RESEARCH

Real Analysis Exchange
Vol. 21(1), 1995–96, pp. 147–153

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UNIFORMLY ANTISYMMETRIC FUNCTIONS AND K_5

Abstract

In [2, Thm 2.5] and [1, Thm 2] it was proved that there is no uniformly antisymmetric function with two- and three-element range by showing that K_3 and K_4 can be embedded into a graph $G(h)$ (defined below) for all appropriate h. In this note we will answer Problem 1 from [1] by showing that under the continuum hypothesis there exists h for which K_5 cannot be embedded into $G(h)$. In particular, the technique used in the proof that there is no uniformly antisymmetric function with three-element range cannot be used for the four-element range proof. Whether there exists a uniformly antisymmetric function with a finite range remains an open problem.

The notion of a uniformly anti-Schwartz function is also defined, and it is proved that there exists a uniformly anti-Schwartz function $f\colon \mathbb{R} \to \mathbb{N}$.

For $S \subset \mathbb{R}$ and $h\colon S \to (0, \infty]$ let

$$E(h) = \left\{ \{a,b\} \in [\mathbb{R}]^2 \colon \frac{a+b}{2} \in S \ \& \ \frac{|a-b|}{2} < h\left(\frac{a+b}{2}\right) \right\}.$$

The graph $G(h) = (\mathbb{R}, E(h))$ is an infinite graph with vertices $\mathbb{R}$ and edges $E(h)$. We will answer the problem from [1] by showing that under the continuum hypothesis there exists a function $h\colon \mathbb{R} \to (0,1)$ such that the graph $G(h)$ does not contain K_5, the complete graph on 5 vertices, as a subgraph.

Key Words: uniformly antisymmetric functions, uniformly anti-Schwartz functions,the Continuum Hypothesis, coloring of infinite graphs.

Mathematical Reviews subject classification: Primary 26A16; Secondary 03E50, 04A20.

Received by the editors July 26, 1994

*The results presented in this paper were in part obtained during the Joint US–Polish Workshop in Real Analysis, Łódź, Poland, July 1994. The Workshop was partially supported by the NSF grant INT–9401673.

The motivation for this question lies in the study the uniformly antisymmetric functions, i.e., functions $f\colon \mathbb{R}\to\mathbb{R}$ for which there exists $h\colon \mathbb{R}\to(0,1)$ such that

$$|f(x-h)-f(x+h)|\geq h(x)$$

for every $x\in\mathbb{R}$ and $0<h<h(x)$. (See [3], [2], [4] and [1].) It is known from [2] that there exists a uniformly antisymmetric function $f\colon\mathbb{R}\to\mathbb{N}$, while it is unknown whether such function can have a finite range [2, Problem 1(a)].

It is not difficult to show [1, Theorem 1] that there exists a uniformly antisymmetric function with n-element range if and only if there exists $h\colon\mathbb{R}\to(0,1)$ such that $G(h)$ is n-colorable. In [1] it was proved that the range of every uniformly antisymmetric function must have at least four elements by showing that $G(h)$ contains K_4 for every $h\colon\mathbb{R}\to(0,1)$. We will show here that this result cannot be extended to K_5.

Theorem 1 *Let M be a subfield of $\mathbb{R}$ of cardinality ω_1. Then there exists $h\colon M\to(0,1)$ such that $G(h)$ does not contain K_5.*

In particular, if the continuum hypothesis holds, then there exists $h\colon\mathbb{R}\to(0,1)$ such that $G(h)$ does not contain K_5.

Proof. Let $\{M_\xi\colon\xi<\omega_1\}$ be an increasing tower of countable subfields of M such that $M=\bigcup_{\xi<\omega_1}M_\xi$ and $M_\lambda=\bigcup_{\xi<\lambda}M_\xi$ for every limit ordinal $\lambda<\omega_1$. We will construct, by transfinite induction on $\xi<\omega_1$, an increasing sequence $\{h_\xi\colon M_\xi\to(0,1)\colon\xi<\omega_1\}$ such that the following inductive conditions are satisfied for every $\xi<\omega_1$:

(A_ξ) $G(h_\xi)$ does not contain K_5, i.e., $[A]^2\not\subset E(h_\xi)$ for every $A\in[\mathbb{R}]^5$;

(B_ξ) for every $\zeta\leq\xi$ and $B\in[M_\zeta]^3$ if

$$S_\xi(B)=\{B+t\in[\mathbb{R}]^3\colon [B+t]^2\subset E(h_\xi)\},$$

then $S_\xi(B)$ is a finite subset of $[M_\zeta]^3$.

The existence of such a sequence immediately implies Theorem 1 if we take $h=\bigcup_{\xi<\omega_1}h_\xi$.

To construct the sequence assume that $\{h_\xi\colon M_\xi\to(0,1)\colon\xi<\eta\}$ is already constructed for some $\eta<\omega_1$. If η is a limit ordinal, then it is easy to see that $h_\eta=\bigcup_{\xi<\eta}h_\xi$ satisfies the inductive hypothesis. So, assume that η is a successor ordinal, say $\eta=\xi+1$.

Let $\{x_n\colon n<\omega\}$ be an enumeration, without repetitions, of $M_{\xi+1}\setminus M_\xi$. Since $h_{\xi+1}$ is already defined on M_ξ, it is enough to define $h_{\xi+1}(x_n)$ for every $n<\omega$.

Before we define it, let us choose $g_\xi \colon M_{\xi+1} \to (0,1)$ such that $G(g_\xi)$ does not contain any odd cycle. Such a function exists since for every countable field $M \subset \mathbb{R}$ there exists a uniformly antisymmetric function $f \colon M \to \{0,1\}$. (See [4, Theorem 1].)

Now, choose $n < \omega$. For each $i < j < n$ consider the family S^n_{ij} of all sets $B = \{a,b,c\} \subset M_\xi$ such that $[B]^2 \subset E(h_\xi)$ and that there exists $d \in \mathbb{R}$ with $a + d = 2x_i$, $b + d = 2x_j$ and $c + d = 2x_n$. Notice that for every $B', B \in S^n_{ij}$ there exists t such that $B' = B + t$. Thus, by (B_ξ), the set S^n_{ij} is finite. In particular, the number

$$\varepsilon_n = \frac{1}{2} \min\{|p - x_n| > 0 \colon p \in B \ \text{ for some } B \in S^n_{ij} \text{ and } i < j < n\}$$

is well defined and positive. Define

$$h_{\xi+1}(x_n) = \min\left\{ g_\xi(x_n), \varepsilon_n, \frac{1}{n} \right\}.$$

To finish the proof it is enough to show that the conditions $(A_{\xi+1})$ and $(B_{\xi+1})$ are satisfied.

We will start with showing $(B_{\xi+1})$. So, choose $\zeta \le \xi+1$, let $B = \{a,b,c\} \in [M_\zeta]^3$ and let $x = (a+b)/2$, $y = (b+c)/2$ and $z = (a+c)/2$. Then

$$a = x - y + z, \quad b = x + y - z, \quad c = -x + y + z. \tag{1}$$

First notice that $S_{\xi+1}(B) \subset [M_{\xi+1}]^3$. This is the case since $B+t \in S_{\xi+1}(B)$ implies $t \in M_{\xi+1}$, as for every $t \in \mathbb{R} \setminus M_{\xi+1}$ we have $x+t = t+(a+b)/2 \notin M_{\xi+1}$ and so, $\{a+t, b+t\} \notin E(h_{\xi+1})$.

If $\zeta < \xi + 1$, then $S_\xi(B)$ is a finite subset of $[M_\zeta]^3$ by (B_ξ). It is enough to show that $S_{\xi+1}(B) = S_\xi(B)$. But if $B + t \in S_{\xi+1}(B) \setminus S_\xi(B)$, then $t \in M_{\xi+1} \setminus M_\xi$. In particular, $\{x+t, y+t, z+t\} \subset M_{\xi+1} \setminus M_\xi$. But this would imply that $[B+t]^2 \subset E(g_\xi)$, contradicting the fact that $G(g_\xi)$ does not contain any odd cycle.

So, assume that $\zeta = \xi + 1$. If there exists a number $t \in \mathbb{R}$ such that $\{x+t, y+t, z+t\} \subset M_\xi$, then $B+t \in [M_\xi]^3$ and $S_{\xi+1}(B+t) = S_{\xi+1}(B) = S_\xi(B)$ is finite subset of $[M_\xi]^3 \subset [M_\zeta]^3$ by the above.

So, assume that $\{x+t, y+t, z+t\} \not\subset M_\xi$ for every $t \in \mathbb{R}$ and choose $B+t \in S_{\xi+1}(B) \subset [M_{\xi+1}]^3$. Then there exist $k < \omega$ such that $x_k \in \{x+t, y+t, z+t\}$. But if $n < \omega$ is such that $1/n < \min\{|a-x|, |b-y|, |c-z|\}$, then $k < n$, since otherwise $[B+t]^2 \not\subset E(h_{\xi+1})$. Therefore, there is at most $3n$ numbers $t \in \mathbb{R}$ such that $[B+t]^2 \subset E(h_{\xi+1})$. Thus, the set $S_{\xi+1}(B)$ is a finite subset of $[M_{\xi+1}]^3 = [M_\zeta]^3$.

To prove $(A_{\xi+1})$ assume, by way of contradiction, that $G(h_{\xi+1})$ contains K_5; i.e., that there is $A=\{a,b,c,d,e\}\in[\mathbb{R}]^5$ such that $[A]^2\subset E(h_{\xi+1})$. Let us identify the edges of $G=(A,[A]^2)$ with their centers. Then all edges are in $M_{\xi+1}$. Let E_0 (E_1) be all edges of G that belong (do not belong) to M_ξ and let $G_i=(A,E_i)$ for $i<2$.

Notice that G_1 does not contain any odd cycle, since it is a subgraph of $G(g_\xi)$. Hence, it is bipartite. As it has 5 vertices, some of the bipartition classes must have at least 3 elements, say a,b,c. This is a triangle in G_0. In particular, by (1), $a,b,c\in M_\xi$.

At least one of the remaining two vertices, say d, is in $M_{\xi+1}\setminus M_\xi$. Then so are $(a+d)/2$, $(b+d)/2$ and $(c+d)/2$, i.e., they are equal to x_i, x_j and x_n, respectively, for some distinct $i,j,n\in\omega$. Assume that $i<j<n$. Then, $\{a,b,c\}\in S^n_{ij}$ and $h_{\xi+1}(x_n)\le\varepsilon_n<|c-x_n|$. Therefore, $\{c,d\}\notin E(h_{\xi+1})$ contradicting our assumption.

Theorem 1 has been proved. □

The above argument, as well as a technique used in [1] and [2], suggest that the problem of existence of uniformly antisymmetric function with finite range has a considerably algebraic content. The next theorem shows that this is indeed the case. That is, in the absence of linear dependency, the problem becomes trivial.

Theorem 2 *If $B\subset\mathbb{R}$ is linearly independent over $\mathbb{Q}$ and $h\colon B\to(0,\infty]$ is such that $h(x)=\infty$ for every $x\in B$, then $G(h)$ is 3-colorable.*

Proof. Let $\mathcal{G}$ be the family of all components of $G(h)$, i.e., of all maximal connected subgraphs of $G(h)$. It is enough to show that every $G\in\mathcal{G}$ is 3-colorable.

Let $\mathcal{G}_0$ be the family of all $G\in\mathcal{G}$ that contain an odd cycle. Evidently every $G\in\mathcal{G}\setminus\mathcal{G}_0$ is 2-colorable. Thus, it is enough to show that the graph $G_0=\bigcup\mathcal{G}_0$ is 3-colorable.

Define

$$\mathcal{F}=\{w\colon B\to\mathbb{Z}\colon 1<|\{b\in B\colon w(b)\neq 0\}|<\omega\ \&\ \Sigma_{b\in B}w(b)=1\}$$

and put $\mathrm{supp}(w)=\{b\in B\colon w(b)\neq 0\}$ for $w\in\mathcal{F}$. Moreover, let $V(G_0)$ denote the set of all vertices of G_0 and define

$$V=\{x\in\mathbb{R}\colon(\exists w_x\in\mathcal{F})(x=\Sigma_{b\in B}w_x(b)\,b\}.$$

We will show first that

$$V(G_0)\subset V. \tag{2}$$

To see this first notice that if $a_0, \dots, a_{2n}$ are the vertices of an odd cycle in G_0 and $b_i = (a_i + a_{i+1})/2$ for all $i \le 2n$ (where $a_{2n+1} = a_0$), then $a_0 = \Sigma_{i=0}^{2n}(-1)^i\, b_i$. It is easy to see that $a_0 \in V$. The proof of (2) is then completed by induction on the distance from an odd cycle if we notice that for every $x = \Sigma_{b \in B} w_x(b)\, b \in V$ and $y \in \mathbb{R}$ connected with x; i.e., such that $b_0 = (x+y)/2 \in B$, we have $y = 2b_0 - x = 2b_0 - (\Sigma_{b \in B} w_x(b)\, b) \in V$.

Now let G_1 be the subgraph of $G(h)$ generated by the vertices of V. Since G_0 is a subgraph of G_1, it is enough to show that G_1 is 3-colorable. To see this first notice that by linear independence of B over $\mathbb{Q}$, for every $x \in V$, there exists precisely one $w_x \in \mathcal{F}$ such that $x = \Sigma_{b \in B} w_x(b)\, b$. Moreover, the vertices $x, y \in V$ are connected, if and only if $b_0 = (x+y)/2$ belongs to B. This is equivalent to saying that there is an edge between $x, y \in V$ if and only if there is $b_0 \in B$ such that $w_x(b_0) + w_y(b_0) = 2$ and $w_x(b) + w_y(b) = 0$ for all $b \in B$, $b \neq b_0$.

Define the 3-coloring $c\colon V \to \{0,1,2\}$ of V the following way. For $x \in V$ let $\mathrm{supp}(w_x) = \{b_0, \dots, b_n\}$, where $b_0 < \dots < b_n$. Notice that $n > 0$. Then put

$$c(x) = \begin{cases} 0 & w_x(0) < 0 \\ 1 & w_x(0) > 0 \;\&\; w_x(n) < 0 \\ 2 & w_x(0) > 0 \;\&\; w_x(n) > 0. \end{cases}$$

It is easy to see that this function is indeed a 3 coloring of G_1.

This finishes the proof of Theorem 2. □

The definition of uniformly antisymmetric function was motivated by the study of the paradoxical behavior of real functions from the point of view of their symmetric continuity. The natural counterpart of the symmetric continuous functions, i.e., functions f for which $\lim_{h \to 0} f(x-h) - f(x+h) = 0$, are Schwartz continuous functions, i.e., functions for which $\lim_{h \to 0} f(x-h) + f(x+h) - 2f(x) = 0$ for all $x \in \mathbb{R}$. (See [5]. [1]) Thus, the following seems to be a natural counterpart for uniformly antisymmetric functions: a function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *uniformly anti-Schwartz* provided for every $x \in \mathbb{R}$ there exists $g(x) > 0$ such that for every $0 < h < g(x)$

$$|f(x-h) + f(x+h) - 2f(x)| \ge g(x).$$

Hajrudin Fejzić asked the author, whether there exists a uniformly anti-Schwartz function. The following theorem gives a positive answer to this question.

[1] Thomson uses terms "nowhere weakly symmetrically continuous function" for uniformly antisymmetric function and "symmetric continuous function" for Schwartz continuous function.

Theorem 3 *There exists a function $f\colon \mathbb{R} \to \mathbb{N}$ which is uniformly anti-Schwartz. Moreover,*

$$|f(x+h)+f(x-h)-2f(x)| \geq 1 \quad \textit{for } \textbf{all } x \in \mathbb{R} \textit{ and } h > 0.$$

PROOF. Let $S \subset \mathbb{N}$ be infinite and such that it does not contain any arithmetic progression of length 3. Let $\{P_n : n \in S\}$ be a partition of $\mathbb{R}$ such that no P_n contains an arithmetic progression of length 3. (The existence of such a partition is well known. For example, the partition from [2, Thm 1.1] has this property.)

Define

$$f(x) = n \text{ if and only if } x \in P_n.$$

Now if $|f(x+h)+f(x-h)-2f(x)| < 1$, then $f(x+h)+f(x-h)-2f(x) = 0$. Thus, the numbers $f(x-h)$, $f(x)$ and $f(x+h)$ form an arithmetic progression. But the assumption on S implies that $f(x-h) = f(x) = f(x+h) = n$ for some n from S.

So $x-h$, x and $x+h$ belong to the same P_n. However since P_n is 3-arithmetic progression free, we conclude that $h = 0$.

This completes the proof. $\square$

The referee and Miroslav Chlebíc [2] noticed that if in the above proof we take $S = \mathbb{N}$, choose a quickly decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ like $a_n = 3^{-n}$ and define

$$f(x) = a_n \text{ if and only if } x \in P_n,$$

then we obtain a uniformly anti-Schwartz function with bounded range. The problem of finding a uniformly antisymmetric function with bounded range ([2, Prob. 1(b)], [5, Prob. 25]) remains open.

Problem 1 *Does there exist a uniformly anti-Schwartz function $f\colon \mathbb{R} \to \mathbb{R}$ with two element range?*

This problem seems to be particularly interesting in light of the fact that it is known that there is no uniformly antisymmetric function with two element range [2, Thm 2.1].

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