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THE HAUSDORFF DIMENSION OF THE GENERALIZED LEVEL SETS OF TAKAGI'S FUNCTION

Abstract

In this note we prove that 1/2 is an upper bound for the Hausdorff dimension of the intersection of the graph of Takagi's function with any line of integer slope.

1 Introduction

The existence of continuous nowhere differentiable functions was an open problem during a part of the 19th century until 1872 when Weierstrass gave the first example. Three decades afterwards, in 1903, Takagi proposed one of the simplest examples of a continuous nowhere differentiable function as follows:

$$T(x) = \sum_{k=0}^{+\infty} \frac{d\left(2^k x\right)}{2^k}, \text{ for all } x \in [0,1],$$

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where $d(t)$ is the distance from t to the nearest integer. For properties and a wide list of references on T we refer the reader to [1, 4]. In the sequel $\tau := \{(x, T(x)) : 0 \leq x \leq 1\}$ will denote the graph of T . In the last years, the level sets and the generalized level sets of T, respectively $L_y := \{x : T(x) = y, 0 \leq x \leq 1\}$, and $L_{m,r} := \{(x, y) \in \tau : y = mx + r\}$, with $m \in \mathbb{Z}$ and $r \in \mathbb{R}$ have been studied; and the following result has been recently proved.

Theorem 1 ([2, Th. 3.4]). *The Hausdorff and upper box-counting dimensions of the level sets L_y of Takagi's function are, at most, 1/2.*

This paper is a sequel to [2], which approaches the study of the Hausdorff dimension of the sets $L_{m,r}$ of the Takagi function via its self-affinity. We show that Theorem 1 in [2] remains true for $L_{m,r}$.

Let us consider the following subsets of $L_{m,r}$:

$$\begin{aligned} L^1_{m,r} &: = \left\{(x, y) \in \tau : 0 \leq x \leq \frac{1}{2}, y = mx + r\right\}, \\ L^2_{m,r} &: = \left\{(x, y) \in \tau : \frac{1}{2} \leq x \leq 1, y = mx + r\right\}, \\ L^3_{m,r} &: = \left\{(x, y) \in \tau : \frac{1}{4} \leq x \leq \frac{1}{2}, y = mx + r\right\}, \end{aligned}$$

2 The result

In order to prove our result, we will use the self-similarity property of the graph of Takagi function together with the following auxiliary results.

Lemma 2. *Let T be the Takagi function. Let us consider the maps $A(x, y) = (2x, 2(y - x))$ and $B(x, y) = (4x - 1, 4y - 2)$. Then we have:*

i. *A maps bijectively $\left\{(x, T(x)) : 0 \leq x \leq \frac{1}{2}\right\}$ onto τ.*

ii. *A maps the straight line $y = mx + r$ to $y = (m - 1)x + 2r$.*

iii. *A is bi-Lipschitz.*

iv. *B^{-1} is a similarity that maps τ onto $\left\{(x, T(x)) : \frac{1}{4} \leq x \leq \frac{1}{2}\right\}$.*

v. *B^{-1} maps the straight line $y = mx + r$ to $y = mx + \frac{r-m+2}{4}$.*

Theorem 3. *The Hausdorff and upper box-counting dimensions of $L_{m,r}$ are, at most, 1/2.*

Proof. We proceed by induction on m. By Theorem 1 the result holds for $m = 0$. Let us suppose that $\dim_{\mathrm{H}}(L_{m-1,r}) \leq 1/2$ for a positive integer m and any real r. Because A is bi-Lipschitz (see Corollary 2.4 in [3]),

$$\dim_{\mathrm{H}}\left(L^1_{m,r}\right) = \dim_{\mathrm{H}}(L_{m-1,2r}) \leq 1/2.$$

If $L^2_{m,r} \neq \emptyset$, then by arguments of self-similarity, applying B^{-1} to τ and to the straight line $y = mx + r$:

$$\dim_{\mathrm{H}}\left(L^2_{m,r}\right) = \dim_{\mathrm{H}}\left(L^3_{m,\frac{r-m+2}{4}}\right) \leq \dim_{\mathrm{H}}\left(L^1_{m,\frac{r-m+2}{4}}\right) \leq 1/2.$$

Since $\dim_{\mathrm{H}}\left(L^1_{m,r}\right) \leq 1/2$ and $\dim_{\mathrm{H}}\left(L^2_{m,r}\right) \leq 1/2$, we conclude that

$$\dim_{\mathrm{H}}(L_{m,r}) \leq 1/2.$$

By symmetry of T with respect to $1/2$, the result follows for negative integers m as well. The same argument applies to the upper box-counting dimension. □

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