INROADS

Real Analysis Exchange
Vol. 37(2), 2011/2012, pp. 477–488

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THE HAKE'S THEOREM AND VARIATIONAL MEASURES

Abstract

We give a characterization of the Henstock-Kurzweil integral on $\mathbb{R}^m$ in terms of variational measures. As an application of this we prove a generalization of the Hake's theorem to $\mathbb{R}^m$.

1 Introduction

The Hake's theorem on $\mathbb{R}$ asserts that in some sense there are no improper Henstock-Kurzweil integrable functions, [4, Theorem 9.21]. More precisely,

Theorem 1.1 (Hake)**.** *A function $f : [0,1] \to \mathbb{R}$ is Henstock-Kurzweil integrable if and only if f is Henstock-Kurzweil integrable over each subinterval $[c,1]$ with $0 < c < 1$ and the following limit exists*

$$\lim_{c \to 0} \int_c^1 f.$$

Some extensions of this theorem have been obtained by Muldowney and Skvortsov in [9] and Faure in [2]. But both of these use an abstract concept of integral convergence over a suitable increasing sequence of figures.

We prove a measure theoretic extension of this theorem on finite dimensional Euclidean spaces using the variational measure W_F, introduced by Schwabik in [13]. This generalizes the following result [13, Theorem 3.11]:

Mathematical Reviews subject classification: Primary: 26A39
Key words: Henstock-Kurzweil Integral, variational measure
Received by the editors October 8, 2011
Communicated by: Luisa Di Piazza
*The research for this paper was supported by Council of Scientific and Industrial Research, India.

Theorem 1.2 (Schwabik)**.** *Let A be a closed subset of $[a,b] \subset \mathbb{R}^1$ and denote by $Comp([a,b],A)$ the family of all non-empty connected components of the set $[a,b] \setminus A$. Let f and F be two real valued functions on $[a,b]$ satisfying*
i) $f.\chi_A$ is Henstock-Kurzweil integrable,
ii) F is continuous on E,
iii) For every interval $[c,d] \subset U \in Comp([a,b],A)$, the function $f.\chi_[c,d]$ is Henstock-Kurzweil integrable with $\int_c^d f = F(d) - F(c)$.

Then f is Henstock-Kurzweil integrable over the interval $[a,b]$ if and only if $W_F(A) = 0$.

Schwabik proved it using Theorem 1.1. The extensions given in [2] and [9] can't be used in that fashion, to prove a similar result for functions on R^m.

We first prove a measure theoretic characterization of the Henstock-Kurzweil integral on $\mathbb{R}^m$. Using that characterization we generalize Theorem 1.2 on $\mathbb{R}^m$ and observe the redundancy of the continuity hypothesis. That may also be considered as a generalization of the Hake's theorem on $\mathbb{R}^m$.

2 Preliminaries

Let $m \geq 1$ be any integer, $\mathbb{R}^m$ be the m-dimensional Euclidean space and μ be the Lebesgue measure on $\mathbb{R}^m$. Let I be a compact interval in $\mathbb{R}^m$ and Ω be the σ-algebra of Lebesgue measurable sets in I. For an interval $J \subset I$, let $Sub(J)$ and $\mathcal{F}(J)$ denote the family of compact subintervals of J and the algebra generated by $Sub(J)$, respectively and let $\mathcal{F} = \mathcal{F}(I)$.

Let d be any given metric on $\mathbb{R}^m$ and $B(x,r)$ denotes the open ball in $(\mathbb{R}^m, d)$ with centre x and radius r, for $x \in \mathbb{R}^m$ and $r > 0$. By a figure E in $\mathbb{R}^m$, we mean a finite union of compact intervals in $\mathbb{R}^m$. The Henstock-Kurzweil integrability of a function $f : I \to \mathbb{R}$ is defined as follows:

Definition 2.1. (i) A collection $\{(t_i, I_i) : i = 1, 2, \ldots, p\}$ of point-interval pairs is said to be a *partial division* in I if I_i's are nonoverlapping intervals in I and $t_i \in I_i$, for each i. If further, $\cup_{i=1}^p I_i = I$, it is called a *division* of I.

(ii) Given a positive valued function $\delta : I \to (0, \infty)$, a partial division $\{(t_i, I_i) : i = 1, 2, \ldots, p\}$ in I is said to be *δ-fine* if $I_i \subset B(t_i, \delta(t_i))$ for each i.

(iii) A function $f : I \to \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* (or simply HK-integrable), with $A \in \mathbb{R}$ as its integral, if for every $\epsilon > 0$

there is a function $\delta : I \to (0,\infty)$ such that the inequality

$$\left|\sum_{i=1}^{p} f(t_i)\mu(I_i) - A\right| < \epsilon$$

is satisfied for all δ-fine divisions $\{(t_i, I_i) : i = 1, \ldots, p\}$ of I.

The Henstock-Kurzweil integral of f over I is denoted by $(HK)\int_I f d\mu$. A function $F : \mathcal{F} \to \mathbb{R}$ is called the *primitive* of f if $F(J) = (HK)\int_J f d\mu$, for each $J \in \mathcal{F}$.

We shall use the following Saks-Henstock Lemma [12, Lemma 3.4.1.], in our proofs.

Lemma 2.2 (Saks-Henstock)**.** *Let $f : I \to \mathbb{R}$ be an HK-integrable function with primitive F. Then for every $\epsilon > 0$ there exists a function $\delta : I \to (0,\infty)$ satisfying*

$$\sum_{i=1}^{p} \big| f(t_i)\mu(J_i) - F(J_i)\big| \le \epsilon$$

for every δ-fine partial division $\{(t_i, J_i) : 1 \le i \le p\}$ of I.

The following proposition is an immediate consequence of the above lemma.

Proposition 2.1. Let $f : I \to \mathbb{R}$ be an HK-integrable function satisfying $(HK)\int_J f d\mu = 0$, for each $J \in Sub(I)$. Then $f = 0$ almost everywhere on I.

We now define the Henstock variational measure V_F and the oscillatory variational measure W_F as follows:

Definition 2.3. Let $F : \mathcal{F} \to \mathbb{R}$ be a finitely additive set function.

(i) For $J \in Sub(E)$, the oscillation of F at J, denoted by $w(F, J)$, is defined as $w(F, J) := sup\{|F(K)| : K \in Sub(J)\}$.

(ii) For $M \subset E$ and a function $\delta : M \to (0,\infty)$, we define

$$V(F, M, \delta) := sup_P \sum_{i=1}^{p} |F(I_i)| \text{ and } W(F, M, \delta) := sup_P \sum_{i=1}^{p} w(F, I_i),$$

where the supremum is taken over all δ-fine partial divisions $P = \{(t_i, I_i) : 1 \le i \le p\}$ in E, such that each $t_i \in M$.

(iii) The Henstock variational measure V_F and the oscillatory variational measure W_F on a subset M of E are defined as follows:

$$V_F(M) := inf_\delta V(F, M, \delta) \text{ and } W_F(M) := inf_\delta W(F, M, \delta)$$

where the infimum is taken over all the functions $\delta : M \to (0, \infty)$.

It can be easily seen that for a compact real interval M, both $V_F(M)$ and $W_F(M)$ are equal to the standard total variation of F over M, [13, Lemma 2.2]. Throught this paper, we adopt the following notations from [5].

Notations 2.4. *(i) For any* $\alpha = (\alpha_1, .., \alpha_m)$ *and* $\beta = (\beta_1, .., \beta_m) \in \mathbb{R}^m$*, define an interval* $[\alpha, \beta] := \{\gamma = (\gamma_1, ..., \gamma_m) : \alpha_i \le \gamma_i \le \beta_i$ *for each* $1 \le i \le m\}$*. In this sense, we write* $I = [a, b]$*, for some* $a, b \in \mathbb{R}^m$*.*

(ii) For a given set function $F : \mathcal{F} \to \mathbb{R}$*, define the corresponding point function* $F_0 : I \to \mathbb{R}$ *as,* $F_0(t) := F([a, t])$ *for all* $t \in I = [a, b]$*.*

(iii) For a given point function $F_0 : I \to \mathbb{R}$*, define the corresponding set function* $F_1 : \mathcal{F} \to \mathbb{R}$ *as follows:*
For $J = [\alpha, \beta] \in Sub(I)$*, we set* $F_1(J) = \sum_\gamma (-1)^{n(\gamma)} F_0(\gamma)$ *where the summation is taken over all vertices* γ *of the interval* J *in* $\mathbb{R}^m$*, and for* $\gamma = (\gamma_1, .., \gamma_m)$ *where* $\gamma_i = \alpha_i$ *or* β_i *for each* $1 \le i \le m$*,* $n(\gamma)$ *is the cardinality of the set* $\{i : \gamma_i = \alpha_i\}$*. Then we extend* F_1 *to* $\mathcal{F}$*, naturally.*

Note that if we start with a finitely additive set function $F : \mathcal{F} \to \mathbb{R}$ then the set function F_1, corresponding to the F_0 function for F, will be identical to the function F itself.

There are several definitions of continuity of additive interval functions. The above notations are aimed at avoiding confusion over that. We shall deal with the continuity of point functions only.

We say that V_F is absolutely continuous with respect to μ, or simply absolutely continuous when there is no ambiguity, if $V_F(N) = 0$, for every $N \subset [0, 1]$ with $\mu(N) = 0$. In that case we write $V_F \ll \mu$. Similarly, we define $W_F \ll \mu$.

Remarks 2.5. (i) In [13], W_F is defined for $m = 1$ and only for the functions F for which the corresponding point function F_0 is continuous. However, we don't assume any continuity hypothesis and shall extend most of the results of [13] in next sections.

(ii) It follows from [3, Proposition 3.3] that V_F is a metric outer measure. The similar arguments prove that W_F too is a metric outer measure. Further, an application of [1, Theorem 3.7] shows that both V_F and W_F are Borel measures.

(iii) It was proved in [6, Theorem 3.7] that if V_F is absolutely continuous then V_F is a measure on Ω. The same result holds true for W_F too.

3 The interdependence of V_F and W_F

In [13], Schwabik had defined V_F and W_F for the case when $m = 1$ and proved their interdependence (see Corollary 2.4, [13]). We here generalize that for any m.

Lemma 3.1. *Let $F_0 : I \to \mathbb{R}$ be a continuous point function, $J \in Sub(I)$ and $t_0 \in J$. Then there exists $J_0 \in Sub(J)$ such that $w(F_0, J) \leq 2|F_1(J_0)|$ and $t_0 \in J_0$.*

PROOF. Write $J = [\alpha, \beta]$; $\alpha, \beta \in \mathbb{R}^m$ and consider a set $A \subset \mathbb{R}^{2m}$, defined as $A := \{(a_1, ..., a_{2m}) : \alpha_i \leq a_i \leq a_{i+m} \leq \beta_i$ for every $1 \leq i \leq m\}$. Now define maps $\phi : A \to \mathbb{R}^{2^m}$ and $\psi : \phi(A) \to \mathbb{R}$ as follows:

For $a \in A$, define $\phi(a) := (F_0(b_1), ..., F_0(b_{2^m}))$, where $b_1, ..., b_{2^m}$ are vertices of the compact interval $[(a_1, ..., a_m), (a_{m+1}, ..., a_{2m})]$ in $\mathbb{R}^m$, arranged in the standard lexicographical order $b_1 \leq b_2 \leq ... \leq b_{2^m}$. We now set

$$\psi(F_0(b_1), ..., F_0(b_{2^m})) = \sum_{1 \leq i \leq 2^m} (-1)^{n(b_i)} F_0(b_i).$$

Since F_0 is continuous, the composite map $\psi o \phi : A \to \mathbb{R}$ is continuous. As $A \subset \mathbb{R}^{2^m}$ is compact, there exists some $a_0 \in A$ where the supremum of $\psi o \phi$ is attained. Note that we have $\psi o \phi(a) = F_1[(a_1, .., a_m), (a_{m+1}, .., a_{2m})]$. Thus if we write $a_0 := (a_1^0, .., a_{2m}^0)$ and $J' := [(a_1^0, .., a_m^0), (a_{m+1}^0, .., a_{2m}^0)]$, we will have $J' \in Sub(J)$ and $w(F_1, J) = |F(J')|$.

Now, if $t_0 \in J'$, we may take $J_0 := J'$. Otherwise choose two subintervals J_1 and J_2 of J such that $J' = J_1 \setminus J_2$ and both J_1 and J_2 contain t_0. Then

$$w(F_1, J) = |F(J')| = |F_1(J_1) - F_1(J_2)| \leq |F_1(J_1)| + |F_1(J_2)|.$$

Now if $|F_1(J_1)| \geq |F_1(J_2)|$ then put $J_0 = J_1$, otherwise take $J_0 = J_2$. This proves the result. □

The above lemma immediately gives us the following.

Proposition 3.1. Let $F_0 : I \to \mathbb{R}$ and $M \subset I$. Then we have

(i) $V_{F_1}(M) \leq W_{F_1}(M)$.

(ii) If F_0 is continuous on I, then $W_{F_1}(M) \leq 2V_{F_1}(M)$.

Note that the first inequality follows directly. Next we present some characterizations of the Henstock-Kurzweil integral.

4 Variational measures and the HK-integral

The following measure theoretic characterization of the Henstock-Kurzweil integral over finite dimensional Euclidean spaces is proved in [6].

Theorem 4.1. *Let $F : \mathcal{F} \to \mathbb{R}$ be a finitely additive set function. Then V_F is absolutely continuous with respect to μ if and only if there exists an Henstock-Kurzweil integrable function $f : I \to \mathbb{R}$ with primitive F.*

We shall prove a similar result for W_F. This will generalize [13, Theorem 3.4] on $\mathbb{R}^m$, proving the redundancy of the continuity hypothesis therein. First we present the following extension of [4, Theorem 9.12] to finite dimensional Euclidean spaces. A proof to this theorem can also be found in [8, Theorem 2.4.10], but our proof below is more direct.

Theorem 4.2. *Let $f : I \to \mathbb{R}$ be a Henstock-Kurzweil integrable function with primitive F. Then the corresponding point function $F_0 : I \to \mathbb{R}$ is continuous.*

Proof. Let $t_0 \in I$ and let $\epsilon > 0$ be given. Choose a function $\delta : I \to (0, \infty)$ so that the conclusion of the Saks-Henstock lemma holds true for this ϵ, that is, the inequality

$$\sum_{i=1}^{p} |f(t_i)\mu(I_i) - F(I_i)| \leq 2\epsilon$$

is satisfied for any $\delta-$fine partial division $\{(t_i, I_i) : 1 \leq i \leq p\}$ of I. We define another function $\delta_0 : I \to (0, \infty)$ as follows:

$$\delta_0(t) = \begin{cases} \min\{\delta(t), \frac{1}{2} \parallel t - t_0 \parallel\} \text{ if } t \neq t_0 \\ \min\{\delta(t_0), \frac{1}{2}(\frac{\epsilon}{|f(t_0)|+1})^{\frac{1}{m}}\} \text{ if } t = t_0 \end{cases}$$

Now for any δ_0-fine interval-point pair $\{(t_0, I_0)\}$ such that $t_0 \in I_0$, we must have $I_0 \subset B(t_0, \delta_0(t_0)) \subset B(t_0, \delta(t_0))$. So, by our choice of δ, we have

$$|f(t_0)\mu(I_0) - F(I_0)| \leq 2\epsilon.$$

Also note that if J_0 is the cube centred at t_0 with each of its sides equal to $2\delta_0(t_0)$, then

$$\mu(I_0) \leq \mu(B(t_o, \delta_0(t_0))) \leq \mu(J_0) < \left[\left(\frac{\epsilon}{|f(t_0)| + 1}\right)^{\frac{1}{m}}\right]^m = \frac{\epsilon}{|f(t_0)| + 1}.$$

In that case we have

$$|F(I_0)| \leq |F(I_0) - f(t_0)\mu(I_0)| + |f(t_0)|\mu(I_0) < 2\epsilon + |f(t_0)|\frac{\epsilon}{|f(t_0)| + 1} < 3\epsilon.$$

Now we choose $\eta > 0$ so small that for any $t \in B(t_0, \eta)$, $\mu([0,t]\triangle[0,t_0]) < \delta_0(t_0)$, where $\triangle$ dentes the usual symmetric difference between the sets. Now for any $t \in B(t_0, \eta)$, we claim that $|F_0(t) - F_0(t_0)| = |F([a,t]) - F([a,t_0])| < n(m)\epsilon$, where $n(m)$ is a fixed number depending only on m.

We illustrate the argument for the case when $m = 2$. Write $t_0 = (t_1^0, t_2^0)$ and $t = (t_1, t_2)$ and consider two cases: case(i): $t_1^0 \leq t_1$ and $t_2^0 \leq t_2$, and case(ii): $t_1^0 \leq t_1$ and $t_2^0 > t_2$. The other two cases can be handled similarly.

In case(i), we write, $|F([a,t]) - F([a,t_0])| = |F(I_1 \cup I_2)|$, where $I_1 := [(a_1, t_2^0), t]$ and $I_2 := [(t_1^0, a_2), t]$. Note that since $t_0 \in I_1 \cap I_2$ and $\mu(I_1 \cup I_2) = \mu([0,t]\triangle[0,t_0]) < \delta_0(t_0)$, we have

$$|F_0(t) - F_0(t_0)| = |F([a,t]) - F([a,t_0])| = |F(I_1 \cup I_2)| =$$
$$|F(I_1)+F(I_2)-F(I_1\cap I_2)| \leq |F(I_1)|+|F(I_2)|+|F(I_1\cap I_2)| < 3\epsilon+3\epsilon+3\epsilon = 9\epsilon.$$

In case(ii) we write, $F([a,t]) - F([a,t_0]) = F(J_1 \setminus J_2) - F(J_3 \setminus J_2)$, where $J_1 := [(t_1^0, a_2), (t_1, t_2^0)]$, $J_2 := [(t_1^0, t_2), (t_1, t_2^0)]$ and $J_3 := [(a_1, t_2), (t_1, t_2^0)]$. Also note that since t_0 lies in each of these J_i's and because of our choice of t, $\mu(J_i) < \delta_0(t_0)$, for each i. Therefore we have

$$|F_0(t) - F_0(t_0)| = |F([a,t]) - F([a,t_0])| \leq |F(J_1 \setminus J_2)| + |F(J_3 \setminus J_2)| \leq$$
$$|F(J_1)| + |F(J_1 \cap J_2)| + |F(J_3)| + |F(J_3 \cap J_2)| < 3\epsilon + 3\epsilon + 3\epsilon + 3\epsilon = 12\epsilon.$$

Thus for $m = 2$ we have established our claim with $n(m) = 12$. The similar arguments would apply for any m. We only need to write $|F([a,t])-F([a,t_0])| \leq \sum_K |F(K)|$ for some finitely many $K \in Sub(I)$ (the maximum number of such K's would depend only on m) such that each such K contains t_0 and $\mu(K) < \delta_0(t_0)$.

This proves continuity of F_0 at t_0. Thence F_0 is continuous on I.

$\square$

Theorem 4.3. *Let $F : \mathcal{F} \to \mathbb{R}$. The following are equivalent:*
(i) W_F is absolutely continuous.
(ii) V_F is absolutely continuous.

PROOF. First assume that W_F is absolutely continuous with respect to μ and take any $M \subset I$ such that $\mu(M) = 0$. Then $W_F(M) = 0$ which implies $V_F(M) = 0$, as $0 \leq V_F(M) \leq W_F(M)$ is always true.

For the converse, assume that V_F is absolutely continuous with respect to μ. Using Theorem 4.1, there exists an HK- integrable function $f : I \to \mathbb{R}$ such that $F(J) = (\mathcal{HK}) \int_J f d\mu$ for $J \in \mathcal{F}$. Now by Theorem 4.2, the corresponding F_0 is a continuous function on I. Further, using Proposition 3.1 we have $V_F(M) \leq W_F(M) \leq 2V_F(M)$ for any $M \subset I$. Then, as above, W_F is absolutely continuous with respect to μ. $\square$

The strong Lusin condition (or SL-condition), which was first considered by Pfeffer in [11], is defined by Schwabik in [13] as follows:

Definition 4.4. A function $F : [a, b] \to \mathbb{R}$ is said to satisfy the SL-condition (strong Lusin condition) if F is continuous and $W_{F_1} \ll \mu$.

As in the previous proof we have seen that $W_F \ll \mu$ implies that the corresponding function F_0 is continuous. So even this continuity hypothesis is not required to define the SL-condition. Thence we present the following characterizations of the HK-integral on R^m.

Theorem 4.5. *Let $F : \mathcal{F} \to \mathbb{R}$ be finitely additive. The following are equivalent:*

(i) F satisfies the SL-condition.

(ii) V_F is absolutely continuous with respect to μ.

(iii) W_F is absolutely continuous with respect to μ.

(iv) There exists an HK-integrable function $f : I \to \mathbb{R}$ with primitive F.

5 The Hake's theorem and variational measures

In this section we generalize the Hake's Theorem for functions on finite dimensional Euclidean spaces. We use Theorem 4.5 in our proofs. First we restate the Hake's theorem as follows:

Theorem 5.1. *Let f and F be real valued functions over $[0, 1]$ such that for each interval $[c, 1]$ with $0 < c < 1$, f is HK-integrable over $[c, 1]$ with $(\mathcal{HK}) \int_c^1 f = F(1) - F(c)$.*

Then f is HK-integrable over $[0, 1]$ if and only if F is continuous at 0. Moreover, in that case we have, $\int_0^1 f = F(1) - F(0)$.

We extend the above version of the Hake's theorem over $\mathbb{R}^m$. That would generalize Theorem 1.2 over $\mathbb{R}^m$ and also show the redundancy of the continuity hypothesis. We first prove a special case, which would be used to prove the main result.

Let E be a compact figure in $\mathbb{R}^m$, $f : E \to \mathbb{R}$ and $\mathcal{F}(E)$ denote the algebra generated by subintervals of E. Let $F : \mathcal{F}(E) \to \mathbb{R}$ be a finitely additive set function.

Theorem 5.2. *Let $I = [a, b] \subset E$ be a compact interval such that for every $J \in Sub(I)$ satisfying $J \cap \partial I = \emptyset$, f is HK-integrable over J with $(\mathcal{HK}) \int_J f d\mu = F(J)$.*

Then f is HK-integrable over I if and only if $W_F(\partial I) = 0$. Moreover, in that case we have, $(\mathcal{HK}) \int_I f d\mu = F(I)$.

PROOF. If f is HK-integrable over I then by Theorem 4.5, W_F is absolutely continuous with respect to μ and thus $W_F(\partial I) = 0$.

For the converse, assume that $W_F(\partial I) = 0$. We write $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_m)$ and choose an increasing sequence of sets $\{A_n\}$ such that for each $n \in \mathbb{N}$,

$$A_n = \left[\left(a_1 + \frac{b_1 - a_1}{n+1}, \dots, a_m + \frac{b_m - a_m}{n+1} \right), \left(b_1 - \frac{b_1 - a_1}{n+1}, \dots, b_m - \frac{b_m - a_m}{n+1} \right) \right]$$

By our hypothesis, f is HK-integrable over each A_n. Using Theorem 4.5, on subsets of A_n, W_F is absolutely continuous with respect to μ. Now choose a subset N of I satisfying $\mu(N) = 0$ and write $N = (N \cap \partial I) \cup (\cup_{n=1}^{\infty} (N \cap A_n))$. Since W_F is an outer measure, we have

$$W_F(N) \leq W_F(N \cap \partial I) + \sum_{n=1}^{\infty} W_F(N \cap A_n) = 0$$

This proves that W_F is absolutely continuous with respect to μ on I and thence by Theorem 4.5 there exists an HK-integrable function $g : I \to \mathbb{R}$ such that $F(J) = (\mathcal{HK}) \int_J g d\mu$ for all $J \in \mathcal{F}$. Further an application of Proposition 2.1 proves this result.

□

In [13, Lemma 3.7] the following is presented for functions on real line. We observe that Lemma 3.1 ensures its validity even over finite dimensional Euclidean spaces.

Lemma 5.3. *Assume that $f : E \to \mathbb{R}$ is HK-integrable with $(HK) \int_J f = F(J)$, for every interval $J \subset I$. Then*

$$W_F(M) \leq 2.\mu(E).sup\{|f(t)| : t \in M\}$$

holds for every $M \subset E$.

Next we prove a more generalized version of Theorem 5.2.

Theorem 5.4. *Let $A \subset E$ be a closed set such that*
(a) f is HK-integrable over A.
(b) For each compact interval $J \subset E \setminus A$, f is HK-integrable over J with integral $F(J)$.

Then $W_F(A) = 0$ if and only if f is HK-integrable over E with

$$(\mathcal{HK}) \int_E f d\mu = F(E) + (\mathcal{HK}) \int_A f d\mu. \tag{1}$$

Proof. Let $\epsilon > 0$ be given. We write $E \setminus A = \cup_{n \in \mathbb{N}} U_n$, where $\{U_n\}$ is a collection of pairwise non-overlapping intervals in E. This is possible since the intervals $[(\alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m)]$ with rational α_i's and β_i's, form a countable base in $\mathbb{R}^m$.

We first consider the case when $f(t) = 0$, for all $t \in A \cup (\cup_n \partial U_n)$. Let $B = A \cup (\cup_n \partial U_n)$. If f is HK-integrable over E, Lemma 5.3 implies $W_F(A) = 0$.

For the converse, we assume that $W_F(A) = 0$. For any $n \in \mathbb{N}$, we write $\partial U_n = (\partial U_n \cap A) \cup (\partial U_n \cap (E \setminus A))$. We find a compact figure $J \subset (E \setminus A)$ such that $\partial U_n \cap (E \setminus A) \subset J$. Using our hypothesis, f is HK-integrable over J. Now by Theorem 4.5, we have $W_F \ll \mu$ on subsets of J and thence $W_F(\partial U_n \cap (E \setminus A)) = 0$. Since W_F is an outer measure we have $W_F(\partial U_n) \leq W_F(\partial U_n \cap A) + W_F(\partial U_n \cap (E \setminus A)) \leq W_F(A) = 0$. Further, we have $W_F(B) \leq W_F(A) + \sum_n W_F(\partial U_n) = 0$.

Now as in Theorem 5.2, by a repeated application of Theorem 4.5, we prove that f is HK-integrable over E with primitive F. Hence we have proved our result for the case when $f(t) = 0$ for all $t \in B$.

For the general case, we define a function $g : E \to \mathbb{R}$ as $g = f - f.\chi_B$, where χ_B denotes the characteristic function of the set B. Then $g(t) = 0$ for all $t \in B$ and $g(t) = f(t)$ for all $t \in E \setminus B$.

Note that for any compact interval $J \subset (E \setminus B) \subset (E \setminus A)$, since f is HK-integrable over J with integral $F(J)$ and $f(t) = g(t)$ for almost all $t \in J$, g is HK-integrable over J with integral $F(J)$.

Now as above, we have $W_F(A) = 0$ if and only if g is HK-integrable over E with $(\mathcal{HK}) \int_E g d\mu = F(E)$, that is, if and only if $f - f.\chi_A$ is HK-integrable over E with $(\mathcal{HK}) \int_E (f - f.\chi_A) d\mu = F(E)$.

Since f is given to be integrable over A we observe that $W_F(A) = 0$ if and only if f is HK-integrable over E with $(\mathcal{HK}) \int_E (f - f.\chi_A) d\mu = F(E)$, that is,

$$(\mathcal{HK}) \int_E f d\mu = F(E) + (\mathcal{HK}) \int_A f d\mu.$$

□

6 Concluding Remarks

Schwabik has presented a general concept of integral in [14], using a functional theoretic approach. He introduces a special class $\mathcal{I}$ of integrals and shows that the Henstock-Kurzweil integral belongs to it. Using Theorem 4.5, we observe that $\mathcal{I}$ is precisely the class of integrals weaker than the Henstock-Kurzweil integral.

We also remark that equation (1) in Theorem 5.4 may appear a bit unintuitive, as one would naturally expect $(\mathcal{HK})\int_E f d\mu = F(E)$ as the conclusion. This happens since we are not given any information about the relationship between f and F, on A. The set function F is given to be the primitive of f, only on the compact intervals inside $E \setminus A$.

We also observe that our main theorems, that is, Theorem 5.2 and Theorem 5.4 hold valid with W_F replaced by V_F. Using similar steps one can also prove the following version of the Hake-type theorem on R^m.

Theorem 6.1. *Let $\mathcal{F}_\infty$ denote the algebra generated by all subintervals of $\overline{\mathbb{R}}^m$ and $F : \mathcal{F}_\infty \to \mathbb{R}$ be a given finitely additive set function. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function such that for each compact interval $I \subset \mathbb{R}^m$, f is HK-integrable over I with integral $F(I)$.*

Then f is HK-integrable over $\mathbb{R}^m$ if and only if $V_F(\overline{\mathbb{R}}^m \setminus \mathbb{R}^m) = 0$. Moreover, in that case we have, $(\mathcal{HK})\int_{\mathbb{R}^m} f d\mu = F(\mathbb{R}^m)$.

In particular, it implies that the improper Riemann integrals over $\mathbb{R}^m$ belong to the class of Henstock-Kurzweil integrable functions over $\mathbb{R}^m$.

The above theorem can also be considered as another way to define the Henstock-Kurzweil integral of a function over multidimensional unbounded intervals.

Acknowledgment. The authors wish to thank the referees for their constructive critique of the first draft.

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