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# IMPROVABLE DISCONTINUOUS FUNCTIONS

**Abstract**

In this paper the class of improvable functions is defined and the pasic properties of such functions is examined. Moreover, a necessary and sufficient condition under which a set $A$ is the set of points of continuity of some $\alpha$-improvable discontinuous function is gives. and it is shown that the classes $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are different is $\alpha \neq \beta$.

## 1 Introduction

If at some point $x$ $\lim_{t\to x} f(t)$ exists and $\lim_{t\to x} f(t) \neq f(x)$, then we can say that $f$ has an improvable discontinuity at the point $x$. If at each such point we change the value $f(x)$ to $\lim_{t\to x} f(t)$, then we obtain a new function $f_{(1)}$ with the "improved" improvable points of discontinuity of the function $f$. Repeating this process for the function $f_{(1)}$ and so on, we can create a sequence (even the transfinite sequence) $\left(f_{(\alpha)}\right)$ in such a way that $f_{(\alpha+1)}$ is obtained from $f_{(\alpha)}$ by "improving" $f_{(\alpha)}$.

## 2 Preliminaries

The word "function" will mean a bounded real function of a real variable. Let $D \subset \mathbb{R}$.

**Definition 1** *For each function $f : D \to \mathbb{R}$, let*

$$C(f) = \left\{x \in D;\ \lim_{t\to x} f(t) = f(x)\right\},$$
$$U(f) = \left\{x \in D;\ \lim_{t\to x} f(t) \neq f(x)\right\},$$
$$L(f) = \left\{x \in D;\ \lim_{t\to x} f(t)\ \textit{exists}\right\}.$$

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**Definition 2** *A point* $x_0 \in U(f)$ *is called* an improvable point of discontinuity of the function $f$.

The following remark can be easily seen.

**Remark 1** *Let* $f : D \to \mathbb{R}$. *Then* $U(f) \cap C(f) = \emptyset$ *and* $L(f) = U(f) \cup C(f)$.

The following proposition is well known. (Compare to [2].)

**Proposition 1** *The set* $U(f)$ *is countable.*

We define the functions $f_{(\alpha)}$ on the class of ordinal numbers.

**Definition 3** *Let* $f : D \to \mathbb{R}$ *and let* $f_{(0)}(x) = f(x)$ *for each* $x \in D$. *For every ordinal number* $\alpha$, *let*

$$f_{(\alpha)}(x) = \begin{cases} f(x) & \text{if } \left\{\gamma < \alpha;\ x \in U\left(f_{(\gamma)}\right)\right\} = \emptyset, \\ \lim_{t \to x} f_{(\gamma_0)}(t) & \text{if } x \in U\left(f_{(\gamma_0)}\right), \\ & \text{where } \gamma_0 = \min\left\{\gamma < \alpha;\ x \in U\left(f_{(\gamma)}\right)\right\}. \end{cases}$$

This theorem will be very useful in the paper.

**Theorem 1** *Let* $f : D \to \mathbb{R}$ *and let* $\alpha > 0$ *be an ordinal number. Then*

**(1,$\alpha$)** *for each* $x \in D$, $\left\{\gamma < \alpha;\ x \in U\left(f_{(\gamma)}\right)\right\}$ *is the empty set or has only one element,*

**(2,$\alpha$)** *for each ordinal number* $\gamma$ ($\gamma < \alpha$),

$$\left\{x \in D;\ f_{(\gamma)}(x) \neq f_{(\alpha)}(x)\right\} = \bigcup_{\gamma \le \beta < \alpha} U\left(f_{(\beta)}\right),$$

**(3,$\alpha$)** *for each ordinal number* $\gamma$ ($\gamma < \alpha$), *if* $x \in L\left(f_{(\gamma)}\right)$, *then*

$$\lim_{t \to x} f_{(\gamma)}(t) = f_{(\alpha)}(x),$$

**(4,$\alpha$)** $\bigcup_{0 \le \beta < \alpha} L\left(f_{(\beta)}\right) \subset C\left(f_{(\alpha)}\right)$.

Proof. It can be easily shown that (1,1), (2,1) and (3,1) hold. Let $x_0 \in L\left(f_{(0)}\right)$. Then, by (3,1), $\lim_{t \to x_0} f_{(0)}(t) = f_{(1)}(x_0)$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that, for each $t \in (x_0 - \delta, x_0 + \delta) \cap D$, if $t \neq x_0$ then $|f_{(0)}(t) - f_{(1)}(x_0)| < \frac{\epsilon}{2}$. We shall show that, for each $t \in (x_0 - \delta, x_0 + \delta) \cap D$, $|f_{(1)}(t) - f_{(1)}(x_0)| < \epsilon$. Let $t \in (x_0 - \delta, x_0 + \delta) \cap D$. It may be assumed that $t \neq x_0$ and $f_{(0)}(t) \neq f_{(1)}(t)$. Then, by (2,1), $t \in U\left(f_{(0)}\right) \subset L\left(f_{(0)}\right)$ and, by (3,1), $\lim_{z\to t} f_{(0)}(z) = f_{(1)}(t)$. Since $t \in (x_0-\delta, x_0+\delta)\cap D$, there exists $z \in (x_0-\delta, x_0+\delta)\cap D$ such that $z \neq x_0$ and $|f_{(0)}(z) - f_{(1)}(t)| < \frac{\epsilon}{2}$. Thus

$$|f_{(1)}(t) - f_{(1)}(x_0)| \leq |f_{(1)}(t) - f_{(0)}(z)| + |f_{(0)}(z) - f_{(1)}(x_0)| < \epsilon.$$

Therefore $x_0 \in C\left(f_{(1)}\right)$ and (4,1) is proved.

Let $\alpha_0 > 1$ be an ordinal number. Assume for each $\alpha$ with $1 \leq \alpha < \alpha_0$, we have (1, $\alpha$), (2, $\alpha$), (3, $\alpha$), (4, $\alpha$). Let $x \in D$ and $\left\{\gamma < \alpha_0;\ x \in U\left(f_{(\gamma)}\right)\right\} \neq \emptyset$. Put $\gamma_0 = \min\left\{\gamma < \alpha_0;\ x \in U\left(f_{(\gamma)}\right)\right\}$. If $\alpha_0 = \gamma_0 + 1$, then

$$\left\{\gamma < \alpha_0;\ x \in U\left(f_{(\gamma)}\right)\right\} = \{\gamma_0\}.$$

If $\gamma_0 + 1 < \alpha_0$ and $\gamma_0 < \gamma_1 < \alpha_0$, then

$$x \in U\left(f_{(\gamma_0)}\right) \subset L\left(f_{(\gamma_0)}\right) \subset \bigcup_{0\leq\beta<\gamma_1} L\left(f_{(\beta)}\right).$$

Since $\gamma_1 < \alpha_0$, by (4,$\gamma_1$), $x \in \bigcup_{0\leq\beta<\gamma_1} L\left(f_{(\beta)}\right) \subset C\left(f_{(\gamma_1)}\right)$ and $x \notin U\left(f_{(\gamma_1)}\right)$. Thus $\gamma_1 \notin \left\{\gamma < \alpha_0;\ x \in U\left(f_{(\gamma)}\right)\right\}$. Hence $\left\{\gamma < \alpha_0;\ x \in U\left(f_{(\gamma)}\right)\right\} = \{\gamma_0\}$ and we have (1,$\alpha_0$).

Let $\gamma < \alpha_0$. First, assume that $x \notin \bigcup_{\gamma\leq\beta<\alpha_0} U\left(f_{(\beta)}\right)$. If $\big\{\beta < \alpha_0;\ x \in U(f_{(\beta)})\big\} \neq \emptyset$, then, by (1,$\alpha_0$), there exists $\beta_0 < \gamma$ such that $\big\{\beta < \alpha_0;\ x \in U\left(f_{(\beta)}\right)\big\} = \{\beta_0\}$. Thus, by the definitions of the functions $f_{(\alpha_0)}$ and $f_{(\gamma)}$, we have $f_{(\alpha_0)}(x) = \lim_{t\to x} f_{(\beta_0)}(t) = f_{(\gamma)}(x)$. If $\left\{\beta < \alpha_0;\ x \in U\left(f_{(\beta)}\right)\right\} = \emptyset$, then $f_{(\alpha_0)}(x) = f(x) = f_{(\gamma)}(x)$.

Now, let $x \in \bigcup_{\gamma\leq\beta<\alpha_0} U\left(f_{(\beta)}\right)$. Then, by (1,$\alpha_0$), there exists $\beta_0$ ($\gamma \leq \beta_0 < \alpha_0$) such that $\left\{\beta < \alpha_0;\ x \in U\left(f_{(\beta)}\right)\right\} = \{\beta_0\}$. Thus

$$f_{(\alpha_0)}(x) = \lim_{t\to x} f_{(\beta_0)}(t) \neq f_{(\beta_0)}(x).$$

If $\beta_0 = \gamma$, then $f_{(\alpha_0)}(x) \neq f_{(\gamma)}(x)$. If $\beta_0 > \gamma$, then $x \notin \bigcup_{\gamma\leq\beta<\beta_0} U\left(f_{(\beta)}\right)$ and, by (2,$\beta_0$), $f_{(\gamma)}(x) = f_{(\beta_0)}(x)$. Therefore $f_{(\alpha_0)}(x) \neq f_{(\gamma)}(x)$ and we have (2,$\alpha_0$). Let $\gamma < \alpha_0$ and $x \in L\left(f_{(\gamma)}\right)$. Then $x \in C\left(f_{(\gamma)}\right) \cup U\left(f_{(\gamma)}\right)$. First, we assume that $x \in U\left(f_{(\gamma)}\right)$. Then, by (1,$\alpha_0$), $\left\{\beta < \alpha_0;\ x \in U\left(f_{(\beta)}\right)\right\} = \{\gamma\}$ and, by the definition of $f_{(\alpha_0)}$, we get $f_{(\alpha_0)}(x) = \lim_{t\to x} f_{(\gamma)}(t)$. Now, let $x \in C\left(f_{(\gamma)}\right)$. Then $\lim_{t\to x} f_{(\gamma)}(t) = f_{(\gamma)}(x)$ and $x \notin U\left(f_{(\gamma)}\right)$. If $\eta$ is an ordinal number such that $\gamma < \eta < \alpha_0$, then $x \in C\left(f_{(\gamma)}\right) \subset L\left(f_{(\gamma)}\right) \subset \bigcup_{0\leq\beta<\eta} L\left(f_{(\beta)}\right)$ and, by (4,$\eta$), $x \in C\left(f_{(\eta)}\right)$ and $x \notin U\left(f_{(\eta)}\right)$. Thus $x \notin \bigcup_{\gamma\leq\eta<\alpha_0} U\left(f_{(\eta)}\right)$. Then, by (2,$\alpha_0$), $f_{(\gamma)}(x) = f_{(\alpha_0)}(x)$. Therefore $\lim_{t\to x} f_{(\gamma)}(t) = f_{(\alpha_0)}(x)$ and we have (3,$\alpha_0$).

Let $x_0 \in \bigcup_{0\leq\beta<\alpha_0} L\left(f_{(\beta)}\right)$. Then there exists $\beta_0 < \alpha_0$ such that $x_0 \in L\left(f_{(\beta_0)}\right)$. Therefore, by (3,$\alpha_0$), $\lim_{t\to x_0} f_{(\beta_0)}(t) = f_{(\alpha_0)}(x_0)$. If $\beta_0 = 0$, then, by (4,1),

$$x_0 \in L\left(f_{(0)}\right) \subset \bigcup_{0 \le \eta < 1} L\left(f_{(\eta)}\right) \subset C\left(f_{(1)}\right) \subset L\left(f_{(1)}\right).$$

Thus it may be assumed that $\beta_0 \ge 1$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that, for each $t \in (x_0 - \delta, x_0 + \delta) \cap D$, if $t \ne x_0$, then $|f_{(\beta_0)}(t) - f_{(\alpha_0)}(x_0)| < \frac{\epsilon}{2}$. We shall show that, for each $t \in (x_0 - \delta, x_0 + \delta) \cap D$, $|f_{(\alpha_0)}(t) - f_{(\alpha_0)}(x_0)| < \epsilon$.

Let $t \in (x_0 - \delta, x_0 + \delta) \cap D$. We may assume that $t \ne x_0$ and $f_{(\alpha_0)}(t) \ne f_{(\beta_0)}(t)$. Then, by (2,$\alpha_0$), $t \in \bigcup_{\beta_0 \le \beta < \alpha_0} U\left(f_{(\beta)}\right)$. Let $\beta_1$ be an ordinal number such that $\beta_0 \le \beta_1 < \alpha_0$ and $t \in U\left(f_{(\beta_1)}\right)$. Then, by (3,$\alpha_0$), $\lim_{z \to t} f_{(\beta_1)}(z) = f_{(\alpha_0)}(t)$. Therefore there exists $\eta > 0$ such that $(t - \eta, t + \eta) \subset (x_0 - \delta, x_0 + \delta)$, $x_0 \notin (t - \eta, t + \eta)$ and, for each $z \in (t - \eta, t + \eta) \cap D$, if $z \ne t$, then $|f_{(\beta_1)}(z) - f_{(\alpha_0)}(t)| < \frac{\epsilon}{2}$. Since either $(t - \eta, t) \cap D \ne \emptyset$ or $(t, t + \eta) \cap D \ne \emptyset$, we may assume that $(t - \eta, t) \cap D \ne \emptyset$. If $\beta_1 = \beta_0$, we choose an arbitrary $t_0 \in (t - \eta, t) \cap D$. Now, we assume that $\beta_0 < \beta_1 < \alpha_0$ and let $J = (t - \eta, t) \cap D$. We suppose that $J \subset \bigcup_{\beta_0 \le \beta < \beta_1} U\left(f_{(\beta)}\right)$ and let $\beta_2 = \min\left\{\beta_0 \le \beta < \beta_1;\ J \cap U\left(f_{(\beta)}\right) \ne \emptyset\right\}$. Then, by (1,$\beta_1$), we have, for each $z \in J$, $\left\{\beta < \beta_2;\ z \in U\left(f_{(\beta)}\right)\right\} = \emptyset$; so $f(z) = f_{(\beta_2)}(z)$. Let $z_0 \in U\left(f_{(\beta_2)}\right) \cap J$. Then $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} f_{(\beta_2)}(z) \ne f_{(\beta_2)}(z_0) = f(z_0)$. Thus $z_0 \in U(f)$ and, by (1,$\beta_1$) and $\beta_2 > 0$, $z_0 \notin U\left(f_{(\beta_2)}\right)$, a contradiction. Therefore $J \setminus \bigcup_{\beta_0 \le \beta < \beta_1} U\left(f_{(\beta)}\right) \ne \emptyset$ and we choose $t_0 \in J \setminus \bigcup_{\beta_0 \le \beta < \beta_1} U\left(f_{(\beta)}\right)$. Then, by (2,$\beta_1$), $f_{(\beta_1)}(t_0) = f_{(\beta_0)}(t_0)$. Thus

$$|f_{(\alpha_0)}(t) - f_{(\alpha_0)}(x_0)| \le |f_{(\alpha_0)}(t) - f_{(\beta_0)}(t_0)| + |f_{(\beta_0)}(t_0) - f_{(\alpha_0)}(x_0)| < \epsilon.$$

Hence $x_0 \in C\left(f_{(\alpha_0)}\right)$.

Thus we have shown that, for each ordinal number $\alpha > 0$, the conjunction of these conditions holds, so the proof of the theorem is complete. □

The following remarks can be easily established.

**Remark 2** *Let $f : D \to \mathbb{R}$ and let $\alpha$ be an ordinal number. Then*

$$f_{(\alpha+1)}(x) = \begin{cases} f_{(\alpha)}(x), & \text{if } x \notin U\left(f_{(\alpha)}\right), \\ \lim\limits_{t \to x} f_{(\alpha)}(t), & \text{if } x \in U\left(f_{(\alpha)}\right). \end{cases}$$

**Remark 3** *Let $f : D \to \mathbb{R}$ and let $\alpha$, $\beta$ be ordinal numbers such that $0 \le \alpha < \beta$. Then $C\left(f_{(\alpha)}\right) \subset C\left(f_{(\beta)}\right)$.*

**Definition 4** *For each ordinal number $\alpha$, we denote*

$$\mathcal{A}_\alpha = \left\{f : D \to \mathbb{R};\ C\left(f_{(\alpha)}\right) = D\right\}.$$

We make the following remark.

**Remark 4** *The family* $(\mathcal{A}_\alpha)_{\alpha\geq 0}$ *has the following properties.*

1. $\mathcal{A}_0$ *is the family of all continuous functions on* $D$.
2. *For each ordinal number* $\alpha > 0$, $\bigcup_{0\leq\beta<\alpha}\mathcal{A}_\beta \subset \mathcal{A}_\alpha$.

**Definition 5** *If a function* $f : D \to \mathbb{R}$ *belongs to* $\mathcal{A}_\alpha \setminus \left(\bigcup_{0\leq\beta<\alpha}\mathcal{A}_\beta\right)$, *then it will be called* an $\alpha$-improvable discontinuous function.

**Theorem 2** *Let* $f : D \to \mathbb{R}$. *Then, for every ordinal number* $\alpha$, $C\left(f_{(\alpha)}\right) \subset \mathrm{cl}\left(L(f)\right)$.

PROOF. Let $H = D \setminus \mathrm{cl}\left(L(f)\right)$. Then $H$ is open in $D$. We suppose that $\left\{\beta < \alpha;\ H \cap U\left(f_{(\beta)}\right) \neq \emptyset\right\} \neq \emptyset$. Let $\beta_0 = \min\left\{\beta < \alpha;\ H \cap U\left(f_{(\beta)}\right) \neq \emptyset\right\}$. Since $U(f) \subset L(f)$, we have $\beta_0 > 0$. Then $H \cap \bigcup_{0\leq\gamma<\beta_0} U\left(f_{(\gamma)}\right) = \emptyset$ and, by Theorem 1 (2,$\beta_0$), $H \subset \left\{x \in D;\ f(x) = f_{(\beta_0)}(x)\right\}$. Thus $H \cap U\left(f_{(\beta_0)}\right) = H \cap U(f) = \emptyset$, a contradiction. Therefore $\left\{\beta < \alpha;\ H \cap U\left(f_{(\beta)}\right) \neq \emptyset\right\} = \emptyset$ and $H \cap \bigcup_{0\leq\beta<\alpha} U\left(f_{(\beta)}\right) = \emptyset$. By Theorem 1 (2,$\alpha$), $H \subset \left\{x \in D;\ f(x) = f_{(\alpha)}(x)\right\}$. Then

$$C\left(f_{(\alpha)}\right) \cap H = C(f) \cap H \subset L(f) \cap H = \emptyset \text{ and } C\left(f_{(\alpha)}\right) \subset \mathrm{cl}\left(L(f)\right).$$

**Corollary 1** *Let* $f : D \to \mathbb{R}$ *and let* $\alpha$ *be an ordinal number such that* $C\left(f_{(\alpha)}\right)$ *is a dense subset of* $D$. *Then* $L(f)$ *is also a dense subset of* $D$.

**Definition 6** *Let* $f : D \to \mathbb{R}$. *For each interval* $I = (a, b) \cap D \neq \emptyset$, *the quantity* $\omega(f, I) = \sup_{x\in I} f(x) - \inf_{x\in I} f(x)$ *is called* the oscillation of $f$ on $I$. *For each fixed* $x$, *the function* $\omega\left(f, (x-\delta, x+\delta) \cap D\right)$ *decreases with* $\delta > 0$ *and approaches a limit* $\omega(f, x) = \lim_{\delta\to 0}\omega\left(f, (x-\delta, x+\delta) \cap D\right)$ *called* the oscillation of $f$ at $x$.

We have shown that if $C\left(f_{(\alpha)}\right)$ is a dense subset of $D$, then $L(f)$ is also a dense subset of $D$. We can ask whether $C(f)$ is a dense subset of $D$. The answer in general is negative.

**Proposition 2** *There exists a subset* $D$ *of* $\mathbb{R}$ *and a function* $f : D \to \mathbb{R}$, *such that* $C(f) = \emptyset$ *and* $C\left(f_{(1)}\right) = D$.

PROOF. Let $D = \mathbb{Q}$ where $\mathbb{Q}$ is the set of all rational numbers. Let $\mathbb{Q} = (x_n)_{n=1}^{\infty}$ and $f(x_n) = \frac{1}{n}$, for each $n \in \mathbb{N}$. We observe that, for each $n \in \mathbb{N}$, $f(x_n) > \lim_{t\to x_n} f(t) = 0$; so $x_n \in U(f)$. Hence $C(f) = \emptyset$ and $f_{(1)}(x) = 0$ for each $x \in D$. $\square$

**Theorem 3** *Let $f : D \to \mathbb{R}$ and let $\alpha$ be an ordinal number. If $C\left(f_{(\alpha)}\right) = D$ and $D$ is closed, then the set $C(f)$ is a dense subset of $D$.*

PROOF. We suppose that the set $C(f)$ is not dense in $D$. Then there exists $(a, b)$ such that $(a, b) \cap D \neq \emptyset$ and $(a, b) \cap D \cap C(f) = \emptyset$. Thus

$$(a, b) \cap D \subset \bigcup_{n=1}^{\infty} \left\{x \in D;\ \omega(f, x) \geq \frac{1}{n}\right\}.$$

Since $(a, b) \cap D$ is a set of the second category in $D \cap [a, b]$, there exist $n_0 \in \mathbb{N}$ and an open interval $(c, d) \subset (a, b)$, such that

$$(c, d) \cap D \neq \emptyset \text{ and } (c, d) \cap D \subset \left\{x \in D;\ \omega(f, x) \geq \frac{1}{n_0}\right\}.$$

Therefore $(c, d) \cap D \cap L(f) = \emptyset$ and, by $C\left(f_{(\alpha)}\right) = D$ and Corollary 1, we have a contradiction. □

**Corollary 2** *If $f : \mathbb{R} \to \mathbb{R}$ and $f \in \mathcal{A}_\alpha$, where $\alpha$ is an ordinal number, then $C(f)$ is a dense subset of $\mathbb{R}$.*

It is interesting whether, for each function $f : D \to \mathbb{R}$ such that $C(f)$ is a dense subset of $D$, there exists an ordinal number $\alpha \geq 0$ such that $f \in \mathcal{A}_\alpha$. The answer suggests the following proposition.

**Proposition 3** *There exists a closed $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ such that $C(f)$ is a dense subset of $D$ and there exist no ordinal number $\alpha$ such that $f \in \mathcal{A}_\alpha$.*

PROOF. Put $D = [0, 1]$. Let $K$ be the Cantor set and let $f : D \to \mathbb{R}$ be the characteristic function of $K$. Note that, for each $x \in K$,

$$1 = f(x) = \limsup_{t \to x} f(t) \neq \liminf_{t \to x} f(t) = 0$$

and, for each $x \in D \setminus K$, $\lim_{t \to x} f(t) = f(x) = 0$. Thus $U(f) = \emptyset$ and $f_{(1)}(x) = f(x)$ for each $x \in D$. By Theorem 1 (2,$\alpha$) and by transfinite induction, we have that $f_{(\alpha)}(x) = f(x)$, for each $x \in D$ and for every ordinal number $\alpha$.

**Theorem 4** *For each closed set $D$, for each function $f : D \to \mathbb{R}$ and for every ordinal number $\alpha$, the set $C\left(f_{(\alpha)}\right) \setminus C(f)$ is of the first category in $D$.*

PROOF. Put $V = C\left(f_{(\alpha)}\right) \setminus C(f)$. We suppose that the set $V$ is of the second category in $D$. Since $V$ has the Baire property in $D$, there exists $(a, b) \subset \mathbb{R}$ such that $(a, b) \cap D \neq \emptyset$ and the set $V$ is a residual subset of $I = [a, b] \cap D$.

Therefore $I \cap C(f) \subset I \setminus V$ is a set of the first category in $I$. Since $C\left(f_{(\alpha)}\right) \supset V$ is a dense subset of $I$, we have, by Theorem 2, that $L(f) \cap I$ is also a dense subset of $I$. Therefore as in Theorem 3 we can prove that $C(f) \cap I$ is also a dense subset of $I$. Thus $C(f) \cap I$ is a residual subset of $I$ and $I \setminus C(f)$ is a set of the first category in $I$. Hence $I = (I \setminus C(f)) \cup (I \cap C(f))$ is a set of the first category in $I$, a contradiction. □

**Definition 7** *Let $K \subset D$. Put $K^{(0)} = K$. Let*

$$K^{(1)} = K^d = \{x \in D;\ x \text{ is an accumulation point of } K \text{ in } D\}$$

*and $K^* = K \setminus K^d$. Let $\alpha \geq 1$ be an ordinal number. Then*

- $K^{(\alpha+1)} = \left(K^{(\alpha)}\right)^d$;
- *if $\alpha$ is a limit ordinal number, then $K^{(\alpha)} = \bigcap_{0 \leq \beta < \alpha} K^{(\beta)}$.*

**Definition 8** *Let $f : D \to \mathbb{R}$. Set*

$$r(f) = \min\left\{\alpha;\ f_{(\alpha)}(x) = f_{(\alpha+1)}(x) \ \text{ for each } x \in D\right\}.$$

Now we show that, for each function $f$, $r(f)$ is countable.

**Theorem 5** *If $f : D \to \mathbb{R}$, then $r(f) < \omega_1$.*

PROOF. Let $\alpha = r(f)$. Then $f_{(\alpha)}(x) = f_{(\alpha+1)}(x)$ for each $x \in D$. Let $\beta > \alpha$ and we assume that, for each $\gamma$ with $\alpha < \gamma < \beta$, $f_{(\gamma)}(x) = f_{(\alpha)}(x)$ for each $x \in D$. We suppose that there exists $x_0 \in D$ such that $f_{(\beta)}(x_0) \neq f_{(\alpha)}(x_0)$. Then, by Theorem 1 (2,$\alpha$), $x_0 \in \bigcup_{\alpha \leq \gamma < \beta} U\left(f_{(\gamma)}\right)$. Therefore there exists $\gamma_0$ with $\alpha \leq \gamma_0 < \beta$ such that $x_0 \in U\left(f_{(\gamma_0)}\right)$ and $f_{(\alpha)}(x_0) = f_{(\gamma_0)}(x_0) \neq f_{(\gamma_0+1)}(x_0)$. If $\gamma_0 + 1 < \beta$, we have a contradiction to our assumption. Thus $\gamma_0 + 1 = \beta$. Since $f_{(\gamma_0)}(x) = f_{(\alpha)}(x)$ for each $x \in D$, we have $U\left(f_{(\gamma_0)}\right) = U\left(f_{(\alpha)}\right) = \emptyset$ and $\left\{x \in D;\ f_{(\beta)}(x) \neq f_{(\gamma_0)}(x)\right\} = U\left(f_{(\gamma_0)}\right) = \emptyset$, a contradiction. Hence $f_{(\beta)}(x) = f_{(\alpha)}(x)$ for each $x \in D$ and, for each $\beta > \alpha$, $C\left(f_{(\beta)}\right) = C\left(f_{(\alpha)}\right)$.

Let $D_1 = C\left(f_{(\alpha)}\right)$ and, let for each $\beta \geq 0$, $F_\beta = \left(D_1 \setminus C\left(f_{(\beta)}\right)\right)^d$. Since, for each $\gamma$ with $0 \leq \gamma < \beta$, by Theorem 1 (4,$\beta$),

$$C\left(f_{(\gamma)}\right) \subset L\left(f_{(\gamma)}\right) \subset \bigcup_{0<\xi<\beta} L\left(f_{(\xi)}\right) \subset C\left(f_{(\beta)}\right),$$

. Therefore we have $F_\beta \subset F_\gamma$. Thus, by Theorem 32 (Cantor-Bendixon) [1], there exists an ordinal number $\alpha_0 < \omega_1$ such that if $\gamma > \alpha_0$, then $F_\gamma = F_{\alpha_0}$.

We assume that $\alpha > \alpha_0$. Then $\emptyset = \left(D_1 \setminus C\left(f_{(\alpha)}\right)\right)^d = \left(D_1 \setminus C\left(f_{(\alpha_0)}\right)\right)^d$. We shall show that $\alpha = \alpha_0 + 1$. Let $x_0 \in D_1 \setminus C\left(f_{(\alpha_0)}\right)$. Then there exists an open interval $(a,b)$ such that $D \cap (a,b) \cap \left(D_1 \setminus C\left(f_{(\alpha_0)}\right)\right) = \{x_0\}$. We suppose that there exists a point $x_1 \in ((D \cap (a,b)) \setminus \{x_0\}) \cap \bigcup_{\alpha_0 \le \xi < \alpha} U\left(f_{(\xi)}\right)$. Then there exists an ordinal number $\xi_0$ with $\alpha_0 \le \xi_0 < \alpha$ such that $x_1 \in U\left(f_{(\xi_0)}\right) \subset C\left(f_{(\xi_0+1)}\right) \subset C\left(f_{(\alpha)}\right) = D_1$ and $x_1 \notin C\left(f_{(\alpha_0)}\right) \subset C\left(f_{(\xi_0)}\right)$. Therefore $((D \cap (a,b)) \setminus \{x_0\}) \cap \left(D_1 \setminus C\left(f_{(\alpha_0)}\right)\right) \neq \emptyset$, a contradiction. Hence, by Theorem 1 (2,$\alpha$),

$$(D \cap (a,b)) \setminus \{x_0\} \subset \left\{x \in D;\ f_{(\alpha_0)}(x) = f_{(\alpha)}(x)\right\}.$$

Since $\lim_{t \to x_0} f_{(\alpha)}(t) = f_{(\alpha)}(x_0)$, we have that $\lim_{t \to x_0} f_{(\alpha_0)}(t) = f_{(\alpha)}(x_0)$ and $x_0 \in U\left(f_{(\alpha_0)}\right)$.

We have shown that $D_1 \setminus C\left(f_{(\alpha_0)}\right) \subset U\left(f_{(\alpha_0)}\right) \subset C\left(f_{(\alpha_0+1)}\right)$. Hence

$$C\left(f_{(\alpha)}\right) = \left(D_1 \setminus C\left(f_{(\alpha_0)}\right)\right) \cup C\left(f_{(\alpha_0)}\right) \subset C\left(f_{(\alpha_0+1)}\right) \subset C\left(f_{(\alpha)}\right)$$

and $\alpha = \alpha_0 + 1$. Hence $\alpha = \alpha_0 + 1 < \omega_1$ and the proof is completed. □

**Definition 9** *Put* $\mathcal{A} = \bigcup_{0 \le \alpha < \omega_1} \mathcal{A}_\alpha$. *If a function* $f \in \mathcal{A}$, *then it will be called* an improvable function.

**Definition 10** *For* $A \subset D \subset \mathbb{R}$, *let*

$$\mathcal{M}(A) = \{f : D \to \mathbb{R};\ f(A) = \{0\} \textit{ and, for each } x \in D, f(x) \ge 0\}.$$

The following theorem will be very useful in the paper.

**Theorem 6** *Let* $A$ *be a dense subset of* $D$ *and let* $f \in \mathcal{A}_\alpha$ *be a function such that* $C(f) = A$. *Then* $g = |f - f_{(\alpha)}| \in \mathcal{M}(A)$ *and for each* $0 \le \beta \le \alpha$, $C\left(f_{(\beta)}\right) = C\left(g_{(\beta)}\right)$, $U\left(f_{(\beta)}\right) = U\left(g_{(\beta)}\right)$ *and* $g_{(\beta)} = |f_{(\beta)} - f_{(\alpha)}|$.

PROOF. Assume that $f \in \mathcal{A}_\alpha$. Let $g = |f - f_{(\alpha)}|$. Of course, for each $x \in$D$, g(x) \ge 0$. Let $x \in A$ and $g(x) = |f(x) - f_{(\alpha)}(x)|$. Since $C(f) = A$, by Theorem 1 (2,$\alpha$), for each $x \in A$, $f_{(\alpha)}(x) = f(x)$; so $g(x) = 0$. Thus $g \in \mathcal{M}(A)$.

Now, we show by the transfinite induction that, for each $\beta$ with $0 \le \beta \le \alpha$,

$$C\left(f_{(\beta)}\right) = C\left(g_{(\beta)}\right),\ U\left(f_{(\beta)}\right) = U\left(g_{(\beta)}\right) \text{ and } g_{(\beta)} = |f_{(\beta)} - f_{(\alpha)}|.$$

First, we show that $L(f) = L(g)$. Since $D = C\left(f_{(\alpha)}\right)$, $L(f) \subset L(g)$. Now, we assume that $x_0 \in L(g)$. Since $A$ is a dense subset of $D$ and $g(A) = \{0\}$, $\lim_{t \to x_0} g(t) = 0$. Therefore, by $x_0 \in C\left(f_{(\alpha)}\right) = D$, and

$$0 = \lim_{t \to x_0} g(t) = \lim_{t \to x_0} |f(t) - f_{(\alpha)}(t)|,$$

we have $\lim_{t\to x_0}\left(f(t)-f_{(\alpha)}(t)\right)=0$ and

$$\lim_{t\to x_0} f(t)=\lim_{t\to x_0}\left(f(t)-f_{(\alpha)}(t)+f_{(\alpha)}(t)\right)=f_{(\alpha)}(x_0).$$

Thus there exists $\lim_{t\to x_0} f(t)$ and $x_0\in L(f)$. Hence $L(f)=L(g)$. It is easy to show that $C(f)=C(g)$. Hence, of course, $U(f)=U(g)$. Now, we assume that, for each ordinal number $\xi$ with $0\le\xi<\beta$, we have shown that $C\left(f_{(\xi)}\right)=C\left(g_{(\xi)}\right)$, $U\left(f_{(\xi)}\right)=U\left(g_{(\xi)}\right)$ and $g_{(\xi)}=|f_{(\xi)}-f_{(\alpha)}|$ for each $x\in D$. First, we show that $g_{(\beta)}=|f_{(\beta)}-f_{(\alpha)}|$. Let $x\in D$ be a point such that $\left\{\xi<\beta;\ x\in U\left(g_{(\xi)}\right)\right\}=\emptyset$. Then $\left\{\xi<\beta;\ x\in U\left(f_{(\xi)}\right)\right\}=\emptyset$; so $f(x)=f_{(\beta)}(x)$. Thus $g(x)=|f(x)-f_{(\alpha)}(x)|=|f_{(\beta)}(x)-f_{(\alpha)}(x)|$. If $\xi_0=\min\left\{\xi<\beta;\ x\in U\left(g_{(\xi)}\right)\right\}$, then $x\in U\left(g_{(\xi_0)}\right)$ and, of course, $x\in U\left(f_{(\xi_0)}\right)$. Therefore $\xi_0=\min\left\{\xi<\beta;\ x\in U\left(f_{(\xi)}\right)\right\}$. Thus

$$\lim_{t\to x} g_{(\xi_0)}(t)=\lim_{t\to x}|f_{(\xi_0)}(t)-f_{(\alpha)}(t)|=|f_{(\beta)}(x)-f_{(\alpha)}(x)|.$$

Since

$$g_{(\beta)}(x)=\begin{cases} g(x), & \text{if } \left\{\xi<\beta;\ x\in U\left(g_{(\xi)}\right)\right\}=\emptyset,\\ \lim_{t\to x} g_{(\xi_0)}(t), & \text{if } x\in U\left(g_{(\xi_0)}\right),\\ & \text{where } \xi_0=\min\left\{\xi<\beta;\ x\in U\left(g_{(\xi)}\right)\right\},\end{cases}$$

we have $g_{(\beta)}(x)=|f_{(\beta)}(x)-f_{(\alpha)}(x)|$. Since $C\left(f_{(\alpha)}\right)=D$ and $C(g)=C(f)=A$, we can show that $L\left(f_{(\alpha)}\right)=L\left(g_{(\alpha)}\right)$ and $C\left(f_{(\alpha)}\right)=C\left(g_{(\alpha)}\right)$. Then $U\left(f_{(\alpha)}\right)=U\left(g_{(\alpha)}\right)$. Thus the proof is complete. □

**Corollary 3** *Let $A$ be a dense subset of $D$ and let $f\in\mathcal{A}_\alpha\setminus\bigcup_{0\le\beta<\alpha}\mathcal{A}_\beta$ be a function such that $C(f)=A$. Then, for each $\beta$ with $0\le\beta\le\alpha$,*

$$g_{(\beta)}\in\mathcal{M}\left(C\left(f_{(\beta)}\right)\right).$$

## 3 $\alpha$-improvable Discontinuous Functions

First we give a necessary and sufficient condition under which a set $A$ is the set of all points of continuity of some $\alpha$-improvable discontinuous function.

**Theorem 7** *Let $A$ be a subset of $D$, where $D\subset\mathbb{R}$ is closed. Then the following are equivalent.*

**(1)** *There exists a function $f:D\to\mathbb{R}$ such that $f\in\mathcal{M}(A)\cap\mathcal{A}_\alpha\setminus\bigcup_{0\le\beta<\alpha}\mathcal{A}_\beta$ and $C(f)=A$.*

**(2)** $\operatorname{cl} A = D$ *and there exist two ascending sequences of sets* $(C_\beta)_{0\le\beta\le\alpha}$ *and* $(F_n)_{n=1}^\infty$ *such that* $C_0 = A$, $C_\alpha = D$ *and, for each ordinal number* $\beta$ *with* $0 \le \beta < \alpha$, $C_\beta \ne C_{\beta+1}$ *and*

$$D \setminus \left( \bigcup_{n=1}^{\infty} (F_n \cap C_{\beta+1}) \cup C_\beta \right) = \bigcup_{n=1}^{\infty} \left( F_n \cap \bigcup_{\beta\le\xi<\alpha} (C_{\xi+1} \setminus C_\xi) \right)^d .$$

PROOF. We assume that condition (1) is satisfied. By Theorem 3, $\operatorname{cl} A = D$. For each ordinal number $\beta$ with $0 \le \beta \le \alpha$, put $C_{(\beta)} = C\left(f_{(\beta)}\right)$ and, for each $n \in \mathbb{N}$, $F_n = \left\{x \in D;\ f(x) \ge \frac{1}{n}\right\}$. Then $C_0 = A$, $C_\alpha = D$ and, for each $\beta$ $(0 \le \beta < \alpha)$, $C_\beta \ne C_{\beta+1}$. It is obvious that $(F_n)_{n=1}^\infty$ is an ascending sequence. By Remark 3, we know that the sequence $(C_\beta)_{0\le\beta<\alpha}$ is ascending, also. Let $\beta (0 \le \beta < \alpha)$ be an ordinal number. Since, for each $x \in D$, $f_{(\alpha)}(x) = 0$, by Theorem 1 (2,$\alpha$), we know that

$$\left\{x \in D;\ f_{(\beta)}(x) > 0\right\} = \left\{x \in D;\ f_{(\beta)}(x) \ne f_{(\alpha)}(x)\right\} = \bigcup_{\beta\le\xi<\alpha} U\left(f_{(\xi)}\right).$$

By Theorem 1 (2,$\beta$) and (4,$\beta$), we have that

$$\left\{x \in D;\ f(x) \ne f_{(\beta)}(x)\right\} = \bigcup_{0\le\xi<\beta} U\left(f_{(\xi)}\right) \subset \bigcup_{0\le\xi<\beta} L\left(f_{(\xi)}\right) \subset C\left(f_{(\beta)}\right).$$

Therefore, for each $x \in D \setminus C_\beta$, $f(x) = f_{(\beta)}(x)$.

We shall show that $L\left(f_{(\beta)}\right) = C_\beta \cup \bigcup_{n=1}^\infty (F_n \cap C_{\beta+1})$. Since $C_\beta \subset L\left(f_{(\beta)}\right)$, we suppose that there exists $x_0 \in D \setminus L\left(f_{(\beta)}\right)$ such that $x_0 \in \bigcup_{n=1}^\infty (F_n \cap (C_{\beta+1} \setminus C_\beta))$. Then $f_{(\beta+1)}(x_0) = f_{(\beta)}(x_0) = f(x_0) > 0$ and $x_0 \notin C\left(f_{(\beta+1)}\right) = C_{\beta+1}$, a contradiction.

Therefore $C_\beta \cup \bigcup_{n=1}^\infty (F_n \cap C_{\beta+1}) \subset L\left(f_{(\beta)}\right)$. If $x_0 \in L\left(f_{(\beta)}\right)$, then $x_0 \in C_\beta$ or $x_0 \in C_{\beta+1} \setminus C_\beta$ and there exists $n \in \mathbb{N}$ such that $f_{(\beta)}(x_0) \ge \frac{1}{n}$. Therefore $x_0 \in C_\beta$ or $x_0 \in C_{\beta+1} \setminus C_\beta$ and $f(x_0) \ge \frac{1}{n}$. Hence $x_0 \in C_\beta \cup \bigcup_{n=1}^\infty (F_n \cap C_{\beta+1})$. Thus

$$L\left(f_{(\beta)}\right) = C_\beta \cup \bigcup_{n=1}^{\infty} (F_n \cap C_{\beta+1}).$$

We fix $n \in \mathbb{N}$. Let $H = \bigcup_{\beta \le \xi < \alpha} (C_{\xi+1} \setminus C_\xi)$. Then

$$\begin{aligned}\left\{x \in D;\ f(x) \ge \frac{1}{n}\right\} \cap H &\subset \left\{x \in D;\ f_{(\beta)}(x) \ge \frac{1}{n}\right\} \\ &\subset \left\{x \in D;\ f_{(\beta)}(x) \ge \frac{1}{n}\right\} \cap \bigcup_{\beta \le \xi < \alpha} U\left(f_{(\xi)}\right) \\ &\subset \left\{x \in D;\ f_{(\beta)}(x) \ge \frac{1}{n}\right\} \cap H \\ &\subset \left\{x \in D;\ f(x) \ge \frac{1}{n}\right\} \cap H.\end{aligned}$$

Therefore $\left(\left\{x \in D;\ f_{(\beta)}(x) \ge \frac{1}{n}\right\}\right)^d = \left(F_n \cap \bigcup_{\beta \le \xi < \alpha} (C_{\xi+1} \setminus C_\xi)\right)^d$ and

$$\begin{aligned}D \setminus \left(C_\beta \cup \bigcup_{n=1}^{\infty} (F_n \cap C_{\beta+1})\right) =& D \setminus L\left(f_{(\beta)}\right) \\ =& \left\{x \in D;\ \limsup_{t \to x} f_{(\beta)}(t) > 0\right\} \\ =& \bigcup_{n=1}^{\infty} \left(\left\{x \in D;\ f_{(\beta)}(x) \ge \frac{1}{n}\right\}\right)^d \\ =& \bigcup_{n=1}^{\infty} \left(F_n \cap \bigcup_{\beta \le \xi < \alpha} (C_{\xi+1} \setminus C_\xi)\right)^d.\end{aligned}$$

Hence we have proved condition (2).

Now, we assume that condition (2) holds. Let

$$f(x) = \begin{cases} 0, & \text{if } \left\{m \in \mathbb{N};\ x \in F_m \cap \bigcup_{0 \le \xi < \alpha} (C_{\xi+1} \setminus C_\xi)\right\} = \emptyset, \\ & \text{otherwise,} \\ \frac{1}{n}, & \text{where } n = \min\left\{m \in \mathbb{N};\ x \in F_m \cap \bigcup_{0 \le \xi < \alpha} (C_{\xi+1} \setminus C_\xi)\right\}. \end{cases}$$

We observe that, for each $\beta$ with $0 \le \beta < \alpha$,

$$\left\{x \in D;\ \limsup_{t \to x} f_{|D \setminus C_\beta}(t) > 0\right\} = \bigcup_{n=1}^{\infty} \left(F_n \cap \bigcup_{\beta \le \xi < \alpha} (C_{\xi+1} \setminus C_\xi)\right)^d.$$

Since $f(A) = f(C_0) = \{0\}$ and since $\operatorname{cl} A = D$, we have that

$$\left\{x \in D;\ \liminf_{t \to x} f(t) = 0\right\} = D.$$

We know that

$$\left\{x \in D;\ \limsup_{t\to x} f(t) > 0\right\} = \left\{x \in D;\ \limsup_{t\to x} f_{|D\setminus C_0}(t) > 0\right\}$$
$$= \bigcup_{n=1}^{\infty}\left(F_n \cap \bigcup_{0\le\xi<\alpha}(C_{\xi+1}\setminus C_\xi)\right)^d .$$

Therefore, by our assumption,

$$L(f) = \left\{x \in D;\ \lim_{t\to x} f(t) = 0\right\} = C_0 \cup \bigcup_{n=1}^{\infty}(F_n \cap C_1)$$
$$= C_0 \cup \bigcup_{n=1}^{\infty}(F_n \cap (C_1 \setminus C_0))$$

and $C(f) = C_0$, $U(f) = \bigcup_{n=1}^{\infty}(F_n \cap (C_1\setminus C_0))$. Let $0 \le \beta \le \alpha$. We assume that, for each $\gamma$ with $0 \le \gamma < \beta$,

$$C\left(f_{(\gamma)}\right) = C_\gamma,\ U\left(f_{(\gamma)}\right) = \bigcup_{n=1}^{\infty}(F_n \cap (C_{\gamma+1}\setminus C_\gamma))$$

and $L\left(f_{(\gamma)}\right) = \left\{x \in D;\ \lim_{t\to t} f_{(\gamma)}(t) = 0\right\}$. Let $x \in C_\beta$.

- If $\left\{\gamma < \beta;\ x \in U\ \left(f_{(\gamma)}\right)\right\} \ne \emptyset$, then $f_{(\beta)}(x) = \lim_{t\to x} f_{(\gamma_0)}(t) = 0$ where $\gamma_0 = \min\left\{\gamma < \beta;\ x \in U\ \left(f_{(\gamma)}\right)\right\}$.
- If $\left\{\gamma < \beta;\ x \in U\ \left(f_{(\gamma)}\right)\right\} = \emptyset$, then, for each $\gamma$ with $0 \le \gamma < \beta$, $x \notin U\left(f_{(\gamma)}\right) = \bigcup_{n=1}^{\infty}(F_n \cap (C_{\gamma+1}\setminus C_\gamma))$ and, by $x \in C_\beta$, we have that $x \notin \bigcup_{n=1}^{\infty}\left(F_n \cap \bigcup_{\beta\le\xi<\alpha}(C_{\xi+1}\setminus C_\xi)\right)$.

Therefore $x \notin \bigcup_{n=1}^{\infty}\left(F_n \cap \bigcup_{0\le\xi<\alpha}(C_{\xi+1}\setminus C_\xi)\right)$ and $x \notin \bigcup_{0\le\xi<\beta} U\left(f_{(\xi)}\right)$. So $f_{(\beta)}(x) = f(x) = 0$. Hence $f_{(\beta)}(C_\beta) = \{0\}$. Since $A = C_0 \subset C_\beta$ and $\operatorname{cl} A = D$, therefore $\left\{x \in D;\ \liminf_{t\to x} f_{(\beta)}(t) = 0\right\} = D$. We observe that

$$\left\{x \in D;\ \limsup_{t\to x} f_{(\beta)}(t) > 0\right\} = \left\{x \in D;\ \limsup_{t\to x} f_{|(D\setminus C_\beta)\ (\beta)}(t) > 0\right\}.$$

By Theorem 1 (2,$\beta$);

$$\left\{x \in D;\ f_{(\beta)}(x) \ne f(x)\right\} = \bigcup_{0\le\xi<\beta} U\left(f_{(\xi)}\right) \subset \bigcup_{0\le\xi<\beta} C_{\xi+1} \subset C_\beta.$$

Therefore

$$\left\{x \in D; \limsup_{t\to x} f_{|(D\setminus C_\beta)\ (\beta)}(t) > 0\right\} = \left\{x \in D;\ \limsup_{t\to x} f_{|D\setminus C_\beta}(t) > 0\right\}$$
$$= \bigcup_{n=1}^{\infty}\left(F_n \cap \bigcup_{\beta\le\xi<\alpha}(C_{\xi+1}\setminus C_\xi)\right)^d.$$

Then, by our assumption, we know that

$$L\left(f_{(\beta)}\right) = \left\{x \in D;\ \lim_{t\to x} f_{(\beta)}(t) = 0\right\} = C_\beta \cup \bigcup_{n=1}^{\infty}(F_n \cap C_{\beta+1})$$
$$= C_\beta \cup \bigcup_{n=1}^{\infty}(F_n \cap (C_{\beta+1}\setminus C_\beta)).$$

Thus $C\left(f_{(\beta)}\right) = C_\beta$ and $U\left(f_{(\beta)}\right) = \bigcup_{n=1}^{\infty}(F_n \cap (C_{\beta+1}\setminus C_\beta))$.

We shall show that, for each $x \in D$, $f_{(\alpha)}(x) = 0$. If there exists $\beta_0$ with $0 \le \beta_0 < \alpha$ such that $x \in U\left(f_{(\beta_0)}\right)$, then $f_{(\alpha)}(x) = \lim_{t\to x} f_{(\beta_0)}(t) = 0$ where $\beta_0 = \min\left\{\beta < \alpha;\ x \in U\left(f_{(\beta)}\right)\right\}$. If $\left\{\beta < \alpha;\ x \in U\left(f_{(\beta)}\right)\right\} = \emptyset$, then, for each $\beta$ with $0 \le \beta < \alpha$, $x \notin \bigcup_{n=1}^{\infty}(F_n \cap (C_{\beta+1}\setminus C_\beta))$. Therefore $x \notin \bigcup_{n=1}^{\infty}\left(F_n \cap \bigcup_{0\le\beta<\alpha}(C_{\beta+1}\setminus C_\beta)\right)$ and $f_{(\alpha)}(x) = f(x) = 0$. Hence $f \in \mathcal{A}_\alpha \setminus \bigcup_{0\le\beta<\alpha}\mathcal{A}_\beta$ and the proof of the theorem is complete. □

**Corollary 4** *Let $(C_\beta)_{0\le\beta\le\alpha}$ be an ascending sequence of sets such that $cl C_0 = \mathbb{R}$, $C_\alpha = \mathbb{R}$. Let $H$ be an arbitrary set such that, for each ordinal number $\beta$ with $0 \le \beta < \alpha$,*

$$D \setminus ((H \cap C_{\beta+1}) \cup C_\beta) = \left(H \cap \bigcup_{\beta\le\xi<\alpha}(C_{\xi+1}\setminus C_\xi)\right)^d$$

*and $C_\beta \ne C_{\beta+1}$. Then the characteristic function of the set $H$ belongs to the class $\mathcal{A}_\alpha \setminus \bigcup_{0\le\beta<\alpha}\mathcal{A}_\beta$.*

Proof. For each $n \in \mathbb{N}$, let $F_n = H$. Then, as in the proof of Theorem 10, we can prove that the characteristic function of the set $H$ belongs to the class $\mathcal{A}_\alpha$ and, for each ordinal number $\beta$ with $0 \le \beta < \alpha$, $C_\beta = C\left(f_{(\beta)}\right)$ and $U\left(f_{(\beta)}\right) = H \cap (C_{\beta+1}\setminus C_\beta)$. Since, by our assumption, for each ordinal number $\beta$ with $0 \le \beta < \alpha$, $C\left(f_{(\beta)}\right) \ne C\left(f_{(\beta+1)}\right)$, we have that $f \notin \bigcup_{0\le\beta<\alpha}\mathcal{A}_\beta$. Thus the proof is complete. □

The following theorem shows that we can construct an $\alpha$-improvable discontinuous function for each $\alpha < \omega_1$. To prove this theorem we need the following lemma.

**Lemma 1** *Let* $A = \bigcup_{n=1}^{\infty} A_n \cup \{0\}$ *where, for each* $n \in \mathbb{N}$, $A_n$ *is a closed set,* $A_n \subset \left[\frac{1}{n+1}, \frac{1}{n}\right]$, $\frac{1}{n+1} \in A_n$ *and* $\frac{1}{n}$ *is a left-side isolated point in the set* $A$. *Then, for each ordinal number* $\alpha$, $A^{(\alpha)} \setminus \{0\} = \bigcup_{n=1}^{\infty} A_n^{(\alpha)}$.

PROOF. If $\alpha = 0$, then the lemma is true.

Let $\alpha > 0$ be an ordinal number and we assume that, for each ordinal number $\beta$ with $0 \leq \beta < \alpha$, $A^{(\beta)} \setminus \{0\} = \bigcup_{n=1}^{\infty} A_n^{(\beta)}$. Consider two possibilities.

1. Let $\alpha = \gamma + 1$, where $\gamma$ is an ordinal number and let $x_0 \in A^{(\alpha)} \setminus \{0\}$. Then there exists $n \in \mathbb{N}$ such that $x_0 \geq \frac{1}{n+1}$ and there exists a sequence $(x_k)_{k=1}^{\infty} \subset A^{(\gamma)}$ such that $\lim_{k\to\infty} x_k = x_0$. Since $\frac{1}{n+1}$ is a left-side isolated point of $A$ and $A^{(\gamma)} \subset A$, there exists $k_0 \in \mathbb{N}$ such that, for each $k > k_0$, $x_k \geq \frac{1}{n+1}$. Hence $(x_k)_{k=1}^{\infty} \subset \bigcup_{i=1}^{n} A_i^{(\gamma)}$; so $x_0 \in \left(\bigcup_{i=1}^{n} A_i^{(\gamma)}\right)^d = \bigcup_{i=1}^{n} A_i^d \subset \bigcup_{n=1}^{\infty} A_n^d$. Thus $A^{(\alpha)} \setminus \{0\} \subset \bigcup_{n=1}^{\infty} A_n^{(\alpha)}$.

   Since, for each $n \in \mathbb{N}$, $A_n \subset A$; so $A_n^{(\alpha)} \subset A^{(\alpha)}$. Hence $\bigcup_{n=1}^{\infty} A_n^{(\alpha)} \subset A^{(\alpha)}$ and since, for each $n \in \mathbb{N}$, $0 \notin A_n$, for each $n \in \mathbb{N}$, $0 \notin A_n^{(\alpha)}$; so $0 \notin \bigcup_{n=1}^{\infty} A_n^{(\alpha)}$. Thus $\bigcup_{n=1}^{\infty} A_n^{(\alpha)} \subset A^{(\alpha)} \setminus \{0\}$.

2. Let $\alpha$ be a limit ordinal number and let $x_0 \in A^{(\alpha)} \setminus \{0\}$. Then there exists $n \in \mathbb{N}$ such that $x_0 \geq \frac{1}{n+1}$. Let $\gamma < \alpha$ be an ordinal number. Then $x_0 \in A^{(\gamma+1)}$. Thus there exists a sequence $(x_k)_{k=1}^{\infty} \subset A^{(\gamma)}$ such that $\lim_{k\to\infty} x_k = x_0$. As above we can show that $x_0 \in A_n^{(\gamma+1)}$. Hence $x_0 \in \bigcap_{\gamma<\alpha} A_n^{(\gamma+1)} \subset \bigcap_{\gamma<\alpha} A_n^{(\gamma)} = A_n^{(\alpha)}$. Thus $x_0 \in \bigcup_{n=1}^{\infty} A_n^{(\alpha)}$. Similarly to the first part, we can show that $\bigcup_{n=1}^{\infty} A_n^{(\alpha)} \subset A^{(\alpha)} \setminus \{0\}$.

Thus the proof is complete. □

**Theorem 8** *For each ordinal number* $\alpha < \omega_1$, *there exists a function* $f \in \mathcal{A}_\alpha \setminus \bigcup_{0\leq\beta<\alpha} \mathcal{A}_\beta$.

PROOF. For each set $A \subset \mathbb{R}$ and $a, b \in \mathbb{R}$, let $aA + b = \{ax + b; x \in A\}$. By transfinite induction, we define a sequence of sets $(W_\alpha)_{0\leq\alpha<\omega_1}$ in the following way: $W_0 = \emptyset$, $W_1 = \{0\}$, $W_2 = \left\{\frac{1}{n};\ n \in \mathbb{N}\right\} \cup \{0\}$ and, for each ordinal number $\alpha$ with $3 \leq \alpha < \omega_1$,

1. if $\alpha = \gamma + 2$, where $\gamma$ is an ordinal number, then put

$$[W_\alpha = \bigcup_{n=1}^{\infty} \left(\frac{1}{n(n+1)} W_{\gamma+1} + \frac{1}{n+1}\right) \cup \{0\},$$

2. if $\alpha$ is a limit ordinal number, then
$$W_\alpha = \bigcup_{n=1}^{\infty} \left( \frac{1}{n(n+1)} W_{\alpha_n} + \frac{1}{n+1} \right) \cup \{0\}$$
where $(\alpha_n)_{n=1}^{\infty}$ is a sequence of ordinal numbers such that $\lim_{n\to\infty} \alpha_n = \alpha$ and, for each $n \in \mathbb{N}$, $\alpha_n < \alpha$ and $\alpha_n$ is not a limit ordinal number,
3. if $\alpha = \gamma + 1$, where $\beta$ is a limit ordinal number, then put $W_\alpha = W_\gamma$.

We shall show that, for each ordinal number $\alpha$ with $0 \le \alpha < \omega_1$,

**(i)** $W_\alpha$ is a closed nowhere dense set and $W_\alpha \subset [0,1]$,

**(ii)** if $\alpha > 1$, then, for each $n \in \mathbb{N}$, $\frac{1}{n} \in W_\alpha$ and there exists $\delta_n^{(\alpha)} > 0$ such that $\left( \frac{1}{n} - \delta_n^{(\alpha)}, \frac{1}{n} \right) \cap W_\alpha = \emptyset$,

**(iii)** if $\alpha > 0$, then, for each $\beta$ with $0 \le \beta < \alpha$, $0 \in W_\alpha^{(\beta)}$,

**(iv)** if $\alpha$ is not a limit ordinal number, then $W_\alpha^{(\alpha)} = \emptyset$ and if $\alpha$ is a limit ordinal number, then $W_\alpha^{(\alpha)} = \{0\}$.

The above conditions are obvious for $\alpha = 0, 1, 2$. Let $\alpha$ with $2 < \alpha < \omega_1$ be an ordinal number. We assume that conditions (i), (ii), (iii), (iv) are satisfied for each ordinal number $\beta < \alpha$.

1. We assume that $\alpha = \gamma + 2$, where $\gamma$ is an ordinal number. Since $W_{\gamma+1}$ is a closed nowhere dense set and $W_{\gamma+1} \subset [0,1]$, for each $n \in \mathbb{N}$, $\frac{1}{n(n+1)} W_{\gamma+1} + \frac{1}{n+1}$ is a closed nowhere dense set and $\frac{1}{n(n+1)} W_{\gamma+1} + \frac{1}{n+1} \subset \left[ \frac{1}{n+1}, \frac{1}{n} \right]$. Therefore $W_\alpha$ is a closed nowhere dense set and $W_\alpha \subset [0,1]$.
Let $n \in \mathbb{N}$. Since $1 \in W_{\gamma+1}$, we obtain
$$\frac{1}{n} = \frac{1}{n(n+1)} + \frac{1}{n+1} \in \frac{1}{n(n+1)} W_{\gamma+1} + \frac{1}{n+1} \subset W_\alpha.$$
By our assumption, there exists $\delta_1^{(\gamma+1)} > 0$ such that $\left( 1 - \delta_1^{(\gamma+1)}, 1 \right) \cap W_{\gamma+1} = \emptyset$. We put $\delta_n^{(\alpha)} = \frac{1}{n(n+1)} \delta_1^{(\gamma+1)}$. Then $\left( \frac{1}{n} - \delta_n^{(\alpha)}, \frac{1}{n} \right) \cap W_\alpha = \emptyset$.
Let $\beta$ be an ordinal number such that $0 \le \beta < \alpha$. By the above, we have that the assumptions of Lemma 1 are satisfied. Therefore
$$\begin{aligned} W_\alpha^{(\beta)} \setminus \{0\} &= \bigcup_{n=1}^{\infty} \left( \frac{1}{n(n+1)} W_{\gamma+1} + \frac{1}{n+1} \right)^{(\beta)} \\ &= \bigcup_{n=1}^{\infty} \left( \frac{1}{n(n+1)} W_{\gamma+1}^{(\beta)} + \frac{1}{n+1} \right). \end{aligned}$$

By our assumption, $0 \in W_{\gamma+1}^{(\gamma)}$. Therefore, for each $n \in \mathbb{N}$, $\frac{1}{n+1} \in \frac{1}{n(n+1)} W_{\gamma+1}^{(\gamma)} + \frac{1}{n+1} \subset W_\alpha^{(\gamma)}$. Thus $0 \in W_\alpha^{(\gamma+1)} \subset W_\alpha^{(\beta)}$. We know that $W_{\gamma+1}^{(\gamma+1)} = \emptyset$. Hence

$$W_\alpha^{(\gamma+1)} \setminus \{0\} = \bigcup_{n=1}^{\infty} \left( \frac{1}{n(n+1)} W_{\gamma+1}^{(\gamma+1)} + \frac{1}{n+1} \right) = \emptyset$$

and $W_\alpha^{(\alpha)} = \emptyset$.

2. Now we assume that $\alpha$ is a limit ordinal number. As to above we may show that conditions (i) and (ii) are satisfied. Additionally, by Lemma 1 , we have that, for each ordinal number $\beta$ with $0 \leq \beta \leq \alpha$,

$$W_\alpha^{(\beta)} \setminus \{0\} = \bigcup_{n=1}^{\infty} \left( \frac{1}{n(n+1)} W_{\alpha_n}^{(\beta)} + \frac{1}{n+1} \right).$$

Let $0 \leq \beta < \alpha$. Then there exists $n_0 \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, $n \geq n_0$, $\alpha_n > \beta$. By our assumption, for each $n \geq n_0$, $0 \in W_{\alpha_n}^{(\beta)}$ and $\frac{1}{n+1} \in \frac{1}{n(n+1)} W_{\alpha_n}^{(\beta)} + \frac{1}{n+1} \subset W_\alpha^{(\beta)}$. Thus $0 \in W_\alpha^{(\beta+1)} \subset W_\alpha^{(\beta)}$ and $0 \in \bigcap_{0 \leq \beta < \alpha} W_\alpha^{(\beta)} = W_\alpha^{(\alpha)}$. We know that, for each $n \in \mathbb{N}$, $W_{\alpha_n}^{(\alpha)} \subset W_{\alpha_n}^{(\alpha_n)} = \emptyset$. Therefore $W_\alpha^{(\alpha)} \setminus \{0\} = \emptyset$. Hence $W_\alpha^{(\alpha)} = \{0\}$.

3. Now we assume that $\alpha = \gamma + 1$, where $\gamma$ is a limit ordinal number. It is obvious that conditions (i), (ii), (iii) are satisfied. Additionally

$$W_\alpha^{(\alpha)} = W_\gamma^{(\alpha)} = \left( W_\gamma^{(\gamma)} \right)^d = (\{0\})^d = \emptyset.$$

Now, we consider the following possibilities.

1. Let $\alpha = \gamma + 2$, where $\gamma$ is an ordinal number. In Corollary 4, we put $H = W_\alpha$ and, for each ordinal number $\beta$ with $0 \leq \beta \leq \alpha$, $C_\beta = \mathbb{R} \setminus W_\alpha^{(\beta)}$. Then

$$\left( H \cap \bigcup_{\beta \leq \xi < \alpha} (C_{\xi+1} \setminus C_\xi) \right)^d = \left( W_\alpha^{(\beta)} \right)^d$$

and $\mathbb{R} \setminus ((H \cap C_{\beta+1}) \cup C_\beta) = W_\alpha^{(\beta+1)}$. Therefore the characteristic function of the set $H$ belongs to the class $\mathcal{A}_\alpha \setminus \bigcup_{0 \leq \beta < \alpha} \mathcal{A}_\beta$.

2. Let $\alpha$ be a limit ordinal number. Put $H = W_\alpha \setminus \{0\}$ and, $C_0 = \mathbb{R} \setminus W_\alpha$, for each ordinal number $\beta$ with $0 \leq \beta < \alpha$, $C_\beta = \mathbb{R} \setminus W_\alpha^{(\beta)}$, $C_\alpha = \mathbb{R}$. We

show that all assumptions of Corollary 4 are satisfied. We observe that

$$\bigcup_{\beta\le\xi<\alpha}(C_{\xi+1}\setminus C_\xi)=\bigcup_{\beta\le\xi<\alpha}\left(W_\alpha^{(\xi)}\setminus W_\alpha^{(\xi+1)}\right).$$

Since $W_\alpha^{(\alpha)}=\{0\}$, we have $\bigcup_{\beta\le\xi<\alpha}\left(W_\alpha^{(\xi)}\setminus W_\alpha^{(\xi+1)}\right)=W_\alpha^{(\beta)}\setminus\{0\}$. Thus

$$\begin{aligned}\left(H\cap\bigcup_{\beta\le\xi<\alpha}(C_{\xi+1}\setminus C_\xi)\right)^d&=\left(\left(W_\alpha\cap W_\alpha^{(\beta)}\right)\setminus\{0\}\right)^d\\&=\left(W_\alpha^{(\beta)}\right)^d=W_\alpha^{(\beta+1)}.\end{aligned}$$

Since $\mathbb{R}\setminus((H\cap C_{\beta+1})\cup C_\beta)=\{0\}\cup W_\alpha^{(\beta+1)}=W_\alpha^{(\beta+1)}$, by Corollary 4, we have that, the characteristic function of the set $H$ belongs to the class $\mathcal{A}_\alpha\setminus\bigcup_{0\le\beta<\alpha}\mathcal{A}_\beta$.

3. Let $\alpha=\gamma+1$, where $\gamma$ is a limit ordinal number. Put $H=W_\alpha$ and, for each ordinal number $\beta$ with $0\le\beta\le\alpha$, $C_\beta=\mathbb{R}\setminus W_\alpha^{(\beta)}$. As in the first part, we can show that the characteristic function of the set $H$ belongs to $\mathcal{A}_\alpha\setminus\bigcup_{0\le\beta<\alpha}\mathcal{A}_\beta$.

Thus the proof is complete. □

## References

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