0. We have thus demonstrated that g cannot achieve a maximum on $[x_1, x_2]$. However, since g is continuous, it must achieve a maximum on $[x_1, x_2]$, a contradiction. $\square$

Lemma 20. *Let F be a measurable function on $[a,b]$, let E' be a countable subset of $[a,b]$, and let $E=[a,b]\setminus E'$. Suppose (i) F is approximately continuous at each point of E' and (ii) each point x_0 of E is a point of p_1-lower right-hand density of the set $\{x\in[a,b]:F(x)\geq F(x_0)\}$ for some $p_1>1/2$, and a point of p_2-lower left-hand density of the set $\{x\in[a,b]:F(x)\leq F(x_0)\}$ for some $p_2>1/2$. Then F is monotone increasing on $[a,b]$.*

Proof. Suppose $x_1,x_2\in[a,b]$ and $F(x_1)<F(x_2)$. We need to show that $x_1<x_2$.

We have that E' is a countable set so that the set $\{y:F(x)=y \text{ for some } x\in E'\}$ is also countable. Therefore, we may choose $\epsilon>0$ so that $F(x_1)<F(x_2)-\epsilon$ and $F(x)\neq F(x_2)-\epsilon$ for any $x\in E'$.

Let $R=\{x\in[a,b]:F(x)\geq F(x_2)-\epsilon\}$ and $L=\{x\in[a,b]:F(x)<F(x_2)-\epsilon\}$. $R\cup L=[a,b]$ where R and L are disjoint measurable sets. Since x_2 is in R and x_1 is in L, both R and L are non-empty.

Let $x_0\in R$. If $x_0\in E$, then x_0 is a point of p_1-lower right-hand density, for some $p_1>1/2$, of $\{x\in[a,b]:F(x)\geq F(x_0)\}\subseteq\{x\in[a,b]:F(x)\geq F(x_2)-\epsilon\}$.

If $x_0\in E'$, then $F(x_0)>F(x_2)-\epsilon$. Choose $\gamma\in(0,F(x_0)-(F(x_2)-\epsilon))$. Then because F is approximately continuous at x_0, we have that x_0 is a point of density of

$$\{x:F(x)\in(F(x_0)-\gamma,F(x_0)+\gamma)\subseteq R\}\,.$$

We have shown that every point of R is a point of p_1-lower right-hand density of R for some $p_1>1/2$. A similar argument shows that every point of L is a point of p_2-lower left-hand density of the set $\{x\in[a,b]:F(x)\leq F(x_0)\}$ for some $p_2>1/2$. This then implies that R and L satisfy the hypotheses of Lemma 19 so that every point of L is to the left of every point of R. Since $x_1\in L$ and $x_2\in R$, it follows that $x_1<x_2$.

Proof of Theorem 15. We have $\underline{D}_r f(x,\phi)\geq 0$ for all $x\in E$, so by Theorem 11 and Chebyshev's inequality [5], we have that $\underline{f}_{app}(x)\geq 0$ for all $x\in E$. Also by Chebyshev's inequality, f is approximately continuous on E'.

The conclusion now follows from Lemma 20.

□

6 $L^{r,\phi}$-ex-major (ex-minor) functions.

In [2], L. Gordon shows that there exists a function f which is an L^r-derivative defined on $[a,b]$, so that if ψ is an L^r-major function of f, then $\underline{\psi}_r(b) = -\infty$. Thus, for a monotone increasing Lipschitz function ϕ, we define $L^{r,\phi}$-ex-major functions and $L^{r,\phi}$-ex-minor functions of f as follows.

Definition 21. *Suppose $f(x)$ is a function defined on $[a,b]$ and ϕ is a monotone increasing Lipschitz function also defined on $[a,b]$. A finite-valued function $\psi(x) \in L^r[a,b], 1 \le r < \infty$, is said to be an $L^{r,\phi}$-ex-major function of f if*

1. $\psi(a) = 0$,
2. *$\psi(x)$ is L^r-continuous on $[a,b]$,*
3. *except for at most a denumerable subset of $[a,b]$, we have*

$$-\infty \ne \underline{D}_r\psi(x;\phi) \ge f(x). \tag{12}$$

A function $\lambda(x)$ is an $L^{r,\phi}$-ex-minor function of f if $-\lambda(x)$ is an $L^{r,\phi}$-ex-major function of $-f$.

Theorem 22. *Suppose that $\psi(x)$ and $\lambda(x)$ are, respectively, $L^{r,\phi}$-ex-major and $L^{r,\phi}$-ex-minor functions of f. The function $u(x) = \psi(x) - \lambda(x)$ is monotone increasing on $[a,b]$.*

Proof. Suppose that ψ is an $L^{r,\phi}$-ex-major function and that λ is an $L^{r,\phi}$-ex-minor function of f on $[a,b]$. We shall show that for nearly every x, we have $\underline{D}_r u(x;\phi) \ge 0$.

Let x be such that $-\infty \ne \underline{D}_r\psi(x;\phi) \ge f(x) \ge \overline{D}_r\lambda(x;\phi) \ne +\infty$, and let $\epsilon > 0$. There exist α, β, with $\alpha \le \beta + \epsilon$, such that

$$\int_0^h [S(x,t)]_-^r \, dt = o(h^{r+1})$$

and

$$\int_0^h [T(x,t)]_+^r \, dt = o(h^{r+1}),$$

where

$$S(x,t) = \psi(x+t) - \psi(x) - \beta(\phi(x+t) - \phi(x))$$

and

$$T(x,t) = \lambda(x+t) - \lambda(x) - \alpha(\phi(x+t) - \phi(x)).$$

Let

$$\begin{aligned} U(x,t) &= u(x+t) - u(x) - (\beta - \alpha)(\phi(x+t) - \phi(x)) \\ &= \psi(x+t) - \lambda(x+t) - (\psi(x) - \lambda(x)) \\ &\quad - (\beta - \alpha)(\phi(x+t) - \phi(x)) \\ &= [\psi(x+t) - \psi(x) - \beta(\phi(x+t) - \phi(x))] \\ &\quad - [\lambda(x+t) - \lambda(x) - \alpha(\phi(x+t) - \phi(x))]. \end{aligned}$$

Therefore, $U(x,t) = S(x,t) - T(x,t)$, and so $[U(x,t)]_- \leq [S(x,t)]_- + [T(x,t)]_+$. By Minkowski's inequality, we have

$$\int_0^h [u(x+t) - u(x) - (\beta - \alpha)(\phi(x+t) - \phi(x))]_-^r \, dt = o(h^{r+1}).$$

So $D_{+,r}u(x;\phi) \geq (\beta - \alpha) \geq -\epsilon$. Since ϵ is arbitrary, we have $D_{+,r}u(x;\phi) \geq 0$. The proof that $D_{-,r}u(x;\phi) \geq 0$ is similar, so we have $\underline{D}_r u(x,\phi) \geq 0$. Since $u(x)$ is L^r-continuous, our conclusion now follows from Theorem 15. □

Definition 23. *Suppose $f(x)$ is a function defined on $[a,b]$ and ϕ is a monotone increasing Lipschitz function also defined on $[a,b]$. If $\inf \psi(b)$ taken over all $L^{r,\phi}$-ex-major functions of f equals $\sup \lambda(b)$ taken over all $L^{r,\phi}$-ex-minor functions of f, then the common value, denoted by*

$$(P_{r,\phi}) \int_a^b f,$$

is called the $P_{r,\phi}$-integral of f on $[a,b]$, and f is said to be $P_{r,\phi}$-integrable on $[a,b]$.

If ϕ is a Lipschitz function defined on $[a,b]$, then it is of bounded variation. We can find monotone increasing Lipschitz functions ϕ_1 and ϕ_2 so that for every $x \in [a,b]$, we have

$$\phi(x) = \phi_1(x) - \phi_2(x).$$

Of course the functions ϕ_1 and ϕ_2 are not unique. However, we have the following theorem.

Theorem 24. *Let ϕ be a Lipschitz function defined on $[a,b]$, and let ϕ_1, ϕ_2, γ_1 and γ_2 be monotone increasing Lipschitz functions so that $\phi(x) = \phi_1(x) - \phi_2(x) = \gamma_1(x) - \gamma_2(x)$ for all $x \in [a,b]$. Suppose that f is P_{r,ϕ_1}-, P_{r,ϕ_2}-, P_{r,γ_1}- and P_{r,γ_2}-integrable on $[a,b]$. Then*

$$(P_{r,\phi_1}) \int_a^b f - (P_{r,\phi_2}) \int_a^b f = (P_{r,\gamma_1}) \int_a^b f - (P_{r,\gamma_2}) \int_a^b f.$$

We first prove the following lemma.

Lemma 25. *Let ϕ_1 and ϕ_2 be monotone increasing Lipschitz functions defined on $[a,b]$ with $\phi = \phi_1 + \phi_2$, and let f be any function defined on $[a,b]$. Suppose ψ_1 is an L^{r,ϕ_1}-ex-major (L^{r,ϕ_1}-ex-minor) function of f and ψ_2 is an L^{r,ϕ_2}-ex-major (L^{r,ϕ_2}-ex-minor) function of f, and let $\psi = \psi_1 + \psi_2$. Then ψ is an $L^{r,\phi}$-ex-major ($L^{r,\phi}$-ex-minor) function of f.*

Proof. We prove the lemma for $L^{r,\phi}$-ex-major functions; the proof for $L^{r,\phi}$-ex-minor functions is similar. Conditions 1 and 2 of the definition of the $L^{r,\phi}$-ex-major function are clearly satisfied by ψ. To prove that condition 3 holds, let us denote by E the set of those $x \in [a,b]$ satisfying

$$-\infty \neq \underline{D}_r \psi_1(x;\phi_1) \geq f(x)$$

and

$$-\infty \neq \underline{D}_r \psi_2(x;\phi_2) \geq f(x).$$

We have that $[a,b] \setminus E$ is countable. Let $x \in E$, and let α be such that $-\infty \neq \alpha < \min\left(\underline{D}_r \psi_1(x;\phi_1), \underline{D}_r \psi_2(x;\phi_2)\right)$. Then

$$\begin{aligned}
&\left(\frac{1}{h}\int_0^h \left[\psi(x+t) - \psi(x) - \alpha\left(\phi(x+t) - \phi(x)\right)\right]_-^r dt\right)^{\frac{1}{r}} \\
&= \left(\frac{1}{h}\int_0^h [\psi_1(x+t) + \psi_2(x+t) - \psi_1(x) - \psi_2(x) \right. \\
&\qquad \left. - \alpha(\phi_1(x+t) + \phi_2(x+t) - \phi_1(x) - \phi_2(x))]_-^r \right)^{\frac{1}{r}} \\
&= \left(\frac{1}{h}\int_0^h [\psi_1(x+t) - \psi_1(x) - \alpha(\phi_1(x+t) - \phi_1(x)) \right. \\
&\qquad \left. + \psi_2(x+t) - \psi_2(x) - \alpha(\phi_2(x+t) - \phi_2(x))]_-^r \right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{h}\int_0^h \left[\psi_1(x+t) - \psi_1(x) - \alpha\left(\phi_1(x+t) - \phi_1(x)\right)\right]_-^r dt\right)^{\frac{1}{r}} \\
&\qquad + \left(\frac{1}{h}\int_0^h \left[\psi_2(x+t) - \psi_2(x) - \alpha\left(\phi_2(x+t) - \phi_2(x)\right)\right]_-^r dt\right)^{\frac{1}{r}}.
\end{aligned}$$

Since both terms on the right side are equal to $o(h)$, we have

$$\left(\frac{1}{h}\int_0^h [\psi(x+t)-\psi(x)-\alpha(\phi(x+t)-\phi(x)]_-^r\,dt\right)^{\frac{1}{r}} \leq o(h).$$

This means that $-\infty \neq \underline{D}_r\psi(x;\phi)$.

Now we show that $\underline{D}_r\psi(x;\phi) \geq f(x)$. If $f(x) = -\infty$, we are done.

But if $f(x) = \infty$, then $P_{\psi_1,\phi_1}(\alpha)$ and $P_{\psi_2,\phi_2}(\alpha)$ hold for all real numbers. So we have $\underline{D}_r\psi(x;\phi) = \infty$ for all real numbers.

Finally, we assume $f(x)$ is finite. Then $P_{\psi_1,\phi_1}(\alpha)$ holds and $P_{\psi_2,\phi_2}(\alpha)$ holds, so that $P_{\psi,\phi}(\alpha)$ holds.

Therefore, $-\infty \neq \underline{D}_r\psi(x;\phi) \geq f(x)$.

□

Lemma 26. *Let ϕ_1 and ϕ_2 be monotone increasing Lipschitz functions defined on $[a,b]$ with $\phi = \phi_1+\phi_2$, and let f be both P_{r,ϕ_1}-integrable and P_{r,ϕ_2}-integrable on $[a,b]$. Then f is $P_{r,\phi}$-integrable on $[a,b]$ and*

$$(P_{r,\phi})\int_a^b f = (P_{r,\phi_1})\int_a^b f + (P_{r,\phi_2})\int_a^b f. \tag{13}$$

Proof. Let $\varepsilon > 0$. For $i \in \{1,2\}$, let ψ_i be an L^{r,ϕ_i}-ex-major function of f on $[a,b]$, and let λ_i be an L^{r,ϕ_i}-ex-minor function of f on $[a,b]$ so that $\psi_i(b) - \lambda_i(b) < \varepsilon/4$. Let $\psi = \psi_1 + \psi_2$ and let $\lambda = \lambda_1 + \lambda_2$. By the lemma above, we have that ψ is an $L^{r,\phi}$-ex-major function of f on $[a,b]$ and that λ is an $L^{r,\phi}$-ex-minor function of f on $[a,b]$ with $\psi(b) - \lambda(b) < \varepsilon/2$. Thus, f is $P_{r,\phi}$-integrable on $[a,b]$. We also have that

$$\begin{aligned}
&\left|(P_{r,\phi})\int_a^b f - \left((P_{r,\phi_1})\int_a^b f + (P_{r,\phi_2})\int_a^b f\right)\right| \\
&\leq \left|\psi(b) - (P_{r,\phi})\int_a^b f\right| + \left|\psi_1(b) - (P_{r,\phi_1})\int_a^b f\right| + \left|\psi_2(b) - (P_{r,\phi_2})\int_a^b f\right| \\
&< \varepsilon,
\end{aligned}$$

so that (13) holds. □

Proof of Theorem 24. By Lemma 26, f is $P_{r,\phi_1+\gamma_2}$-integrable and $P_{r,\gamma_1+\phi_2}$-integrable on $[a,b]$ with

$$(P_{r,\phi_1+\gamma_2})\int_a^b f = (P_{r,\gamma_1+\phi_2})\int_a^b f$$

and

$$(P_{r,\phi_1})\int_a^b f + (P_{r,\gamma_2})\int_a^b f = (P_{r,\gamma_1})\int_a^b f + (P_{r,\phi_2})\int_a^b f.$$

We now define the P_r-integral with respect to an arbitrary Lipschitz function.

Definition 27. *Suppose $f(x)$ is a function defined on $[a,b]$ and ϕ is a Lipschitz function also defined on $[a,b]$. Let ϕ_1 and ϕ_2 be monotone increasing Lipschitz functions such that $\phi = \phi_1 - \phi_2$. If f is P_{r,ϕ_1}-integrable and P_{r,ϕ_2}-integrable on $[a,b]$, then f is $P_{r,\phi}$-integrable on $[a,b]$ and we get*

$$(P_{r,\phi})\int_a^b f = (P_{r,\phi_1})\int_a^b f - (P_{r,\phi_2})\int_a^b f.$$

This value is well-defined by Theorem 24.

References

[1] A. P. Calderon and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math., **20** (1961), 171–225.

[2] L. Gordon, *Perron's integral for derivatives in L^r*, Studia Math., **28** (1966), 295–316.

[3] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Grad. Stud. Math., **4**, Amer. Math. Soc., 1994.

[4] P. Musial and Y. Sagher, *The L^r Henstock-Kurzweil integral*, Studia. Math., **160(1)** (2004), 53–81.

[5] H. L. Royden and P. M. Fitzpatrick, *Real Analysis*, 4th ed., Pearson, 2010.

[6] S. Saks, *Theory of the Integral*, 2nd English ed., Warszawa, 1937, Reprint, Dover, New York, 1964.

[7] S. A. Tikare and M. S. Chaudhary, *The Henstock-Stieltjes integral in L^r*, J. Adv. Res. Pure Math., **4(1)** (2012), 59–80.

[8] R. Wheeden and A. Zygmund, *Measure and Integral*, Pure and Applied Mathematics. Vol. 43, Marcel Dekker, Inc., New York-Basel, 1977.