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Vasile Ene, Ovidius University Constanţa, Romania
Current address: 23 August 8717, Jud. Constanţa, Romania
e-mail: ene@univ-ovidius.ro

RIESZ TYPE THEOREMS FOR GENERAL INTEGRALS

Abstract

The author gives a general descriptive definition for integration, denoted by $\mathcal{P}$, which has as special cases the Lebesgue integral for bounded measurable functions, the Lebesgue integral, the Denjoy-Perron integral $\mathcal{D}^*$, the wide Denjoy integral $\mathcal{D}$, the Foran integral, the Iseki integral and the $S\mathcal{F}$-integral ([5]). This $\mathcal{P}$-integral will admit Riesz type representation theorems (introducing an Alexiewicz norm, and identifying f with g whenever $f = g$ *a.e.* on $[a, b]$). The classical Riesz representation theorem for the linear and continuous functionals on $(C([a, b]), \|\cdot\|_\infty)$ is a consequence of Theorem 2.

In addition it is shown that the space of $\mathcal{P}$-integrable functions is of the first category in itself (see Section 5). Also a characterization of the weak convergence on this space is given.

1 Introduction

Our purpose is to define a suitable general descriptive definition for integration, denoted by $\mathcal{P}$, which has as special cases the Lebesgue integral for bounded measurable functions, the Lebesgue integral, the Denjoy- Perron integral $\mathcal{D}^*$, the wide Denjoy integral $\mathcal{D}$, the Foran integral, the Iseki integral and the $S\mathcal{F}$-integral ([5]). This $\mathcal{P}$-integral will admit Riesz type representation theorems, i.e., introducing an Alexiewicz norm on the space of all $\mathcal{P}$-integrable functions, and identifying f with g whenever $f = g$ *a.e.* on $[a, b]$ (see Section 3) we obtain a characterization of the linear and continuous functionals on this space, see

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Lemma 4 and Theorem 2. The proof of Theorem 2 is based on Theorem 1, and to prove Theorem 1 we use a technique of [12], p. 75. As a consequence of Theorem 2 it follows the classical Riesz representation theorem for the linear and continuous functionals on $(C([a,b]), \|\cdot\|_\infty)$.

Further, we also prove that the space of $\mathcal{P}$-integrable functions is of the first category in itself (see Section 5) and we give a characterization of the weak convergence on it.

2 Essentially Bounded Variation and the Bounded Slope Variation

Definition 1. ([14]). Let $P \subset [a,b]$ be a set of positive measure, and let $f : P \to \overline{\mathbb{R}}$ be a measurable function, finite *a.e.*.

- f is said to be essentially upper bounded if there exists a real number M such that the set $\{x \in P \,:\, f(x) > M\}$ has measure zero.
- f is said to be essentially lower bounded if the function $-f$ is essentially upper bounded.
- f is said to be essentially bounded if it is simultaneously essentially upper bounded and essentially lower bounded, i.e., there exists $M > 0$ such that the set $\{x \in P \,:\, |f(x)| > M\}$ is of measure zero.
- Let $\sup_{ess}(f;P) = \inf\{M \,:\, \{x \in P \,:\, f(x) > M\}$ has measure zero$\}$ if f is essentially upper bounded and $\sup_{ess}(f;P) = +\infty$ if not. Define $\inf_{ess}(f;P)$ similarly .
- Let $\mathcal{O}_{ess}(f;P) = \sup_{ess}(f;P) - \inf_{ess}(f;P)$.
- Let $\mathcal{O}_{ess}(f;X) = 0$, whenever X is a null subset of P.
- f is said to be of essentially bounded variation (abbreviated $f \in EVB$) on P, if there exists $M > 0$ such that $\sum_{i=1}^{n} \mathcal{O}_{ess}(f;[a_i,b_i] \cap P) < M$, whenever $[a_i,b_i], i = 1,2,\ldots,n$ are nonoverlapping closed intervals with endpoints in P.
- Let $EV(f;P) = \inf\{M \,:\, M$ is as above$\}$ if $f \in EVB$ on P and let $EV(f;P) = +\infty$ if not.
- Let $V(f;P) = \inf\{M \,:\, \sum_{i=1}^{n}(f(b_i) - f(a_i))/(b_i - a_i) < M$, whenever $[a_i,b_i], i = 1,2,\ldots,n$ are nonoverlapping closed intervals with endpoints in $P\}$ if $f \in VB$ on P and let $V(f;P) = +\infty$ otherwise.

Lemma 1. *Let P be a dense subset of $[a,b]$ and let $f : P \to \mathbb{R}$, $f \in VB$. Then there exists $\tilde{f} : [a,b] \to \mathbb{R}$ such that $\tilde{f} \in VB$ on $[a,b]$, $\tilde{f}_{|P} = f$ and $V(\tilde{f};[a,b]) = V(f;P)$.*

PROOF. Let $x \in [a,b) \setminus P$. Then $\lim_{y \searrow x, y \in P} f(y)$ exists and is finite (because f is bounded on P). Suppose that the above limit does not exist; then there exists $\epsilon_o > 0$ such that whenever $\delta > 0$ there exist $x^{'}, x^{''} \in (x, x+\delta) \cap P$ such that $|f(x^{'}) - f(x^{''})| \geq \epsilon_o$. For $\delta = 1$ there exist $a_1, b_1 \in (x, x+1) \cap P$, $a_1 < b_1$ such that $|f(a_1) - f(b_1)| \geq \epsilon_o$. For $\delta = a_1 - x$ there exist $a_2, b_2 \in (x, x+\delta) \cap P$, $a_2 < b_2$ such that $|f(a_2) - f(b_2)| \geq \epsilon_o$. Inductively we obtain a sequence $\{[a_n, b_n]\}$, $n = 1, 2, \ldots$, of nonoverlapping closed intervals with endpoints in P such that $b_1 > a_1 > b_2 > a_2 > \cdots b_n > a_n \ldots$ and $|f(a_n) - f(b_n)| \geq \epsilon_o$. Therefore $\sum_{n=1}^{\infty} |f(a_n) - f(b_n)| = \infty$, which contradicts the fact that f is VB on P. Similarly we can prove that $\lim_{y \nearrow x, y \in P} f(y)$ exists and is finite whenever $x \in (a,b]$. Let $\lim_{y \searrow x, y \in P} f(y) = f(x+)$ and $\lim_{y \nearrow x, y \in P} f(y) = f(x-)$. Define $\tilde{f} : [a,b] \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in P \\ f(x+) & \text{if } x \in [a,b) \setminus P \\ f(b-) & \text{if } b \notin P \end{cases}$$

Then $\tilde{f}_{|P} = f$ and $V(f;P) \leq V(\tilde{f};[a,b])$. Suppose on the contrary that $V(f;P) < V(\tilde{f};[a,b])$. Let $\epsilon > 0$ be such that $\epsilon + V(f;P) < V(\tilde{f};[a,b])$. Then there exists $a = t_0 < t_1 < \cdots < t_n = b$ such that $\sum_{i=1}^{n} |\tilde{f}(t_i) - \tilde{f}(t_{i-1})| > \epsilon + V(f;P)$. We may suppose without loss of generality that each t_i does not belong to P. Then for each t_i with $i = 0, 1, \cdots n-1$ it follows that there exists $t_i^{'} \in (t_i, t_{i+1}) \cap P$ such that $|\tilde{f}(t_i) - \tilde{f}(t_i{'})| < \epsilon/(4n)$ and for t_n there exists $t_n^{'} \in (t_{n-1}, t_n) \cap P$ such that $|\tilde{f}(t_n) - \tilde{f}(t_n{'})| < \epsilon/(4n$. Therefore $\epsilon + V(f;P) < \sum_{i=1}^{n} |\tilde{f}(t_i) - \tilde{f}(t_{i-1})| = \sum_{i=1}^{n} |\tilde{f}(t_i) - f(t_i{'}) + f(t_i{'}) - f(t_{i-1}^{'}) + f(t_{i-1}^{'}) - \tilde{f}(t_{i-1})| < 2n \cdot \epsilon/(4n) + \sum_{i=1}^{n} |f(t_i{'}) - f(t_{i-1}^{'}| < \epsilon/2 + V(f;P)$, a contradiction. $\square$

Lemma 2. *Let $f : [a,b] \to \overline{\mathbb{R}}$ be a measurable function. The following assertions are equivalent:*

(i) $f \in EVB$ *on* $[a,b]$,

(ii) There exists $\tilde{f} : [a,b] \to \mathbb{R}$, *such that* $\tilde{f} \in VB$ *and* $\tilde{f} = f$ *a.e. on* $[a,b]$. *Moreover* $EV(f;[a,b]) \leq V(\tilde{f};[a,b]) \leq 2 \cdot EV(f;[a,b])$.

PROOF. $(i) \Rightarrow (ii)$ We may suppose that $[a,b] = [0,1]$. For $n \geq 2$ let

$$\pi_n^{'} = \left\{ \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right] \right\}, \quad i = 0, 1, 2, \ldots, 2^n - 1;$$

$$\pi_n^{''} = \left\{ \left[0, \frac{1}{2^n} \right], \left[\frac{2i-1}{2^n}, \frac{2i+1}{2^n} \right], \left[\frac{2^n - 1}{2^n}, 1 \right] \right\}, \quad i = 1, 2, \ldots, 2^{n-1} - 1;$$

$$M_{n,i}^{'} = \sup_{ess} \left(f; \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right] \right), \quad i = 0, 1, 2, \ldots, 2^n - 1;$$

$$m_{n,i}^{'} = \inf_{ess} \left(f; \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right] \right), \quad i = 0, 1, 2, \ldots, 2^n - 1;$$

$$M_{n,i}^{''} = \sup_{ess} \left(f; \left[\frac{2i-1}{2^n}, \frac{2i+1}{2^n} \right] \right) \quad i = 1, 2, \ldots, 2^{n-1} - 1;$$

$$m_{n,i}^{''} = \inf_{ess} \left(f; \left[\frac{2i-1}{2^n}, \frac{2i+1}{2^n} \right] \right), \quad i = 1, 2, \ldots, 2^{n-1} - 1;$$

$$A_{n,i}^{'} = \left\{ x \in \left(\frac{i}{2^n}, \frac{i+1}{2^n} \right) : m_{n,i}^{'} \le f(x) \le M_{n,i}^{'} \right\}, \quad i = 0, 1, 2, \ldots, 2^n - 1\,;$$

$$A_{n,i}^{''} = \left\{ x \in \left(\frac{2i-1}{2^n}, \frac{2i+1}{2^n} \right) : m_{n,i}^{''} \le f(x) \le M_{n,i}^{''} \right\}, \ i = 1, 2, \ldots, 2^{n-1} - 1;$$

$$A_n^{'} = \cup_{i=1}^{2^n - 1} A_{n,i}^{'};$$

$$A_n^{''} = A_{n,0}^{'} \cup A_{n,2^n-1}^{'} \cup \left(\cup_{i=1}^{2^{n-1}-1} A_{n,i}^{''} \right).$$

For example, each set

$$\left\{ x \in \left(\frac{i}{2^n}, \frac{i+1}{2^n} \right) : f(x) > M_{n,i}^{'} \right\} =$$

$$\bigcup_{k=1}^{\infty} \left\{ x \in \left(\frac{i}{2^n}, \frac{i+1}{2^n} \right) : f(x) \ge M_{n,i}^{'} + \frac{1}{k} \right\},$$

$i = 1, 2, \ldots 2^n - 1$, is therefore a countable union of null sets. Thus $A_n^{'}$, $A_n^{''}$ are measurable sets and $|A_n^{'}| = |A_n^{''}| = 1$. Let $A = \cap_{n=2}^{\infty} (A_n^{'} \cup A_n^{''})$. Then A is measurable and $|A| = 1$.

We show that $F \in VB$ on A. Since $F \in EVB$ on $[a, b]$, there exists $M > 0$ such that

$$\sum_{i=0}^{2^n - 1} \mathcal{O}_{ess} \left(f; \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right] \right) = \sum_{i=0}^{2^n - 1} (M_{n,i}^{'} - m_{n,i}^{'}) < M, \ n \ge 2 \text{ and}$$

$$\sum_{i=1}^{2^{n-1}-1} \mathcal{O}_{ess}\left(f; \left[\frac{2i-1}{2^n}, \frac{2i+1}{2^n}\right]\right) = \sum_{i=1}^{2^{n-1}-1} (M''_{n,i} - m''_{n,i}) < M,\ n \geq 2.$$

Let $\{[\alpha_k, \beta_k]\}$, $k = 1, 2, \ldots, p$ be a finite set of nonoverlapping closed intervals with endpoints in A. Then there exists a positive integer n_o such that each α_k and β_k is contained in the interior of exactly one component interval of the partition π'_{n_o}. Let $x_0 < x_1 < x_2 < \ldots < x_{2^{n_o-1}}$ be such that

$$x_i \in A \cap \left(\frac{i}{2^{n_o}}, \frac{i+1}{2^{n_o}}\right),\ i = 0, 1, \ldots, 2^{n_o} - 1, \text{ and}$$

$$\{x_0, x_1, \cdots x_{2^{n_o}-1}\} \supseteq \{\alpha_1, \beta_1, \alpha_2, \beta_2 \cdots \alpha_n, \beta_n\}.$$

Clearly

$$\sum_{k=1}^{p} |f(\beta_k) - f(\alpha_k)| \leq \sum_{k=1}^{p} \sum_{[x_{i-1}, x_i] \subseteq [\alpha_k, \beta_k]} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{2^{n_o}-1} |f(x_i) - f(x_{i-1})|.$$

But

$$\sum_{i=0}^{2^{n_o}-1} |f(x_{2i+1}) - f(x_{2i})| < \sum_{i=0}^{2^{n_o-1}-1} (M'_{n_o-1,i} - m'_{n_o-1,i}) < M$$

(because $2i/2^{n_o} < x_{2i} < (2i+1)/2^{n_o} < x_{2i+1} < (2i+2)/2^{n_o}$, so $i/(2^{n_o-1}) < x_{2i} < x_{2i+1} < (i+1)/(2^{n_o-1})$) and

$$\sum_{i=1}^{2^{n_o-1}-1} |f(x_{2i}) - f(x_{2i-1})| < \sum_{i=1}^{2^{n_o-1}-1} (M''_{n_o,i} - m''_{n_o,i}) < M$$

(because $(2i-1)/2^{n_o} < x_{2i-1} < 2i/2^{n_o} < x_{2i} < (2i+1)/2^{n_o}$). It follows that $\sum_{k=1}^{p} |f(b_k) - f(a_k)| < 2M$, hence $f \in VB$ on A. Moreover $V(f; A) \leq 2 \cdot EV(f; [a, b])$. By Lemma 2, it follows that there exists $\tilde{f} : [a, b] \to \mathbb{R}$ such that $\tilde{f} = f$ on A and $V(\tilde{f}; [a, b]) = V(f; A)$. Therefore $V(\tilde{f}; [a, b]) \leq 2 \cdot EV(f; [a, b])$.

$(ii) \Rightarrow (i)$ Let $M > 0$ be given by the fact that $\tilde{f} \in VB$ on $[a, b]$. Let $\{[a_i, b_i]\}$, $i = 1, 2, \ldots, n$ be a set of nonoverlapping closed subintervals of $[a, b]$. Then $M > \sum_{i=1}^{n} \mathcal{O}(f; A \cap [a_i, b_i]) \geq \sum_{i=1}^{n} \mathcal{O}_{ess}(f; [a_i, b_i])$. It follows that $f \in EVB$ on $[a, b]$. Moreover $EV(f; [a, b]) \leq V(\tilde{f}; [a, b])$. $\square$

Remark 1. Lemma 2 is in fact an observation of [14] (p. 81). It was used for example in the proof of Sargent's Theorem 50 (see [3], p. 45) without demonstration, but with the warning of Peter Bullen (see [3], p. 309) that a more complete proof of it is in [16].

Definition 2. A function $F : [a,b] \to \mathbb{R}$ is said to be of bounded slope variation (abbreviated $F \in BSV$) on a subset P of $[a,b]$, if there exists $M > 0$ such that

$$\sum_{i=1}^{n} \left| \frac{F(b_{2i}) - F(a_{2i})}{b_{2i} - a_{2i}} - \frac{F(b_{2i-1}) - F(a_{2i-1})}{b_{2i-1} - a_{2i-1}} \right| < M$$
$$\text{whenever } a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_{2n} < b_{2n} \text{ are points in } P. \quad (1)$$

Let $SV(F;P) = \inf\{M : (1) \text{ holds.}\}$ If $F \notin BSV$ on P let $SV(F;P) = +\infty$.

Lemma 3. *Let $F : [a,b] \to \mathbb{R}$. The following assertions are equivalent:*

(i) $F \in BSV$ *on* $[a,b]$.

(ii) There exists $M > 0$ such that

$$\sum_{i=0}^{n-2} \left| \frac{F(x_{i+2}) - F(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} \right| < M,$$

whenever $a = x_o < x_1 < x_2 < \ldots < x_n = b$.

PROOF. $(i) \Rightarrow (ii)$ Let M be given by the fact that $F \in BSV$ on $[a,b]$. We have

$$\begin{aligned}
&\sum_{i=0}^{n-2} \left| \frac{F(x_{i+2}) - F(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} \right| \\
&= \sum_{\substack{i=0 \\ i=even}}^{n-2} \left| \frac{F(x_{i+2}) - F(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} \right| \\
&= \sum_{\substack{i=0 \\ i=odd}}^{n-2} \left| \frac{F(x_{i+2}) - F(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} \right| < M + M = 2M.
\end{aligned}$$

$(ii) \Rightarrow (i)$ We may suppose without loss of generality that $a < a_1 < b_1 < a_2 < b_2 < \ldots < a_{2n} < b_{2n} < b$. Let's rename these points $a = x_0 < x_1 < x_2 <$

$\ldots < x_{4n+1} = b$. Then we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left|\frac{F(b_{2i})-F(a_{2i})}{b_{2i}-a_{2i}}-\frac{F(b_{2i-1})-F(a_{2i-1})}{b_{2i-1}-a_{2i-1}}\right| \\
&\le \sum_{i=1}^{n}\left|\frac{F(b_{2i})-F(a_{2i})}{b_{2i}-a_{2i}}-\frac{F(a_{2i})-F(b_{2i-1})}{a_{2i}-b_{2i-1}}\right| \\
&\quad + \sum_{i=1}^{n}\left|\frac{F(a_{2i})-F(b_{2i-1})}{a_{2i}-b_{2i-1}}-\frac{F(b_{2i-1})-F(a_{2i-1})}{b_{2i-1}-a_{2i-1}}\right| \\
&\le \sum_{i=0}^{4n-1}\left|\frac{F(x_{i+2})-F(x_{i+1})}{x_{i+2}-x_{i+1}}-\frac{F(x_{i+1})-F(x_{i})}{x_{i+1}-x_{i}}\right| < M\,.
\end{aligned}$$

□

Remark 2. Lemma 3, (ii) is in fact Definition 12.5 of [12] (p. 74) for the condition BSV on $[a,b]$.

Theorem 1. *With the above notations we have the following results:*

(i) Let $f : [a,b] \to \mathbb{R}$, $f \in EBV$ and let $F(x) = (\mathcal{L})\int_a^x f(t)\,dt$. Then $F \in BSV$ on $[a,b]$ and $SV(F;[a,b]) \le EV(f;[a,b])$.

(ii) Let $F : [a,b] \to \mathbb{R}$, $F \in BSV$ and let

$$F^*(x) = \begin{cases} F'(x) & \text{where } F \text{ is derivable} \\ 0 & \text{elsewhere.} \end{cases}$$

Then F satisfies the Lipschitz condition L, $F^* \in EBV$ on $[a,b]$ and $EV(F^*;[a,b]) \le SV(F;[a,b])$.

PROOF. (i) Clearly f is essentially bounded on $[a,b]$; so f is summable on $[a,b]$ and $F(x) = (\mathcal{L})\int_a^x f(t)\,dt$ is Lipschitz. Let $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_{2n} < b_{2n} \le b$. We have

$$\inf_{ess}(f;[a_{2i-1},b_{2i}]) \le \frac{F(b_{2i})-F(a_{2i})}{b_{2i}-a_{2i}} \le \sup_{ess}(f;[a_{2i-1},b_{2i}])$$

and

$$\inf_{ess}(f;[a_{2i-1},b_{2i}]) \le \frac{F(b_{2i-1})-F(a_{2i-1})}{b_{2i-1}-a_{2i-1}} \le \sup_{ess}(f;[a_{2i-1},b_{2i}]).$$

Hence

$$\left|\frac{F(b_{2i})-F(a_{2i})}{b_{2i}-a_{2i}}-\frac{F(b_{2i-1})-F(a_{2i-1})}{b_{2i-1}-a_{2i-1}}\right|<\mathcal{O}_{ess}(f;[a_{2i-1},b_{2i}]).$$

Let $\epsilon>0$. Then

$$\sum_{i=1}^{n}\left|\frac{F(b_{2i})-F(a_{2i})}{b_{2i}-a_{2i}}-\frac{F(b_{2i-1})-F(a_{2i-1})}{b_{2i-1}-a_{2i-1}}\right|<$$

$$\sum_{i=1}^{n}\mathcal{O}_{ess}(F;[a_{2i-1},b_{2i})<(\epsilon+EV(f;[a,b])).$$

Hence $F\in BSV$ on $[a,b]$. Since ϵ was arbitrary, $SV(F;[a,b])\le EV(f;[a,b])$.

(ii) We show that F is bounded on $[a,b]$. Suppose for example that F is upper unbounded. Then there exists a sequence $\{x_n\}_n$ such that $F(x_n)>n$ for each n. For $F(x_n)>\max\{|F(a)|,|F(b)|\}$ we have

$$\left|\frac{F(b)-F(x_n)}{b-x_n}-\frac{F(x_n)-F(a)}{x_n-a}\right|>\frac{F(x_n)-F(a)}{x_n-a}>\frac{n-a}{b-a}\to+\infty.$$

Hence $F\notin BSV$ on $[a,b]$, a contradiction.

Suppose on the contrary that $F\notin L$ on $[a,b]$. For each positive integer n, there exist x_n, $y_n\in[a,b]$, $x_n<y_n$, such that $|F(y_n)-F(x_n)|/(y_n-x_n)>n$. Since F is bounded, $y_n-x_n\to 0$. But $\{x_n\}_n$ is a bounded sequence; so it contains a convergent subsequence. Hence, we may suppose without loss of generality that $\{x_n\}_n$ converges to x_o. Then $\{y_n\}_n$ converges to x_o too. We have two cases:

1) If $x_o=a$, then there exists n_o such that $y_n<(a+b)/2$ for each $n\ge n_o$. It follows that $[x_n,y_n]$ and $[(a+b)/2,b]$ are nonoverlapping closed intervals for each $n\ge n_o$. We have

$$\left|\frac{F(b)-F((a+b)/2)}{(b-a)/2}-\frac{F(y_n)-F(x_n)}{y_n-x_n}\right|\to+\infty,\quad n\to\infty.$$

This contradicts the fact that $F\in BSV$ on $[a,b]$.

2) If $x_o\ne a$, then there exists n_o such that $x_n>(a+x_o)/2$, for each $n>n_o$. It follows that $[a,(a+x_o)/2]$ and $[x_n,y_n]$ are nonoverlapping closed intervals for each $n\ge n_o$. We have

$$\left|\frac{F(y_n)-F(x_n)}{y_n-x_n}-\frac{F((a+x_o)/2)-F(a)}{(x_o-a)/2}\right|\to+\infty,\quad n\to\infty.$$

This contradicts the fact that $F \in BSV$ on $[a, b]$.

Therefore we have obtained that $F \in L$ on $[a, b]$. It follows that F is derivable a.e. on $[a, b]$. Let $A = \{x \in [a, b] : F$ is derivable at $x\}$. Clearly $F^* = F'$ on A. We show that $F' \in VB$ on A. Let $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n$ be points in A. For $\epsilon > 0$ let $[c_i, d_i] \subset (a_i, b_i)$ such that

$$\left|\frac{F(c_i) - F(a_i)}{c_i - a_i} - F'(a_i)\right| < \frac{\epsilon}{2n} \text{ and } \left|\frac{F(b_i) - F(d_i)}{b_i - d_i} - F'(b_i)\right| < \frac{\epsilon}{2n}.$$

We have

$$\begin{aligned}
&\sum_{i=1}^{n} |F'(b_i) - F'(a_i)| \le \sum_{i=1}^{n} \left|\frac{F(c_i) - F(a_i)}{c_i - a_i} - F'(a_i)\right| \\
&+ \sum_{i=1}^{n} \left|\frac{F(b_i) - F(d_i)}{b_i - d_i} - F'(b_i)\right| + \sum_{i=1}^{n} \left|\frac{F(c_i) - F(a_i)}{c_i - a_i} - \frac{F(b_i) - F(d_i)}{b_i - d_i}\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} + (\epsilon + SV(F; [a, b])).
\end{aligned}$$

Therefore $F^* \in VB$ on A. Since ϵ was arbitrary, $V(F^*; A) \le SV(F; [a, b])$. Let $\{[a_i, b_i]\}$, $i = 1, 2, \cdots n$ be a set of nonoverlapping closed intervals of $[a, b]$. Then $V(F^*; A) \ge \sum_{i=1}^{n} \mathcal{O}(F^*; A \cap [a_i, b_i]) \ge \sum_{i=1}^{n} \mathcal{O}_{ess}(F^*; [a_i, b_i])$. Therefore $EV(F^*; [a, b]) \le SV(F; [a, b])$. □

Corollary 1. *A function $F : [a, b] \to \mathbb{R}$ is the indefinite Lebesgue integral of a VB function $f : [a, b] \to \mathbb{R}$, if and only if $F \in BVS$ on $[a, b]$.*

Proof. See Theorem 1 and Lemma 2. □

Remark 3. If in Corollary 1 "$F \in BVS$" is replaced by "$F \in BSV \cap L$" we obtain a result of Riesz (Lemma 12.6 of [12], p.75). As we see from our Theorem 1 "$F \in BSV \cap L$" is superfluous, because "$BSV \in L$". Let's mention that in the prove of Theorem 1 we used some techniques of Riesz' lemma.

3 A General Descriptive definition for Integration

Definition 3. A class of functions $\mathcal{P}([a, b]) \subset \{F : [a, b] \to \mathbb{R} : F$ is continuous on $[a, b]$ and approximately derivable $a.e.$ on $[a, b]\}$ is called a general class of primitives on $[a, b]$ if it satisfies the following properties :

(i) $\mathcal{P}([a, b])$ is a real linear space ;

(ii) If $F'_{ap} = G'_{ap}$ $a.e.$ on $[a, b]$ and $F, G \in \mathcal{P}([a, b])$, then $F - G$ is a constant on $[a, b]$;

(iii) If $F \in \mathcal{P}([a,b])$ and $g : [a,b] \to \mathbb{R}$ is a VB function on $[a,b]$, then $H \in \mathcal{P}([a,b])$, where $H(x) = F(x) \cdot g(x) - (\mathcal{RS}) \int_a^x F(t)\, dg(t)$ and $(\mathcal{RS})$ stands for the Riemann-Stieltjes integral.

(iv) $\mathcal{P}([a,b])$ contains the class $Lip([a,b]) = \{F : [a,b] \to \mathbb{R} : F \text{ is Lipschitz}\}$.

Definition 4. A function $f : [a,b] \to \overline{\mathbb{R}}$ is said to be $\mathcal{P}$-integrable on $[a,b]$ if there exists a function $F : [a,b] \to \mathbb{R}$ such that $F'_{ap}(x) = f(x)$ *a.e.* on $[a,b]$. We will write $(\mathcal{P}) \int_a^b f(t)dt = F(b) - F(a)$. We refer to F as $\mathcal{P}$-primitive of f on $[a,b]$.

Remark 4. Note the following:

(i) From Definition 3 (ii) it follows that the $\mathcal{P}$-integral is well defined.

(ii) By Definition 3 (i) it follows that the set of all $\mathcal{P}$-integrable functions on $[a,b]$ is a real linear space.

(iii) If $f : [a,b] \to \mathbb{R}$ is $\mathcal{P}$-integrable, then f is measurable (see [15], p. 299).

(iv) We will define on the set of all $\mathcal{P}$-integrable functions on $[a,b]$ an equivalence relation : $f \sim g$ if $f(x) = g(x)$ *a.e.* on $[a,b]$.

(v) We denote the set of all classes of equivalence with $\mathcal{P}_{int}([a,b])$. With the usual operations with classes the set $\mathcal{P}_{int}([a,b])$ becomes a real linear space. We shall denote the equivalence class of f also by f.

(vi) Let $\mathcal{P}_o([a,b]) = \{F : [a,b] \to \mathbb{R} \,:\, F \in \mathcal{P}([a,b]), F(a) = 0\}$.

(vii) Formula $\|F\|_\infty = \sup_{x\in[a,b]} |F(x)|$ defines a norm on each of the following linear spaces: $\mathcal{P}_o([a,b])$, $\mathcal{P}([a,b])$, $C([a,b])$, $C_o([a,b])$ (here $C([a,b]) = \{f : [a,b] \to \mathbb{R} \,:\, f \text{ is continuous}\}$ and $C_o([a,b]) = \{f : [a,b] \to \mathbb{R} \,:\, f$ is continuous and $f(a) = 0\}$).

(viii) Let $f \in \mathcal{P}_{int}([a,b])$ and let $F \in \mathcal{P}_o([a,b])$ be the unique $\mathcal{P}$ primitive of f. The formula $\|f\| = \|F\|_\infty$ defines a norm on $\mathcal{P}_{int}([a,b])$.

(ix) We denote by $VB([a,b]) = \{g : [a,b] \to \mathbb{R} \,:\, g \in VB \text{ on } [a,b]\}$. With the usual operations with functions and with the norm $\|g\|_{VB} = |g(b)| + V_a^b(g)$, the set $VB([a,b])$ becomes a real Banach space.

4 Riesz representation theorems for the $\mathcal{P}$ integration

Definition 5. Let $\langle \cdot\,,\,\cdot\rangle : \mathcal{P}_{int}([a,b]) \times VB([a,b]) \to \mathbb{R}$ be defined by the formula $\langle f,g\rangle = (\mathcal{P})\int_a^b f(t)g(t)dt$. (That $f\cdot g$ is $\mathcal{P}$-integrable on $[a,b]$ follows by Definition 3 (iii) and the fact that $H^{'}_{ap}(x) = f(x)g(x)$ *a.e.* on $[a,b]$, see the proof of Theorem 5.23.2 of [5].)

Lemma 4. *Let $f \in \mathcal{P}_{int}([a,b])$ and $g \in VB([a,b])$. Then we have :*

(i) $\langle\cdot,\cdot\rangle$ *is bilinear*

(ii) $|\langle f,g\rangle| \le \|f\|\cdot\|g\|_{VB}$

(iii) $T : \mathcal{P}_{int}([a,b]) \to \mathbb{R}$, $T(f) = \langle f,g\rangle$ *is a continuous linear functional and* $\|T\| \le \|g\|_{VB}$.

PROOF. By Definitions 3 and 5, $\langle f,g\rangle = F(b)g(b) - (\mathcal{RS})\int_a^b F(t)\,dg(t)$, where $F \in \mathcal{P}_o([a,b])$ is the unique $\mathcal{P}$-primitive of f.

(i) This follows by the fact that the $\mathcal{RS}$-integral is linear in the first argument and in the second argument.

(ii) We have $|\langle f,g\rangle| = |F(b)g(b) - (\mathcal{RS})\int_a^b F(t)\,dg(t)| \le |F(b)|\cdot|g(b)| + \|F\|_\infty \cdot V(g;[a,b]) \le \|F\|_\infty\cdot(|g(b)| + V(g;[a,b])) = \|f\|\cdot\|g\|_{VB}$.

(iii) This follows by (i) and (ii). □

Lemma 5. *Let $(X,\|\cdot\|_1)$ and $(Y,\|\cdot\|_2)$ be normed real spaces and let $\langle\cdot\,,\,\cdot\rangle : X \times Y \to \mathbb{R}$ be such that:*

a) $\langle\cdot,y\rangle$ *is linear in the first variable, for each* $y \in Y$*;*

b) $|\langle x,y\rangle| \le \|x\|_1\cdot\|y\|_2$, *whenever* $x \in X$, $y \in Y$.

If $f : X \to \mathbb{R}$ is a continuous linear functional and if there exist $y_o \in Y$ and a dense subset X_o of X such that $f(x) = \langle x,y_o\rangle$ for each $x \in X_o$, then $f(x) = \langle x,y_o\rangle$ on X and $\|f\| \le \|y_o\|_2$.

PROOF. Since $\overline{X}_o = X$, for $x \in X$ there exists a sequence $\{x_n\}_n \subset X_o$ such that $\|x_n - x\|_1 \to 0$, for $n \to \infty$. But $|\langle x_n,y_o\rangle - \langle x,y_o\rangle| = |\langle x_n - x,y_o\rangle| \le \|x_n - x\|_1\cdot\|y_o\|_2$ (see a)and b)). Since f is continuous, $f(x) = \lim_{n\to\infty}\langle x_n,y_o\rangle = \langle x,y_o\rangle$. Hence $f(x) = \langle x,y_o\rangle$, for each $x \in X$ and $\|f\| \le \|y_o\|_2$ (see a) and b)). □

Theorem 2. *Let $T : \mathcal{P}_{int}([a,b]) \to \mathbb{R}$ be a continuous linear functional. Then there exists $g \in VB$ such that*

$$T(f) = \langle f,g\rangle = (\mathcal{P})\int_a^b f(t)g(t)\,dt \quad \textit{and} \tag{2}$$

$$\frac{1}{2}V(g;[a,b]) \le \|L\| \le \|g\|_{VB}. \tag{3}$$

PROOF. Let

$$\mathcal{S}([a,b]) = \{s : [a,b] \to \mathbb{R} : s \text{ is a step function of the form } s(t) = \sum_{i=1}^{n-1} \alpha_i K_{[t_{i-1},t_i)} + \alpha_n K_{[t_{n-1},t_n]} \text{ for some positive integer } n, \text{ where each } \alpha_i \in \mathbb{R},\ a = t_0 < t_1 < \ldots < t_n = b\}.$$

(Here K_E denotes the characteristic function of the set E.) We show that $\overline{\mathcal{S}([a,b])} = \mathcal{P}_{int}([a,b])$. Let $f \in \mathcal{P}_{int}([a,b])$ and let $F \in \mathcal{P}_o([a,b]$ the unique primitive of f. Then $F(x)$ is continuous on $[a,b]$. Let $a = x_0 < x_1 < \ldots < x_n = b$, $x_i - x_{i-1} = (b-a)/n$ for each $i = 1,2,\ldots,n$. Let $F_n(x_i) = F(x_i)$, $i = 0,1,\ldots,n$ and let F_n be linear on each closed interval $[x_{i-1},x_i]$. Then $F_n \xrightarrow{[unif]} F$ on $[a,b]$ and each F_n is Lipschitz. By Definition 3 (iv), each F_n is in $\mathcal{P}_o([a,b])$. Let

$$s_n(x) = \begin{cases} \frac{F(x_i)-F(x_{i-1})}{x_i - x_{i-1}} & \text{if } x \in [x_{i-1},x_i),\ i = 1,2,\ldots,n-1 \\ \frac{F(x_n)-F(x_{n-1})}{x_n - x_{n-1}} & \text{if } x \in [x_{n-1},x_n] \end{cases}$$

Then $s_n \in \mathcal{S}([a,b])$ and $\|s_n - f\| = \|F_n - F\|_\infty \to 0$ (because $F_n \xrightarrow{[unif]}$ on $[a,b]$). Therefore $\mathcal{S}([a,b])$ is dense in $\mathcal{P}_{int}([a,b])$.

Let $G(t) = T(K_{[a,t]})$ and let $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_{2n} < b_{2n} \le b$. Since T is linear and continuous,

$$\sum_{i=1}^{n} \left| \frac{G(b_{2i}) - G(a_{2i})}{b_{2i} - a_{2i}} - \frac{G(b_{2i-1}) - G(a_{2i-1})}{b_{2i-1} - a_{2i-1}} \right| =$$

$$\sum_{i=1}^{n} |T(\varphi_i)| = \sum_{i=1}^{n} \epsilon_i T(\varphi_i) = T(\sum_{i=1}^{n} \epsilon \varphi_i) \le \|T\| \cdot \|\sum_{i=1}^{n} \epsilon_i \varphi_i\| \le \|T\|$$

where $\epsilon_i = \operatorname{sign} T(\varphi_i)$ and

$$\varphi_i = \frac{1}{b_{2i} - a_{2i}} \cdot K_{(a_{2i},b_{2i}]} - \frac{1}{b_{2i-1} - a_{2i-1}} \cdot K_{(a_{2i-1},b_{2i-1}]}.$$

It follows that $G \in BSV$ and

$$SV(G;[a,b]) \le \|T\|. \tag{4}$$

By Theorem 1, (ii) $G^* \in EBV$ and

$$EV(G^*,[a,b]) \leq SV(G;[a,b]). \tag{5}$$

By Lemma 2 it follows that there exists a function $g : [a,b] \to \mathbb{R}$ such that $g \in VB$, $g = G^*$ *a.e.* on [a,b] and

$$EV(G^*;[a,b]) \leq V(g;[a,b]) \leq 2 \cdot EV(G^*;[a,b]) \tag{6}$$

Clearly $G(t) = (\mathcal{L}) \int_a^t G^*(x)dx = (\mathcal{L}) \int_a^b K_{[a,t]}(x)G^*(x)dx = T(K_{[a,t]})$. Since T is linear, it follows that $T(s) = \langle s, g\rangle$ whenever $s \in \mathcal{S}([a,b])$. By Lemma 5 we have $T(f) = \langle f, g\rangle$ for every $f \in \mathcal{P}_{int}([a,b])$ and $\|T\| \leq \|g\|_{VB}$. By (5) and (4), $EV(G^*;[a,b]) \leq \|T\|$. Hence $EV(G^*;[a,b]) \leq \|T\| \leq \|g\|_{VB}$. Now by (6) it follows that $\frac{1}{2} \cdot V(g;[a,b]) \leq \|T\| \leq \|g\|_{VB}$ □

Remark 5. Theorem 2 extends Alexiewicz' Theorem 12.7 of [12] (see also [1]).

Lemma 6. *The normed spaces* $(\mathcal{P}_{int}([a,b]), \|\cdot\|)$ *and* $(\mathcal{P}_o([a,b]), \|\cdot\|_\infty)$ *are isomorph.*

PROOF. Let $\Phi : (\mathcal{P}_{int}([a,b]) \to (\mathcal{P}_o([a,b])$, $\Phi(f) = F$ where F is the unique $\mathcal{P}$-primitive of f which is contained in $(\mathcal{P}_o([a,b])$. It is easy to verify that Φ is well defined, linearly, bijective and $\|\Phi(f)\|_\infty = \|f\|$ □

Lemma 7. *We have the following results:*

(i) The completion of $(\mathcal{P}([a,b]); \|\cdot\|_\infty)$ *is* $(C([a,b]), \|\cdot\|_\infty)$.

(ii) The completion of the isomorphic spaces $(\mathcal{P}_{int}([a,b]), \|\cdot\|)$ *and* $(\mathcal{P}_o([a,b]), \|\cdot\|_\infty)$ *is* $(C_o([a,b]), \|\cdot\|)$.

PROOF. We prove only (ii). Let $F \in C_o([a,b])$. By the Weierstrass Theorem, there exists a sequence $\{P_n\}_n$ of polynomials on $[a,b]$ such that $\|P_n - F\|_\infty \to 0$ if $n \to \infty$. Let $Q_n(x) = P_n(x) - P_n(a)$. Then for each n, $Q_n(a) = 0$, Q_n is Lipschitz (hence $Q_n \in \mathcal{P}_o([a,b])$), and $\|Q_n - F\|_\infty \to 0$. □

Remark 6. In the proof of Lemma 7 instead of Q_n we can use B_n the Bernstein polynomial of degree n for the function F on $[a,b]$, i.e.,

$$B_n(x) = \sum_{k=0}^{n} F(a + \frac{k}{n}(b-a))C_n^k \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}$$

(see [13], Definition 1 and Theorem 1, p. 108 and the proof of Theorem 2, p. 109).

Theorem 3. *We have the following results:*

(i) Let $T : (\mathcal{P}_o([a,b]), \|\cdot\|_\infty) \to \mathbb{R}$ be a continuous linear functional. Then there exists $g \in VB$ on $[a,b]$ such that $T(F) = (\mathcal{RS}) \int_a^b F(t)\, dg(t)$, whenever $F \in \mathcal{P}_o([a,b])$.

(ii) Assertion (i) remains true if $\mathcal{P}_o([a,b])$ is replaced by $\mathcal{P}([a,b])$.

PROOF. (i) Let $T^* : \mathcal{P}_{int}([a,b]) \to \mathbb{R}$, $T^* = T \circ \Phi$, where Φ is the isomorphicism defined in the proof of Lemma 6. Since T is a continuous linear functional, it follows that T^* is also a continuous linear functional. By Theorem 2, there exists $G : [a,b] \to \mathbb{R}$, $G \in VB$, such that $T^*(f) = (\mathcal{P}) \int_a^b f(t)G(t)\, dt$. Let $F \in \mathcal{P}_o([a,b])$ and $f = \Phi^{-1}(F)$. Then

$$T(F) = T^*(f) = (\mathcal{P}) \int_a^b f(t)G(t)\, dt = F(b)G(b) - (\mathcal{RS}) \int_a^b F(t)\, dG(t)$$

$$= G(b) \cdot (\mathcal{RS}) \int_a^b F(t)\, dK_{\{b\}}(t) - (\mathcal{RS}) \int_a^b F(t)\, dG(t) = (\mathcal{RS}) \int_a^b F(t)\, dg(t)$$

, where $g(t) = G(b) \cdot K_{\{b\}}(t) - G(t)$ (clearly $g \in VB$).

(ii) Let $\mathbb{I} : [a,b] \to \mathbb{R}$, $\mathbb{I}(x) = 1$. Let $F \in \mathcal{P}([a,b])$ and $F_o(x) = F(x) - F(a) \cdot \mathbb{I}(x)$. Then $F_o \in \mathcal{P}_o([a,b])$. By (i),

$$T(F) = T(F_o) + F(a) \cdot T(\mathbb{I}) = (\mathcal{RS}) \int_a^b F_o(t)\, dg(t) + F(a)T(\mathbb{I})$$

$$= (\mathcal{RS}) \int_a^b F(t)\, dg(t) - F(a)(g(b) - g(a) - T(\mathbb{I}))$$

$$= (\mathcal{RS}) \int_a^b F(t)\, dg(t) + (g(b) - g(a) - T(\mathbb{I})) \cdot (\mathcal{RS}) \int_a^b F(t) dK_{\{a\}}(t)$$

$$= (\mathcal{RS}) \int_a^b F(t)\, dG(t),$$

where $G(x) = g(x) + (g(b) - g(a) - T(\mathbb{I})) \cdot K_{\{a\}}(x)$ (clearly $G \in VB$). □

Corollary 2 (The Riesz Representation Theorem [12])**.**
Let $T : (C([a,b]), \|\cdot\|_\infty) \to \mathbb{R}$ be a continuous linear functional. Then there exists $g \in VB$ on $[a,b]$ such that $T(F) = (\mathcal{RS}) \int_a^b F(t)\, dg(t)$, whenever $F \in C([a,b])$.

PROOF. Since $\mathcal{P}([a,b])$ is dense in $C([a,b])$ (see for example Lemma 7 (i)), for each $F \in C([a,b])$ there exists a sequence $\{F_n\}_n$, $F_n \in \mathcal{P}([a,b])$, such that $F_n \xrightarrow{[unif]} F$ on $[a,b]$. Applying the Uniform Convergence Theorem for the $\mathcal{RS}$-integral and Theorem 3 (ii), we obtain $T(F) = \lim_{n\to\infty} T(F_n) = \lim_{n\to\infty}(\mathcal{RS})\int_a^b F_n(t)\,dg(t) = (\mathcal{RS})\int_a^b F(t)\,dg(t)$. □

Remark 7. In Corollary 1 we may replace the linear space $(C([a,b]), \|\cdot\|_\infty)$ by $(C_o([a,b]), \|\cdot\|_\infty)$.

5 The Category of $\mathcal{P}_{int}([a,b])$

Lemma 8. *([7], p. 49). Let (X,τ) be a topological space and let X_o be a dense subset of X. Let $\tau_o = \tau_{|X_o}$. If X_o is of the second category in (X_o,τ_o), then X_o is of the second category in (X,τ).*

Lemma 9 (Jarnik)**.** *([2], p.224). Let $(C([a,b]), \|\cdot\|_\infty)$ and let $\mathcal{A} = \{f : [a,b] \to \mathbb{R} : f$ is continuous and f is nowhere approximately differentiable$\}$. Then $C([a,b]) \setminus \mathcal{A}$ is of the first category in $C([a,b])$.*

Theorem 4. *We have the following results:*

(i) $(\mathcal{P}_o([a,b]), \|\cdot\|_\infty)$ is of first category on itself.

(ii) $(\mathcal{P}_{int}([a,b]), \|\cdot\|$ is of first category on itself.

PROOF. It suffices to prove only (ii) (because the proof of (i) is contained in (ii)). Suppose to the contrary that $(\mathcal{P}_{int}([a,b]), \|\cdot\|)$ is of the second category in itself. Since the spaces $(\mathcal{P}_{int}([a,b]), \|\cdot\|)$ and $(\mathcal{P}_o([a,b]), \|\cdot\|_\infty)$ are isomorphic (see Lemma 6), they are also homeomorphic. It follows that $(\mathcal{P}_o([a,b]), \|\cdot\|_\infty)$ is of the second category in itself. By Lemma 7 (ii), $\mathcal{P}_o([a,b])$ is dense in $C_o([a,b])$. By Lemma 8, $(\mathcal{P}_o([a,b]), \|\cdot\|_\infty)$ is of second category in $(C_o([a,b]), \|\cdot\|_\infty)$, and by Lemma 9, $\mathcal{P}_o([a,b])$ is of first category in $(C_o([a,b]), \|\cdot\|_\infty)$. This contradicts the fact that $(C_o([a,b]), \|\cdot\|_\infty)$ is a Banach space. □

6 Weak Convergence in $\mathcal{P}_{int}([a,b])$

Theorem 5. *([11], p. 259). Let $f, f_n : [a,b] \to \mathbb{R}$, $n = 1,2,\dots$ be such that f, f_n are continuous and $|f_n(x)| < M$ for some M, for every $x \in [a,b]$ and each $n = 1,2,\dots$. Let $g : [a,b] \to \mathbb{R}$, $g \in VB$. If $f_n \to f$ on $[a,b]$, then $(\mathcal{RS})\int_a^b f(t)\,dg(t) = \lim_{n\to\infty}(\mathcal{RS})\int_a^b f_n(t)\,dg(t)$.*

Lemma 10 ([4] or [10], Theorem 2, # 1 of Chapter VIII)**.**
$x_n \to x$ weakly in a normed space if and only if $\sup_n \|x_n\| < +\infty$ and $\{f : f(x_n) \to f(x)\}$ is a dense set of functionals in X^*.

Theorem 6. *Let $f, f_n \in \mathcal{P}_{int}([a,b])$, and let $F, F_n \in \mathcal{P}_o([a,b])$ be the unique $\mathcal{P}$-primitives of f, f_n, $n = 1, 2, \ldots$. The following assertions are equivalent:*

(i) $f_n \to f$ weakly on $(\mathcal{P}_{int}([a,b]), \|\cdot\|)$;

(ii) 1) $|F_n(x)| \le M$ for some M, for every $x \in [a,b]$ and each $n = 1, 2, \ldots$;
2) $F_n(x) \to F(x)$ for every $x \in [a,b]$.

Proof. (i) $\Rightarrow$ (ii) Since $f_n \to f$ weakly, by Lemma 10 we obtain $\|f_n\| = \|F_n\|_\infty \le M$ for some positive number M. So we have 1) of (ii). For $x \in [a,b]$ let $T_x : \mathcal{P}_{int}([a,b]) \to \mathbb{R}$ be a continuous linear functional defined by $T_x(f) = F(x)$ (because clearly T_x is linear and $|T_x(f)| = |F(x)| \le \|F\|_\infty = \|f\|$). Since $f_n \to f$ weakly it follows that $T_x(f_n) \to T_x(f)$, hence $F_n(x) \to F(x)$. Therefore we have condition 2) of (ii).

(ii) $\Rightarrow$ (i) Let $T : \mathcal{P}_{int}([a,b]) \to \mathbb{R}$ be a continuous linear functional. By Theorem 2 there exists $g_T \in VB$ on $[a,b]$ such that $T(f) = (\mathcal{P}) \int_a^b f(t) g_T(t)\, dt$, for every $f \in \mathcal{P}_{int}([a,b])$. We show that $T(f_n) \to T(f)$. Indeed, $|T(f_n) - T(f)| = |(\mathcal{P}) \int_a^b (f_n - f)(t) g_T(t)\, dt| = |(F_n - F)(b) \cdot g_T(b) - (\mathcal{RS}) \int_a^b (F_n - F)(t)\, dg_T(t)| \to 0$ (see Theorem 5). Therefore we have (i). $\square$

Remark 8. We observe the following:

(i) Our proof parallels the proof of Theorem 3, # 3, Chapter VIII of [10].

(ii) Using Theorem 3 (i) (respectively Remark 7) instead of Theorem 2 and $T_x : \mathcal{P}_o([a,b]) \to \mathbb{R}$ (respectively $T_x : C_o([a,b]) \to \mathbb{R}$), $T_x(F) = F(x)$, we can prove also the following theorem .
Let $F, F_n \in (\mathcal{P}_o([a,b]), \|\cdot\|_\infty)$ (respectively $(C_o([a,b]), \|\cdot\|_\infty)$ $n = 1, 2, \cdots$. Then $F_n \to F$ weakly if and only if $|F_n(t)| < M$ for some M, whenever $t \in [a,b]$, $n = 1, 2, \cdots$ and $F_n(t) \to F(t)$ for every $t \in [a,b]$.

7 Applications

In what follows we shall use the definitions given in [5] for the following classes of functions: $\mathcal{C}$, AC, AC^*, AC_n, SAC_n, AC^*G, $SACG$, $\mathcal{F}$, $S\mathcal{F}$. We set

$$\begin{aligned} \Delta_{ae} &= \{F : [a,b] \to \mathbb{R} \,:\, F \text{ is derivable } a.e. \text{ on } [a,b]\} \Delta_{ap\, ae} \\ &= \{F : [a,b] \to \mathbb{R} \,:\, F \text{ is approximately derivable } a.e. \text{ on } [a,b]\}. \end{aligned}$$

Lemma 11 (A slight reformulation of Lemma 1 of [3], p. 31)**.** *Let $g : [a,b] \to \mathbb{R}$, $g \in VB$, and let $F : [a,b] \to \mathbb{R}$ be a bounded function which is $\mathcal{RS}$-integrable on $[a,b]$ with respect to g. Let $H : [a,b] \to \mathbb{R}$, $H(x) = F(x)g(x) - (\mathcal{RS}) \int_a^x F(t)\, dg(t)$. Then* *(i)* $|H(\beta)-H(\alpha)| \le \sup_{x\in[a,b]} |g(x)|\cdot|F(\beta)-F(\alpha)|+V(g;[\alpha,\beta])\mathcal{O}(F;[\alpha,\beta])$ *whenever* $a \le \alpha < \beta \le b$.

(ii) $\mathcal{O}(H;P) \le \sup_{x\in[a,b]} |g(x)|\cdot\mathcal{O}(F;P)+V(g;[\alpha,\beta])\cdot\mathcal{O}(F;[\alpha,\beta])$, *whenever* $P \subseteq [\alpha,\beta] \subseteq [a,b]$.

PROOF. (i) $|H(\beta) - H(\alpha)| = |(F(\beta) - F(\alpha)) \cdot g(\beta) + (g(\beta) - g(\alpha)) \cdot F(\alpha) - (\mathcal{RS}) \int_\alpha^\beta F(t)\, dg(t)| = |(F(\beta) - F(\alpha)) \cdot g(\beta) + (\mathcal{RS}) \int_\alpha^\beta (F(\alpha) - F(t))\, dg(t)| \le |F(\beta) - F(\alpha)| \cdot \sup_{x\in[a,b]} |g(x)| + V(g;[\alpha,\beta]) \cdot \mathcal{O}(F;[\alpha,\beta])$.

(ii) This follows by the definition of the oscillation and applying (i) to each $\alpha^{'}, \beta^{'} \in P$, where $\alpha \le \alpha^{'} < \beta^{'} \le \beta$. □

Lemma 12. *Let* $F, g, H : [a,b] \to \mathbb{R}$ *be such that* F *is continuous,* $g \in VB$ *and* $H(x) = F(x)g(x) - (\mathcal{RS}) \int_a^x F(t)\, dg(t)$. *Let* $P \subseteq [a,b]$ *and let* n *be a positive integer. Theneach of the following hold.*

(i) H *is continuous on* $[a,b]$.

(ii) If F *is Lipschitz on* $[a,b]$, *then* H *is Lipschitz on* $[a,b]$.

(iii) If $F \in AC$ *on* P, *then* $H \in AC$ *on* P.

(iv) If $F \in AC^*$ *on* P, *then* $H \in AC^*$ *on* P.

(v) If $F \in AC_n$ *on* P, *then* $H \in AC_n$ *on* P. *(This is a slight extension of Lemma 5.23.1 of [5].)*

(vi) If $F \in SAC_n$ *on* P, *then* $H \in SAC_n$ *on* P. *(This is a slight extension of Lemma 5.24.1 of [5].)*

PROOF. (i) This follows immediately from Lemma 11.

(ii) Let $c > 0$ be a constant given by the fact that F is Lipschitz on $[a,b]$. Let $[\alpha,\beta] \subseteq [a,b]$. Since F is continuous on $[a,b]$, there exists $[\alpha_o,\beta_o] \subseteq [\alpha,\beta]$ such that

$$\mathcal{O}(F;[\alpha,\beta]) = |F(\beta_o) - F(\alpha_o)| \le c(\beta_o - \alpha_o) \le c(\beta - \alpha)\,. \tag{7}$$

By Lemma 11 (i) and (7), we have

$$\begin{aligned}|H(\beta) - H(\alpha)| \le &\sup_{x\in[a,b]} |g(x)| \cdot c(\beta - \alpha) + V(g;[\alpha,\beta]) \cdot c(\beta - \alpha)\\ \le&(\beta - \alpha) \cdot c \cdot (\sup_{x\in[a,b]} |g(x)| + V(g;[\alpha,\beta])).\end{aligned}$$

Therefore H is Lipschitz on $[a,b]$.

(iii) and (iv) follow by Lemma 11 (see also Lemma 2 of [3], p. 31).

(v) Let $M = \sup_{x\in[a,b]} |g(x)| + V(g;[\alpha,\beta])$ and let $\epsilon > 0$. Since $F \in AC_n$ on P, by Proposition 2.28.1 of [5], it follows that there exists a $\delta > 0$ such that if $\{I_k\}$, $k = 1, 2, \ldots, s$ are nonoverlapping closed intervals with each $P \cap I_k \neq \emptyset$, and $\sum_{k=1}^{s} |I_k| < \delta$, then for each k there exists $\{P_{kj}\}$, $j = 1, 2, \ldots, n$ such that $P \cap I_k = \cup_{j=1}^{n} P_{kj}$ and $\sum_{k=1}^{s} \sum_{j=1}^{n} \mathcal{O}(F; P_{kj}) < \epsilon/(2M)$. Let $\eta > 0$ such that $\mathcal{O}(F; I) < \epsilon/(2nM) = \epsilon_1$, whenever I is a closed subinterval of $[a,b]$ with $|I| < \eta$. (This is possible because F is continuous on $[a,b]$.) Let $\delta_1 = \min\{\delta, \eta\}$. Then $\mathcal{O}(F; I_k) < \epsilon_1$, for each k. By Lemma 11 (ii) it follows that $\mathcal{O}(H; P_{kj}) \leq M \cdot \mathcal{O}(F; P_{kj}) + V(g; I_k) \cdot \mathcal{O}(F; I_k)$. Hence

$$\begin{aligned}\sum_{k=1}^{s}\sum_{j=1}^{n} \mathcal{O}(H; P_{kj}) &\leq M \cdot \sum_{k=1}^{s}\sum_{j=1}^{n} \mathcal{O}(F; P_{kj}) + n\epsilon_1 \cdot \sum_{k=1}^{s} V(g; I_k)\\ &\leq M\epsilon/(2M) + n\epsilon_1 M < \epsilon.\end{aligned}$$

Therefore $H \in AC_n$ on P.

(vi) The proof is similar to that of (v) using Proposition 2.34.1 of [5] instead of Proposition 2.28.1 of [5]. □

Theorem 7. *Let $F, g, H : [a,b] \to \mathbb{R}$ be such that F is continuous, $g \in VB$ and $H(x) = F(x)g(x) - (\mathcal{RS}) \int_a^x F(t)\, dg(t)$. Then each of the following hold.*

(i) H is continuous on $[a,b]$.

(ii) If F is Lipschitz on $[a,b]$, then H is Lipschitz on $[a,b]$ and $H'(x) = g(x)F'(x)$ a.e. on $[a,b]$.

(iii) If $F \in AC$ on $[a,b]$, then $H \in AC$ on $[a,b]$ and $H'(x) = g(x)F'(x)$ a.e. on $[a,b]$.

(iv) If $F \in ACG^*$ on $[a,b]$, then $H \in ACG^*$ on $[a,b]$ and $H'(x) = g(x)F'(x)$ a.e. on $[a,b]$.

(v) If $F \in ACG$ on $[a,b]$ and is derivable a.e. on $[a,b]$, then $H \in ACG$ on $[a,b]$ and $H'(x) = g(x)F'(x)$ a.e. on $[a,b]$.

(vi) If $F \in ACG$ on $[a,b]$, then $H \in ACG$ on $[a,b]$ and $H'_{ap}(x) = g(x)F'_{ap}(x)$ a.e. on $[a,b]$.

(vii) If $F \in \mathcal{F}$ on $[a,b]$, then $H \in \mathcal{F}$ on $[a,b]$ and $H'_{ap}(x) = g(x)F'_{ap}(x)$ a.e. on $[a,b]$.

(viii) If $F \in S\mathcal{F}$ on $[a,b]$, then $H \in S\mathcal{F}$ on $[a,b]$ and $H'_{ap}(x) = g(x)F'_{ap}(x)$ a.e. on $[a,b]$.

(ix) *If* $F \in SACG$ *on* $[a,b]$*, then* $H \in SACG$ *on* $[a,b]$ *and* $H'_{ap}(x) = g(x)F'_{ap}(x)$ *a.e. on* $[a,b]$.

PROOF. That $H'_{ap}(x) = g(x)F'_{ap}(x)$ *a.e.* on $[a,b]$ or $H'(x) = g(x)F'(x)$ *a.e.* on $[a,b]$ follows easily (see for example [5], Theorem 5.23.3). Now the other assertions follow by the linearity of the $\mathcal{RS}$-integral in the second argument, and by Lemma 12. □

Remark 9. Here are some special cases of $\mathcal{P}_{int}([a,b])$.

(i) $AC_{int}([a,b]) = \{f : [a,b] \to \overline{\mathbb{R}} : f$ is Lebesgue integrable on $[a,b]\}$.

(ii) $Lip_{int}([a,b]) = \{f : [a,b] \to \overline{\mathbb{R}} : f$ is measurable and bounded *a.e.* on $[a,b]\}$.

(iii) $(AC^*G \cap \mathcal{C})_{int}([a,b]) = \{f : [a,b] \to \overline{\mathbb{R}} : f$ is $\mathcal{D}^*$-integrable on $[a,b]\}$.

(iv) $(ACG \cap \mathcal{C} \cap \Delta_{a.e.})_{int}([a,b]) = \{f : [a,b] \to \overline{\mathbb{R}} : f$ is Khintchine-integrable on $[a,b]\}$.

(v) $(ACG \cap \mathcal{C})_{int}([a,b]) = \{f : [a,b] \to \overline{\mathbb{R}} : f$ is $\mathcal{D}$-integrable on $[a,b]\}$.

(vi) $(S\mathcal{F} \cap \mathcal{C} \cap \Delta_{ap\,a.e.})_{int}([a,b]) = \{f : [a,b] \to \overline{\mathbb{R}} : f$ is $S\mathcal{F}$-integrable on $[a,b]\}$. (For the definition of the $S\mathcal{F}$-integral see [5], p. 210.)

(vii) $(\mathcal{F} \cap \mathcal{C} \cap \Delta_{ap\,a.e.})_{int}([a,b]) = \{f : [a,b] \to \overline{\mathbb{R}} : f$ is Foran-integrable on $[a,b]\}$. (For the Foran integral $\mathcal{F}$, see [6] or [5], p. 207.)

(viii) $(SACG \cap \mathcal{C} \cap \Delta_{ap\,a.e.})_{int}([a,b]) = \{f : [a,b] \to \overline{\mathbb{R}} : f$ is $SACG$-integrable on $[a,b]\}$ ($SACG$ is the Iseki sparse integral, see [8] and [9]).

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