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ON THE CHORD SET OF CONTINUOUS FUNCTIONS

It is well-known that for a given continuous function f, $f(0) = f(1)$ and for any natural number n there exist x_n, $y_n = x_n + 1/n$ such that $f(x_n) = f(y_n)$. It is also known that if the graph of f (or more generally a planar curve connecting the point 0 and 1) does not have a horizontal chord of length a and b respectively then there is no horizontal chord of length $a + b$ either (see [1]). It is almost immediate that the lengths of possible horizontal chords of f form a closed set F of the unit interval [0,1], and according to the remark above its complement $G = [0, 1] \setminus F$ is an additive set: $a \in G$, $b \in G$, $a + b \leq 1$ imply $a + b \in G$. C. Ryll-Nardzewski, Z. Romanowicz and M. Morayne raised the problem whether this additive property is not just necessary but also sufficient for a set to be the complement of the chord-set of some continuous function.

In this paper we answer their question affirmatively by proving the following theorem.

Theorem 1 *Let $F \subset [0, 1]$ be a closed set, and put $G = [0, 1] \setminus F$. Suppose that $0, 1 \in F$ and if $x, y \in G$, $x + y \leq 1$, then $x + y \in G$. Then there is a continuous function f defined on $[0, 1]$ such that $\{y - x : x, y \in [0, 1],\ x < y,\ f(x) = f(y)\} = F$.*

Proof. Let
$$[0,1] = \left(\bigcup_n G_n\right) \cup \left(\bigcup_k F_k\right) \cup (\partial F)\,,$$
where G_n and F_k are disjoint open intervals, $\cup_n G_n = G$, $\cup_k F_k = \operatorname{int} F$ and ∂F is the boundary of F.

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We define $f(x) = 0$ if $x \in \partial F$ (in particular, $f(0) = f(1) = 0$), $f(x) = \operatorname{dist}(x, [0,1] \setminus F_k)$ if $x \in F_k$, and $f(x) = -\operatorname{dist}(x, [0,1] \setminus G_n)$ if $x \in G_n$. We claim that f satisfies the requirements.

f is clearly continuous (moreover, Lipschitz 1) on $[0,1]$. Let $x, y \in [0,1]$, $x < y$, $f(x) = f(y)$. We prove that $y - x \in F$. First we show that

$$\text{if} \quad x, y \in \partial F \quad \text{and} \quad x < y \quad \text{then} \quad y - x \in F. \tag{$*$}$$

Indeed, if $x = 0$ then $y - x = y \in \partial F \subset F$. If $0 < x < 1$ then let $x_n \to x$, $x_n \in G$. Then $y - x_n \notin G$ (since $x_n \in G$, $y - x_n \in G$ would imply $y \in G$). Thus $y - x_n \in F$ and $y - x = \lim(y - x_n) \in F$, as F is closed.

If $f(x) = f(y) = 0$ then $x, y \in \partial F$ and thus $y - x \in F$ by $(*)$. Therefore we may assume that $f(x) = f(y) \neq 0$. Since $f > 0$ in $\operatorname{int} F$ $f < 0$ in G, and $f = 0$ in ∂F, this implies that either $x, y \in \operatorname{int} F$ or $x, y \in G$.

Suppose first that $x, y \in F_k$ for some k. If $h = y - x \in G$ then $n \cdot h \in G$ for every $n \leq 1/h$, which is impossible, since $h < |F_k|$ and thus $n \cdot h \in F_k$ for some n.

Next suppose that $x, y \in G_n = (u, v)$ for some n. Then $h = y - x < |G_n|$ and thus $v - h \in G_n \subset G$. If $h \in G$ then $v = h + (v - h) \in G$ which is impossible, since $v \in \partial F \subset F$.

Thus we may assume that $x \in (a, b)$ and $y \in (c, d)$, where (a, b) and (c, d) are different components of $\operatorname{int} F$ or G. We shall consider the following cases separately.

(i) $x \leq (a+b)/2$, $y \leq (c+d)/2$ and (a,b), $(c,d) \subset G$;

(ii) $x \leq (a+b)/2$, $y > (c+d)/2$ and (a,b), $(c,d) \subset G$;

(iii) $x > (a+b)/2$, $y > (c+d)/2$ and (a,b), $(c,d) \subset G$;

(iv) $x > (a+b)/2$, $y \leq (c+d)/2$ and (a,b), $(c,d) \subset G$;

(v) $x \leq (a+b)/2$, $y \leq (c+d)/2$ and (a,b), $(c,d) \subset \operatorname{int} F$;

(vi) $x \leq (a+b)/2$, $y > (c+d)/2$ and (a,b), $(c,d) \subset \operatorname{int} F$;

(vii) $x > (a+b)/2$, $y > (c+d)/2$ and (a,b), $(c,d) \subset \operatorname{int} F$;

(viii) $x > (a+b)/2$, $y \leq (c+d)/2$ and (a,b), $(c,d) \subset \operatorname{int} F$.

If (i), (iii), (v) or (vii) holds then $y - x = c - a$ or $y - x = d - b$. Since $a, b, c, d \in \partial F$, this implies $y - x \in F$ by $(*)$. In the sequel we shall denote

$$u = \begin{cases} a, & \text{if } x \leq (a+b)/2, \\ b, & \text{if } x > (a+b)/2 \end{cases}; \qquad v = \begin{cases} c, & \text{if } x \leq (c+d)/2, \\ d, & \text{if } x > (c+d)/2 \end{cases}.$$

Let $\delta = |x-u| = |y-v| = |f(x)| = |f(y)|$.

Case (ii): $v \in F$, $u+2\delta-\varepsilon \in G \Longrightarrow v-(u+2\delta-\varepsilon) \in F$; $v-(u+2\delta-\varepsilon) \to v-u-2\delta = y-x \in F$. Case (iv): $v \in F$, $u-2\delta+\varepsilon \in G \Longrightarrow v-(u-2\delta+\varepsilon) \in F$; $v-(u-2\delta+\varepsilon) \to v-u+2\delta = y-x \in F$. Case (vi): Either $u=0$, and then $y-x = v-2\delta \in F$; or $\exists a_n \to u$, $a_n \in G$, and then $v-2\delta \in F$, $\Longrightarrow v-2\delta-a_n \in F$, $(v-2\delta)-a_n \to v-u-2\delta = y-x \in F$. Case (viii) $\exists a_n \to u$, $a_n \in G$, $v+2\delta \in F \Longrightarrow (v+2\delta)-a_n \in F$, $(v+2\delta)-a_n \to v+2\delta-u = y-x \in F$.

This completes the first part of the proof ($f(x)=f(y) \Longrightarrow y-x \in F$). Next we show that for every $d \in F$ there are $x, y \in [0,1]$ such that $x<y$, $y-x=d$ and $f(x)=f(y)$.

This is clear if $G=\emptyset$; so that we may assume $G \neq \emptyset$. If $(a,b)=G_n$ then for every $0 \le c \le b-a$ there are points $a \le x \le y \le b$ such that $y-x=c$ and $f(x)=f(y)$. As we proved above, this implies $c \in F$ for every $c \in [0,b-a]$. Therefore $g = \inf G > 0$. Then $(0,g)$ is (one of the) longest component of $\operatorname{int} F$, since there are elements of G arbitrarily close to g, and the integer multiples of these elements also belong to G.

If $d \in \partial F$ then $x=0$, $y=d$ satisfy the requirements. Next let $d \in \operatorname{int} F$, $d \in (a,b) = F_k$. We have $f(d)-f(0) = f(d) > 0$ and $f(b)-f(b-d) = -f(b-d) < 0$, since $b-d < b-a \le g$ and f is positive on $(0,g)$. Now f is continuous, and thus $f(y)-f(y-d)$ must vanish for a $y \in [d,b]$, completing the proof. $\square$

References

[1] A. M. Yaglom and I. M. Yaglom, *Non-elementary problems in elementary presentation*, GITTL, Moscow, 1954, Problem 118, p. 60 (in Russian).