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Vasile Ene, Ovidius University Constanţa, Romania
Current address: 23 August 8717, Jud. Constanţa, Romania.

A NOTE ON AN ASSUMPTION OF P. Y. LEE AND T. S. CHEW

Abstract

In [3] (p. 224), P. Y. Lee and T. S. Chew use Corollary 1 of our paper essentially without proof, and without stating it explicitly, claiming that "it is easy to verify". The same result is also used by P. Y. Lee in [2] (Theorem 10.2, p. 59). The aim of this article is to prove Corollary 1.

In what follows we shall use several classes of functions: $\mathcal{C}$, (N), VB, VB^*, AC^*, bAC^*, AC^*G (see [1]).

Definition 1 ([1], p. 41.) *Let $F : [a,b] \mapsto \mathbb{R}$ and let $P, Q \subseteq [a,b]$ such that $\{(x,y) \in P \times Q : x < y\} \neq \emptyset$. F is said to be $VB(P \wedge Q)$ if there exists $M \in (0, +\infty)$ such that*

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| < M,$$

whenever $\{[a_k, b_k]\}$, $k = \overline{1,n}$ is a finite set of nonoverlapping closed intervals with $a_k \in P$, $b_k \in Q$. For $P \subseteq Q \subseteq [a,b]$ we define $VB(P;Q) = VB(P \wedge Q) \cap VB(Q \wedge P)$.

Lemma 1 ([1], pp. 45-46.) *Let $F : [a,b] \mapsto \mathbb{R}$, $P \subseteq [a,b]$, $c = \inf(P)$, $d = \sup(P)$. The following assertions are equivalent:*

(i) $F \in VB^*$ *on* P;

(ii) $F \in VB(P; [c,d])$.

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Lemma 2 *Let $F : [a, b] \mapsto \mathbb{R}$. If F is bounded on $[a, b]$ and VB^* on a subset E of $[a, b]$ then F is VB^* on $E \cup \{a, b\}$.*

Proof. Let $M > 0$ such that $|F(x)| < M$, $x \in [a, b]$ and let M_1 be the constant given by the fact that $F \in VB^*$ on E. Then $F \in VB^*$ on $E \cup \{a, b\}$ with the constant $M_1 + 4M$. □

Theorem 1 *Let $F : [a, b] \mapsto \mathbb{R}$ and let $E_i \subset [a, b]$, $i = \overline{1, n}$. If $F \in VB^*$ on each $E_i \cup \{a, b\}$ then $F \in VB^*$ on $\cup_{i=1}^n E_i \cup \{a, b\}$.*

Proof. By Lemma 1, $F \in VB((E_i \cup \{a, b\}) \wedge [a, b])$ with the constant M_i, $i = \overline{1, n}$. Let $[\alpha_j, \beta_j]$, $j = \overline{1, m}$ be a finite set of nonoverlapping closed intervals, with $\alpha_j \in \cup_{i=1}^n E_i \cup \{a, b\}$ and $\beta_j \in [a, b]$. Let $\mathcal{A}_i = \{a_j : a_j \in E_i \setminus (\cup_{k=1}^{i-1} E_k)\}$. Then $\sum_{j=1}^m |F(\beta_j) - F(\alpha_j)| = \sum_{i=1}^n \sum_{j \in \mathcal{A}_i} |F(\beta_j) - F(\alpha_j)| < \sum_{i=1}^n M_i$. Therefore $F \in VB(\cup_{i=1}^n E_i \cup \{a, b\} \wedge [a, b])$. Similarly, we can prove that $F \in VB([a, b] \wedge \cup_{i=1}^n E_i \cup \{a, b\})$. By Lemma 1 it follows that $F \in VB^*$ on $\cup_{i=1}^n E_i \cup \{a, b\}$. □

Lemma 3 *Let $F : [a, b] \mapsto \mathbb{R}$ and E_i, $i = \overline{1, n}$ be closed subsets of $[a, b]$. If F is continuous on $[a, b]$ and F is AC^* on each E_i then F is AC^* on $\cup_{i=1}^n E_i \cup \{a, b\}$.*

Proof. Since F is continuous on $[a, b]$, F is bounded on $[a, b]$. It follows that $F \in bAC^* = VB^* \cap AC^*$ on each E_i (see Proposition 2.12.1. (v) of [1]). By Lemma 2, $F \in VB^*$ on each $E_i \cup \{a, b\}$, and by Theorem 1, $F \in VB^* \subset VB$ on $\cup_{i=1}^n E_i \cup \{a, b\}$. It follows that $F \in VB \cap \mathcal{C} \cap AC^*G \subset VB \cap \mathcal{C} \cap (N)$ on the closed set $\cup_{i=1}^n E_i \cup \{a, b\}$. By the Banach-Zarecki theorem (see [1], p.75), $F \in AC$ on $\cup_{i=1}^n E_i \cup \{a, b\}$. Therefore $F \in AC \cap VB^* = bAC^* = AC^*$ on $\cup_{i=1}^n E_i \cup \{a, b\}$ (see Theorem 2.12.1., (i), (ii) of [1]). □

Remark 1 In Theorem 1, VB^* cannot be replaced by VB, and in Lemma 3, AC^* cannot be replaced by AC. Indeed:
Let $F : [0, 1] \mapsto \mathbb{R}$,

$$F(x) = \begin{cases} x \cdot \sin \frac{2\pi}{x} & , \quad x \in (0, 1] \\ 0 & , \quad x = 0 \end{cases}$$

Let $E_1 = \{0\} \cup \{1/n : n = \overline{2, \infty}\}$ and $E_2 = \{0\} \cup \{4/(4n + 1) : n = \overline{1, \infty}\}$. Then E_1 and E_2 are closed subsets of $[0, 1]$, $F(x) = 0$ if $x \in E_1$ and $F(x) = x$ if $x \in E_2$. Therefore $F \in AC \subset VB$ on E_1 and $F \in AC \subset VB$ on E_2. Since $[4/(4n + 1), 1/n]$, $n = \overline{1, \infty}$ are nonoverlapping closed intervals, with $4/(4n+1) \in E_2$ and $1/n \in E_1$ we obtain that $\sum_{n=1}^\infty |F(1/n) - F(4/(4n+1))| =$

$\sum_{n=1}^{\infty} 4/(4n+1) = +\infty$. It follows that $F \notin VB$ on $E_1 \cup E_2$, hence $F \notin AC$ on $E_1 \cup E_2$.

Corollary 1 *Let $F : [a,b] \mapsto \mathbb{R}$ and let E_i, $i = \overline{1,n}$ be closed subsets of $[a,b]$. Let $F_n : [a,b] \mapsto \mathbb{R}$, such that $F_n(x) = F(x)$, for $x \in \cup_{i=1}^{n} E_i \cup \{a,b\}$, and F_n is linear on the closure of each interval contiguous to $\cup_{i=1}^{n} E_i \cup \{a,b\}$. If $F \in \mathcal{C}$ on $[a,b]$ and $F \in AC^*$ on each E_i then F_n is derivable a.e. on $[a,b]$ and F_n' is summable on $[a,b]$.*

PROOF. By Lemma 3, $F \in AC^* \subset AC$ on $\cup_{i=1}^{n} E_i \cup \{a,b\}$. By Theorem 2.11.1. (xviii) of [1], $F_n \in AC$, and by Corollary 2.14.2. of [1], F_n is derivable a.e. on $[a,b]$ and $F_n^{'}$ is summable on $[a,b]$. □

Remark 2 In [3] (p. 224), P. Y. Lee and T. S. Chew use Corollary 1 essentially without proof, claiming that "it is easy to verify". The same result is also used by P. Y. Lee in [2] (see Theorem 10.2, p. 59).

Remark 3 A different proof of this result has been given by P. Y. Lee and C. S. Ding, using Lemma 6.4 (iii) of [2]. C. S. Ding and P. Y. Lee, Generalized Riemann integral, Scientific Press, Beijing, 1989, (in Chinese).

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