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ON EQUI-DERIVATIVES

Abstract

The notion of equi-derivatives is introduced and is compared with approximate equicontinuity. Moreover, it is proved that a function f of two variables whose sections f_x are equi-derivatives and sections f^y are measurable (derivatives) [have the Baire property] is measurable (a strong derivative) [has the Baire property].

1 Preliminaries and Notations

Let $\mathbb{R}$ be the set of all reals and let μ_e (μ) denote outer Lebesgue measure (Lebesgue measure) in $\mathbb{R}$. Let

$$d_u(A,x) = \limsup_{h\to 0^+} \mu_e(A\cap(x-h,x+h))/2h$$
$$(d_l(A,x) = \liminf_{h\to 0^+} \mu_e(A\cap(x-h,x+h))/2h)$$

be the upper (lower) density of a set $A \subset \mathbb{R}$ at x. A point $x \in \mathbb{R}$ is called a density point of a set $A \subset \mathbb{R}$ if there exists a (Lebesgue) measurable set $B \subset A$ such that $d_l(B,x) = 1$. The family $\mathcal{T}_d = \{A \subset \mathbb{R};\ A$ is measurable and every point $x \in A$ is a density point of $A\}$ is a topology called the density topology [1].

A function $f : \mathbb{R} \to \mathbb{R}$ is called $\mathcal{T}_d$-continuous or approximately continuous at a point x if it is continuous at x as a function from $(\mathbb{R},\mathcal{T}_d))$ into $(\mathbb{R},\mathcal{T}_e)$, where $\mathcal{T}_e$ denotes the Euclidean topology in $\mathbb{R}$.

A family of functions $f_s : \mathbb{R} \to \mathbb{R}$, $s \in S$, is called $\mathcal{T}_d$-equicontinuous or approximately equicontinuous at a point x if the functions f_s, $s \in S$, are

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equicontinuous at x as the functions from $(\mathbb{R}, \mathcal{T}_d)$ into $(\mathbb{R}, \mathcal{T}_e)$, i.e. for every $\eta > 0$ there is a set $B \in \mathcal{T}_d$ such that $x \in B$ and for all $t \in B$ and $s \in S$ the inequality $|f_s(t) - f_s(x)| < \eta$ holds.

A family of locally Henstock-Kurzweil integrable functions $f_s : \mathbb{R} \to \mathbb{R}$, $s \in S$, is called a family of equi-derivatives at a point $x \in \mathbb{R}$ if for every positive η there is a $r > 0$ such that for every real h with $0 < |h| < r$ and for every $s \in S$ we have

$$\left| \frac{1}{h} \int_x^{x+h} f_s(t)\,dt - f_s(x) \right| < \eta.$$

2 Equi-derivatives and Approximate Equicontinuity

It is well known [1] that every locally bounded (Lebesgue) measurable function f which is approximately continuous at a point x is also a derivative at x, i.e.

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = f(x).$$

By a similar proof we obtain:

Remark 1 *If locally integrable functions $f_s : \mathbb{R} \to \mathbb{R}$, $s \in S$, are approximately equicontinuous at x and if there are $M > 0$, $r > 0$ such that for all $s \in S$ and for all $t \in (x - r, x + r)$ the inequality $|f_s(t)| < M$ is true, then the functions f_s, $s \in S$, are equi-derivatives at x.*

In the above remark the existence of the constant M is important. Indeed, if $(a_n)_n$ is a sequence of positive reals such that $a_1 > a_2 > \cdots > a_n > \cdots \searrow 0$ and

$$d_u \left(\bigcup_{n=1}^{\infty} [a_{2n}, a_{2n-1}], 0 \right) = 0,$$

then let f_n, $n = 1, 2, \ldots$, be a continuous function such that $f_n(x) = 0$ for $x \in \mathbb{R} \setminus [a_{2n}, a_{2n-1}]$ and $\int_{a_{2n}}^{a_{2n-1}} f_n(t)\,dt = n a_{2n-1}$. Then the functions f_n, $n = 1, 2, \ldots$, are continuous, bounded and approximately equicontinuous, but they are not equi-derivatives at 0.

From Lipiński's theorem in [6] it follows that if for all reals a, b with $a < b$ the functions $\min(b, \max(a, f))$ are derivatives, then f is approximately continuous. So, we obtain the following question:

Suppose that for all reals a, b with $a < b$ the functions $\min(b, \max(a, f_s))$ are equi-derivatives. Must the functions f_s, $s \in S$, be approximately equicontinuous?

Example 1 shows that the answer is "no".

Example 1 *For every positive integer n let $J_n \subset (1/(n+1), 1/n)$ be a closed interval such that $n(n+1)|J_n| > 1 - 1/n$, where $|J_n|$ denotes the length of J_n. Define the continuous function f_n to be 1 on J_n, 0 on $\mathbb{R} \setminus (1/(n+1), 1/n)$ and linear otherwise on $\mathbb{R}$. The functions f_n, $n = 1, 2, \ldots$, are continuous everywhere on $\mathbb{R}$ and approximately equicontinuous (even equicontinuous) at all points $x \neq 0$. Since $d_u(\bigcup_n J_n, 0) = 1/2$, the functions f_n, $n \geq 1$, are not approximately equicontinuous at 0. To prove that for all $a < b$ the functions $\min(b, \max(a, f_n))$, $n \geq 1$, are equi-derivatives, it suffices to show that they are equi-derivatives at 0. Fix a, b such that $a < b$ and let $g_n = \min(b, \max(a, f_n))$ for $n \geq 1$. Fix $\eta > 0$. There is a positive integer k such that $1/k < \eta$. Let $r = 1/(k+1)$ and let real h be such that $0 < |h| < r$. If $a \geq 1$ or $b \leq 0$ or if $b > 0$, $a < 1$ and $h < 0$, then for every $n \geq 1$ we have*

$$\left| \frac{1}{h} \int_0^h g_n(t)\, dt - g_n(0) \right| = |g_n(0) - g_n(0)| = 0 < \eta.$$

We proceed similarly in the case $a < 1$, $b > 0$ and $h > 0$ for $n < 1/h - 1$. If $a < 1$, $b > 0$, $h > 0$ and $n \geq 1/h - 1$, then

$$\int_0^h g_n(t)\, dt \leq \min(b, 1)/(n(n+1)) + g_n(0)h \leq 1/(n(n+1)) + g_n(0)h,$$

whence

$$\left| \frac{1}{h} \int_0^h g_n(t)\, dt - g_n(0) \right| < 1/n < \eta.$$

So, the functions g_n, $n \geq 1$, are equi-derivatives.

Remark 2 *It is well known* ([1, Th. 5.8]) *that every lower semi-continuous locally bounded derivative is approximately continuous. Meanwhile the functions f_n, $n \geq 1$, from Example 1 are not approximately equicontinuous at 0, although they are equi-derivatives bounded by a common constant and they are lower semi-equicontinuous at 0, i.e. for every $\eta > 0$ there is a positive real r such that $f_n(0) - f_n(t) < \eta$ for all points $t \in (-r, r)$ and $n \geq 1$.*

The next theorem gives some sufficient condition for the approximate equicontinuity of families of equi-derivatives.

Theorem 1 *Let measurable functions $f_s : \mathbb{R} \to \mathbb{R}$, $s \in S$, be such that there is a $M > 0$ with $|f_s| < M$ for all $s \in S$. Suppose that for every $\eta > 0$ there* *is an approximately continuous positive function* $r : \mathbb{R} \to \mathbb{R}$ *such that for all* $x \in \mathbb{R}$, $s \in S$ *and* h *with* $0 < |h| < r(x)$ *the inequality*

$$\left|\frac{1}{h}\int_x^{x+h} f_s(t)\,dt - f_s(x)\right| < \eta$$

holds. Then the functions f_s, $s \in S$, *are approximately equicontinuous.*

PROOF. Suppose, to the contrary, that the functions f_s, $s \in S$, are not approximately equicontinuous at a point x. Then there is a $\eta > 0$ such that for every $A \in \mathcal{T}_d$ containing x there are $s \in S$ and $t \in A$ such that $|f_s(t) - f_s(x)| \geq \eta$. Let r be a positive function corresponding to $\eta/4$ by hypothesis of our theorem and let $A \in \mathcal{T}_d$ be a set containing x such that $|r(t) - r(x)| < r(x)/4$ for every $t \in A$. Assume that $I \subset (x - r(x)/4, x + r(x)/4)$ is an open interval containing x such that for every $t \in I$ we have $2M|t - x|/r(x) < \eta/8$ and $(Mr(x)/2)(1/((r(x)/2) - |t - x|) - 2/r(x)) < \eta/8$. There are an index $s \in S$ and a point $u \in A \cap I$ with $|f_s(u) - f_s(x)| \geq \eta$. We can assume that $u > x$, since in the case $u < x$ the proof is analogous. Observe that $x < u < h = x + r(x)/2 < u + r(u)$ and

$$\begin{aligned}
&\left|(1/(h-u))\int_u^h f_s(t)\,dt - (2/r(x))\int_x^h f_s(t)\,dt\right| \\
=&\left|(1/(h-u))\int_u^h f_s(t)\,dt - (2/r(x))\int_u^h f_s(t)\,dt - (2/r(x))\int_x^u f_s(t)\,dt\right| \\
\leq&|(1/(h-u) - (2/r(x))|\int_u^h |f_s(t)|\,dt + (2/r(x))\int_x^u |f_s(t)|\,dt \\
\leq&(Mr(x)/2)(1/((r(x)/2) - |u-x|) - 2/r(x)) + 2M|u-x|/r(x) \\
<&\frac{\eta}{8} + \frac{\eta}{8} = \frac{\eta}{4}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
|f_s(u) - f_s(x)| \leq &\left|f_s(u) - (1/(h-u))\int_u^h f_s(t)\,dt\right| \\
&+ \left|(1/(h-u))\int_u^h f_s(t)\,dt - (1/(h-x))\int_x^h f_s(t)\,dt\right| \\
&+ \left|(1/(h-x))\int_x^h f_s(t)\,dt - f_s(x)\right| < \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} < \eta,
\end{aligned}$$

a contradiction. $\square$

We say that a function $f : \mathbb{R} \to \mathbb{R}$ has property SAC if for every $\eta > 0$ there is an approximately continuous positive function $r : \mathbb{R} \to \mathbb{R}$ such that for every x and h with $0 < |h| < r(x)$ we have

$$\left| \frac{1}{h} \int_x^{x+h} f(t)\, dt - f(x) \right| < \eta.$$

It follows from Theorem 1 applied to the family containing one function that every function having the property SAC is also approximately continuous.

Problem 1 *Does every approximately continuous function have property SAC?*

Remark 3 *There is a function $f : \mathbb{R} \to [0,1]$ having property SAC whose set of discontinuities is of positive measure.*

Proof. Let $C \subset (0,1)$ be a Cantor set of positive measure and let $(I_n)_n$ be an enumeration of all components of the set $(0,1) \setminus C$ such that $I_n \neq I_m$ for $n \neq m$, $n, m = 1, 2, \ldots$. In every interval I_n, $n \geq 1$, we find closed intervals $I_{n,1}$, $I_{n,2} = [c_n, d_n]$ having the same center as I_n and such that

$$\max((|I_{n,1}|/|I_{n,2}|), (|I_{n,2}|/|I_n|)) < 4^{-n}.$$

Let f be a function which is continuous at every point $x \in \mathbb{R} \setminus C$, equal to 0 at every $x \in \mathbb{R} \setminus \bigcup_n I_{n,1}$ and such that $f(I_{n,1}) = [0,1]$ for $n \geq 1$. Since f is discontinuous at every point $x \in C$, the set of discontinuities of f is of positive measure. Now we will prove that f has property SAC. Fix $\eta > 0$. There is a positive integer k with $4^{-k} + 2(4^{-2k+1})/(1 - 16^k) < \eta$. Let

$$A = \bigcup_{n \leq k} I_{n,2}.$$

Since for every $n > k$ the function f is uniformly continuous on the interval $I_{n,2}$, there are positive reals $r_n < |I_{n,1}|$, $n > k$, such that for all $x, y \in I_{n,2}$ with $|x - y| < r_n$ we have $|f(x) - f(y)| < \eta$. Similarly there is a positive real $r_0 < \min_{j \leq k} |I_{j,1}|/4$ such that for all $x, y \in A$ with $|x - y| < r_0$ we have $|f(x) - f(y)| < \eta$. Put $r_n = r_0$ for $n \leq k$ and $a = \min(r_0, \operatorname{dist}(A, C)/4)$, where $\operatorname{dist}(A, C) = \inf\{|x - y|; x \in A, y \in C\}$. Moreover, let $\operatorname{dist}(x, A) = \inf\{|x - y|; y \in A\}$ and let

$$g(x) = a + \min(\operatorname{dist}(x, A), \operatorname{dist}(x, C))/4 \text{ for } x \in I_n \setminus \operatorname{int}(I_{n,2}), \quad n \geq 1,$$

where $\mathrm{int}(A)$ denotes the interior of A. For $x \in C \cup \bigcup_{n\leq k} I_n$ we put $r(x) = a$. For the definition of the function r on the intervals I_n, $n > k$, we observe that $r(c_n) = r(d_n)$. Fix a positive integer $n > k$. If $r_n \geq g(c_n)$, then we put $r(x) = g(c_n)$ for $x \in I_{n,2}$ and $r(x) = g(x)$ for $x \in I_n \setminus I_{n,2}$. If $r_n < g(c_n)$, then we find a closed interval $I_{n,3}$ having the same center as I_n and such that $I_{n,2} \subset \mathrm{int}(I_{n,3})$ and $|I_{n,3}|/|I_n| < 4^{-n}$. Then we define $r(x) = r_n$ for $x \in I_{n,2}$, $r(x) = g(x)$ for $x \in I_n \setminus \mathrm{int}(I_{n,3})$ and r is linear on the components of $I_{n,3} \setminus \mathrm{int}(I_{n,2})$. The function r is already defined on the interval $(0, 1)$. Observe $u = \lim_{x\to 0+} r(x)$ and $v = \lim_{x\to 1-} r(x)$ exists and are positive. Put $r(x) = u$ for $x \leq 0$ and $r(x) = v$ for $x \geq 1$. The positive function r is defined on $\mathbb{R}$, and continuous at each point $x \in \mathbb{R} \setminus C$. Since the function g is continuous at each $x \in C$, $r(x) = g(x)$ for all $x \in C$ and every $x \in C$ is a density point of the set $\{x; g(x) = r(x)\}$, the function r is approximately continuous at all points of the set C. If $x \in I_{n,2}$ for some integer n, and h is such that $0 < h < r(x) = r_n < |I_{n,1}|$, then $[x, x + h] \subset I_n$ and $|f(t) - f(x)| < \eta$ for all $t \in [x, x + h]$. Consequently,

$$\begin{aligned}\left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt\right| \\ &\leq \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dt < \frac{1}{h}h\eta = \eta.\end{aligned}$$

If x is such that there is not an integer n for which $x \in I_{n,2}$ and if h is such that $0 < h < r(x)$, then we put $K = \{i; I_i \subset [x, x + h]\}$ and let L be the set of such indexes ℓ which are not in K and for which $\mathrm{int}(I_\ell) \cap [x, x + h] \neq \emptyset$. Since $r(x) < \mathrm{dist}(A, C)$ for all x, we obtain $i > k$ for every $i \in K$. The set L contains at most two elements. From the construction of the function r we obtain that if $n \in L$ and $n \leq k$, then $f(t) = 0$ for each $t \in [x, x + h] \cap I_n$. We have:

$$\begin{aligned}\left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt\right| \\ &= \left|\frac{1}{h}\left(\sum_{n\in K}\int_{I_n} f(t)\,dt + \sum_{l\in L}\int_{I_l} f(t)\,dt\right)\right| \\ &\leq \frac{1}{h}\left(\sum_{n\in K}|I_{n,1}| + \sum_{l\in L}|I_{l,1} \cap [x, x + h]|\right) \\ &< \frac{1}{h}(4^{-k-1}h + 2(4^{-2l+1}h)/(1 - 16^{-l})) < \eta.\end{aligned}$$

If $-r(x) < h < 0$ the proof is analogous. So, the function f has the property SAC and the proof is finished. $\square$

3 Equi-derivatives and Some Properties of Functions of Two Variables

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. It is well known ([4]) that if all sections $f_x(t) = f(x,t)$, $t, x \in \mathbb{R}$, are approximately equicontinuous and if all sections $f^y(t) = f(t,y)$, $t, y \in \mathbb{R}$, are (Lebesgue) measurable [have the Baire property], then f is measurable [has the Baire property] as a function of two variables. These theorems are also true if we suppose that the sections f_x, $x \in \mathbb{R}$, are equi-derivatives.

Theorem 2 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a locally bounded function with all sections f^y, $y \in \mathbb{R}$, being measurable (having the Baire property). Suppose that there is a set $B \subset \mathbb{R}$ of measure zero (of the first category) such that the sections f_x, $x \in \mathbb{R} \setminus B$, are equi-derivatives at every point $y \in \mathbb{R}$. Then the function f is measurable (has the Baire property).*

Proof. It suffices to prove that for every bounded closed interval $I \subset \mathbb{R}$ the restricted function $f|(I \times I)$ is measurable. Assume that $I = [a,b]$. Since the set $I \times I$ is compact, the function $f|(I \times I)$ is bounded. Let $g(x,y) = f(x,y)$ for $x \in I \setminus B$ and and let $g(x,y) = 0$ otherwise on $I \times I$. Observe that the restricted function $f|(I \times I)$ is measurable if and only if the function g is measurable. All sections g_x, $x \in I$, are derivatives. So, by Lipiński's Theorem 3 from [7], for the measurability of g it suffices to prove that for every $t \in I$ the function

$$h(x) = \int_a^t g(x,y)\,dy, \quad x \in I,$$

is measurable. Fix $t \in I$. We will prove that the function h satisfies the hypothesis of Davies' Lemma from [3]. Let η be a positive real and let $C \subset I$ be a measurable set of positive measure. For every $y \in I$ there is a positive number $r(y)$ such that for every h with $0 < |h| < r(y)$ and for every $x \in I \setminus B$ we have

$$\left| \frac{1}{h} \int_y^{y+h} g(x,v)\,dv - g(x,y) \right| < \eta/(4(t-a)).$$

The family $\{(y - r(y), y + r(y)); y \in I\}$ is an open covering of the compact $[a,t]$. So, there are points

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$$

such that for every $x \in I \setminus B$ and $i = 1, \dots, n$ we have

$$\left|(1/(t_i - t_{i-1})) \int_{t_{i-1}}^{t_i} g(x,y)\,dy - g(x,t_{i-1})\right| < \eta/(4(t-a)).$$

Since all sections g^{t_i}, $i = 0, 1, \dots, n$, are measurable, there is a density point $u \in C$ at which all sections g^{t_i}, $i = 0, 1, \dots, n$, are approximately continuous. Thus there is a measurable set $E \subset C$ of positive measure such that $|g(v,t_i) - g(w,t_i)| < \eta/(2n)$ for all $v, w \in E$ and $i = 0, \dots, n$. Fix $v, w \in E$. Then

$$\begin{aligned}
|h(v)-h(w)| &= \left|\int_a^t g(v,y)\,dy - \int_a^t g(w,y)\,dy\right| \\
&= \left|\int_a^t (g(v,y) - g(w,y))\,dy\right| = \left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (g(v,y) - g(w,y))\,dy\right| \\
&= \left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (g(v,y) - g(v,t_{i-1}))\,dy + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (g(v,t_{i-1}) - g(w,t_{i-1}))\,dy\right. \\
&\quad \left. + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (g(w,t_{i-1}) - g(w,y))\,dy\right| \\
&= \left|\sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} g(v,y)\,dy - g(v,t_{i-1})(t_i - t_{i-1})\right)\right. \\
&\quad + \sum_{i=1}^n (g(v,t_{i-1}) - g(w,t_{i-1}))(t_i - t_{i-1}) \\
&\quad \left. + \sum_{i=1}^n (g(w,t_{i-1})(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} g(w,y)\,dy)\right| \\
&\le \sum_{i=1}^n (t_i - t_{i-1}) \left[\left|(1/(t_i - t_{i-1})) \int_{t_{i-1}}^{t_i} g(v,y)\,dy - g(v,t_{i-1})\right|\right. \\
&\quad \left. + \left|(1/(t_i - t_{i-1})) \int_{t_{i-1}}^{t_i} g(w,y)\,dy - g(w,t_{i-1})\right|\right] + n\eta/(2n) \\
&\le \sum_{i=1}^n (t_i - t_{i-1})(\eta/(4(t-a)) + \eta/(4(t-a))) + \eta/2 = \eta.
\end{aligned}$$

So, $\operatorname{osc}(h) \le \eta$ on the set E and by Davies' lemma from [3] the function h is measurable. This completes the proof of the first part of our theorem for the measurability. The proof of the second part is similar. Instead of Lipiński's theorem from [7] we apply an analogous theorem for the property of Baire from [4] and instead of Davies' lemma from [3] we apply an analogous theorem for the Baire property from [5]. □

In [7] Ślezak proved that if all sections f_x, $x \in \mathbb{R}$, are approximately continuous and if all sections f^y, $y \in \mathbb{R}$, are of Baire class $\alpha \geq 1$, then f is also of Baire class α. So, we obtain the following:

Problem 2 *Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a function such that all sections f_x are equi-derivatives and all sections f^y are of Baire class α. Is the function f of Baire class α?*

By a standard proof we observe that if all sections f_x, $x \in \mathbb{R}$, are approximately equicontinuous and if all sections f^y, $y \in \mathbb{R}$, are approximately equicontinuous, then f is $(\mathcal{T}_d \times \mathcal{T}_d)$-continuous as a function of two variables. For the equi-derivatives we obtain the following:

Theorem 3 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a locally bounded function such that all its sections f_x, $x \in \mathbb{R}$, are equi-derivatives at every point $y \in \mathbb{R}$ and all its sections f^y, $y \in \mathbb{R}$, are derivatives. Then f is a strong derivative at every point $(x, y) \in \mathbb{R}^2$, i.e. for every (x, y) the equality*

$$\lim_{h,k \to 0} \left(\int_{x-h}^{x+h} \int_{y-k}^{y+k} f(u, v)\, du\, dv \right) / (4hk) = f(x, y).$$

PROOF. Fix a point $(x, y) \in \mathbb{R}^2$ and a $\eta > 0$. Since all sections f_x, $x \in \mathbb{R}$, are equi-derivatives at the point y, there is a $r > 0$ such that for every h with $0 < |h| < r$ and for every $u \in \mathbb{R}$ we have

$$\left| \frac{1}{h} \int_y^{y+h} f(u, v)\, dv - f(u, y) \right| < \frac{\eta}{4}.$$

By the hypothesis the section f^y is a derivative at the point x. Thus there is a $s > 0$ such that for every k with $0 < |k| < s$ the inequality

$$\left| \frac{1}{k} \int_x^{x+k} f(u, y)\, du - f(x, y) \right| < \frac{\eta}{4}$$

is true. Fix h, k such that $0 < h < r$ and $0 < k < s$. Then for every $u \in (x-s, x+s)$ we obtain:

$$\begin{aligned}
&\left|\frac{1}{2h}\int_{y-h}^{y+h} f(u,v)\,dv - f(u,y)\right| \\
\leq &\left|\frac{1}{2h}\int_{y-h}^{y} f(u,v)\,dv - f(u,y)/2\right| + \left|\frac{1}{2h}\int_{y}^{y+h} f(u,v)\,dv - f(u,y)/2\right| \\
= &\frac{1}{2}\left[\left|\frac{1}{-h}\int_{y}^{y-h} f(u,v)\,dv - f(u,y)\right| + \left|\frac{1}{h}\int_{y}^{y+h} f(u,v)\,dv - f(u,y)\right|\right] \\
< &\frac{1}{2}\left(\frac{\eta}{4}+\frac{\eta}{4}\right) = \frac{\eta}{4}.
\end{aligned}$$

Since f is locally bounded, we can assume that it is bounded on the set $D = [x-k, x+k] \times [y-h, y+h]$. By Theorem 2 the function f is measurable, so it is integrable on the rectangle D. For $u \in (x-s, x+s)$ we have

$$2h(f(u,y) - \eta/4) < \int_{y-h}^{y+h} f(u,v)\,dv < 2h(f(u,y) + \eta/4).$$

Consequently,

$$\begin{aligned}
2h\int_{x-k}^{x+k} (f(u,y) - \eta/4)\,du &\leq \int_{x-k}^{x+k}\int_{y-h}^{y+h} f(u,v)\,dv\,du \\
&\leq 2h\int_{x-k}^{x+k} (f(u,y) + \eta/4)\,du.
\end{aligned}$$

As above we can prove that

$$2k(f(x,y) - \eta/4) < \int_{x-k}^{x+k} f(u,y)\,du < 2k(f(x,y) + \eta/4).$$

From the above we obtain

$$2h\int_{x-k}^{x+k} (f(u,y) - \eta/4)\,du \geq 4hkf(x,y) - 2hk\eta = 4hk(f(x,y) - \eta/2)$$

and

$$2h\int_{x-k}^{x+k} (f(u,y) + \eta/4)\,du \leq 4hk(f(x,y) + \eta/2).$$

So,

$$\left|\frac{1}{4hk}\int_{x-k}^{x+k}\int_{y-h}^{y+h} f(u,v)\,du\,dv - f(x,y)\right| \leq \frac{\eta}{2} < \eta,$$

and the proof is finished. □

Remark 4 *Observe that in the above theorem the hypothesis that f is locally bounded can be replaced by the hypothesis that f is locally integrable. Then the proof is the same, but we needn't rely on Theorem 2 for the measurability of the function f.*

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