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LUSIN'S CONDITION (N) AND FORAN'S CONDITION (M) ARE EQUIVALENT FOR BOREL FUNCTIONS THAT ARE VBG ON A BOREL SET

Abstract

In this paper we show that Lusin's condition (N) and Foran's condition (M) are equivalent for Borel functions that are VBG on a Borel set. Also new characterizations of conditions (M) and $\underline{M}$ are given.

Lusin's condition (N) plays an important role in the theory of integration, since the classes of primitives for many nonabsolutely convergent integrals (Denjoy–Perron, Denjoy, α-Ridder, β-Ridder [6], Sarkhel-De-Kar [11], [9], [10], [12], etc.) are contained in $(N)\cap VBG$. In [2], we showed that $(N)\cap VBG$ is a real linear space for Borel functions on Borel sets. However Foran's condition (M), which strictly contains condition (N), seems to be more relevant to the theory of integral (see [1]). In this paper we show that Lusin's condition (N) and Foran's condition (M) are equivalent for Borel functions that are VBG on a Borel set (see Theorem 2, (ii)). In fact we prove stronger results (see Theorem 2, (i), (iii)), using conditions $\underline{M}$ and $(\underline{N})$. These results are very useful proving theorems of Hake-Alexandroff-Looman type (see for example [1], p. 199). In the present paper we give some new characterizations of conditions (M) and $\underline{M}$.

1 Preliminaries

We denote by $m^*(X)$ the outer measure of a set X and by $m(A)$ the Lebesgue measure of A, whenever $A \subset \mathbb{R}$ is Lebesgue measurable. For the definitions of

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VB and AC see [8]. Let $\mathcal{C}$ denote the class of continuous functions. For two classes $\mathcal{A}_1$, $\mathcal{A}_2$ of real functions on a set P let

$$\mathcal{A}_1 \boxplus \mathcal{A}_2 = \left\{\alpha_1 F_1 + \alpha_2 F_2 : F_1 \in \mathcal{A}_1,\ F_2 \in \mathcal{A}_2,\ \alpha_1, \alpha_2 \geq 0\right\} \text{ and}$$

$$\mathcal{A}_1 \oplus \mathcal{A}_2 = \left\{\alpha_1 F_1 + \alpha_2 F_2 : F_1 \in \mathcal{A}_1,\ F_2 \in \mathcal{A}_2,\ \alpha_1, \alpha_2 \in \mathbb{R}\right\}.$$

Definition 1. Let $P \subseteq [a,b]$, $x_0 \in P$ and $F : P \to \mathbb{R}$. F is said to be $\mathcal{C}_i$ at x_0 if $\limsup_{x \nearrow x_0, x \in P} F(x) \leq F(x_0)$, whenever x_0 is a left accumulation point for P, and $F(x_0) \leq \liminf_{x \searrow x_0, x \in P} F(x)$, whenever x_0 is a right accumulation point for P. F is said to be $\mathcal{C}_i$ on P, if F is so at each point $x \in P$.

Definition 2. ([7]). Let P be a bounded real set and let $F : P \to \mathbb{R}$. Put

- $\mathcal{O}(F;P) = \sup\{|F(y) - F(x)| : x, y \in P\}$ the oscillation of F on P.
- $\mathcal{O}_-(F;P) = \inf\{F(y) - F(x) : x, y \in P,\ x \leq y\}$.
- $\mathcal{O}_+(F;P) = \sup\{F(y) - F(x) : x, y \in P,\ x \leq y\}$.

Definition 3. ([1], p. 6). Let $F : [a,b] \to \mathbb{R},\ P \subseteq [a,b]$. Put

- $\mathcal{O}^\infty(F;P) = \inf\{\sum_{i=1}^\infty \mathcal{O}(F;P_i) : \cup_{i=1}^\infty P_i = P\}$.
- $\mathcal{O}_+^\infty(F;P) = \inf\{\sum_{i=1}^\infty \mathcal{O}_+(F;P_i) : \cup_{i=1}^\infty P_i = P\}$.
- $\mathcal{O}_-^\infty(F;P) = \sup\{\sum_{i=1}^\infty \mathcal{O}_-(F;P_i) : \cup_{i=1}^\infty P_i = P\}$.

Definition 4. ([6], p. 236). A function $F : P \to \mathbb{R}$ is said to be $\underline{AC}$ (respectively $\overline{AC}$) if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} \big(F(b_k) - F(a_k)\big) > -\epsilon\,, \tag{1}$$

$$\text{(respectively } \sum_{k=1}^{n} \big(F(b_k) - F(a_k)\big) < \epsilon\,), \tag{2}$$

whenever $\{[a_k, b_k]\}$, $k = 1, 2, \ldots, n$ is a finite set of nonoverlapping closed intervals with endpoint in P and $\sum_{k=1}^n (b_k - a_k) < \delta$. Clearly $AC = \underline{AC} \cap \overline{AC}$.

Proposition 1. *Let $F : P \to \mathbb{R}$, $F \in \underline{AC}$ and let $\epsilon > 0$. For $\epsilon/2$ let $\delta > 0$ be given by the fact that $F \in \underline{AC}$ on P. Let $\{(a_i, b_i)\}_i$ be a sequence of nonoverlapping open intervals such that $\sum_{i=1}^\infty (b_i - a_i) < \delta$. Then*

$$\sum_{i=1}^{\infty} \mathcal{O}_-\big(F; P \cap (a_i, b_i)\big) > -\epsilon\,.$$

PROOF. We may suppose without loss of generality that for each i

$$(a_i, b_i) \cap P \neq \emptyset \quad \text{and} \quad \mathcal{O}_-\big(F; P \cap (a_i, b_i)\big) < 0\,. \tag{3}$$

Since $F \in \underline{AC}$ the oscillations in (3) are always finite. Then, for each i, there exist $a_i^{'}, b_i^{'} \in P \cap (a_i, b_i)$, $a_i^{'} < b_i^{'}$ such that

$$F(b_i^{'}) - F(a_i^{'}) < \frac{2}{3} \cdot \mathcal{O}_-\big(F; P \cap (a_i, b_i)\big)\,.$$

It follows that for each positive integer n we have

$$\sum_{i=1}^{n} \mathcal{O}_-\big(F; P \cap (a_i, b_i)\big) > \frac{3}{2} \cdot \sum_{i=1}^{n} \big(F(b_i^{'}) - F(a_i^{'})\big) > -\frac{3}{4}\epsilon\,.$$

Therefore $\sum_{i=1}^{\infty} \mathcal{O}_-\big(F; P \cap (a_i, b_i)\big) > -\epsilon$. □

Definition 5. A function $F : P \to \mathbb{R}$ is said to be VBG (respectively ACG, $\underline{AC}G$, $\overline{AC}G$) on P if there exists a sequence of sets $\{P_n\}$ with $P = \cup_n P_n$, such that F is VB (respectively AC, $\underline{AC}$, $\overline{AC}$) on each P_n. If in addition the sets P_n are assumed to be closed, we obtain the classes $[VBG]$, $[ACG]$, $[\underline{AC}G]$ and $[\overline{AC}G]$. Note that condition ACG used here differs from that of [8] (because in our definition the continuity is not assumed).

Definition 6. ([8], p. 224). A function $F : P \to \mathbb{R}$ is said to satisfy Lusin's condition (N) on P if $m^*(F(Z)) = 0$ whenever Z is a null subset of P.

Definition 7. Let $F : [a, b] \to \mathbb{R}$, $P \subset [a, b]$. F is said to be $\underline{M}$ on P if $F \in \underline{AC}$ on Q, whenever $Q = \overline{Q} \subset P$ and $F \in VB \cap \mathcal{C}$ on Q. A function F is said to satisfy Foran's condition (M) on P if F is simultaneously $\underline{M}$ and $\overline{M}$ (i.e., F is AC on Q whenever Q is a closed subset of P and $F \in VB \cap \mathcal{C}$ on Q, see [3]).

Definition 8. ([1], p. 78). Let $F : [a, b] \to \mathbb{R}$, $P \subseteq [a, b]$. F is said to be $(\overline{N})$ on P if $\mathcal{O}_+^{\infty}(F; Z) = 0$, whenever $Z \subset P$ and $m(Z) = 0$. F is said to be $(\underline{N})$ on P if $-F$ is $(\overline{N})$ on P; i.e., $\mathcal{O}_-^{\infty}(F; Z) = 0$.

Remark 1. In [1] (p. 84), there is given an equivalent definition for $\underline{M}$ (i.e., condition 4) of Theorem 3). By Corollary 2.21.1 (iii) of [1], we have $(\underline{N}) \subset \underline{M}$ on a set P.

2 Conditions (N), ($\underline{\text{N}}$), (M), $\underline{\text{M}}$ and VB on Closed Sets

Lemma 1. *Let P be a closed subset of $[a,b]$. Then we have*

(i) $VB \cap (\underline{N}) \subseteq VB \cap \underline{M} \subseteq (VB \cap \underline{M}) \boxplus (VB \cap \underline{M}) \subseteq VB \cap (\underline{N})$ *on* P*;*

(ii) $VB \cap (N) \subseteq VB \cap (M) \subseteq (VB \cap (M)) \oplus (VB \cap (M)) \subseteq VB \cap (N)$ *on* P*.*

PROOF. (i) By Remark 1 the first two inclusions are evident. We prove the last inclusion. Let $F_1, F_2 : P \to \mathbb{R}$ such that $F_1, F_2 \in VB \cap \underline{M}$. It is sufficient to show that $F = F_1 + F_2$ is $VB \cap (\underline{N})$ on P. Let A_1 and A_2 be the sets of points of discontinuity for F_1 respectively F_2. Then A_1, A_2 are countable and

$$A_1 \cup A_2 = \{d_1, d_2, d_3, \dots, d_n, \dots\}$$

contains all discontinuity points of F. Given $\epsilon > 0$, for each d_n we can find some intervals $I_n = (p_n, d_n)$ and $J_n = (d_n, q_n)$ such that

$$\mathcal{O}(F; P \cap I_n) + \mathcal{O}(F; P \cap J_n) < \frac{\epsilon}{2^n}.$$

Let $Q = P \setminus \cup_{n=1}^{\infty}(I_n \cup J_n)$. Then Q is a compact set and $F_1, F_2 \in VB \cap \mathcal{C}$ on Q. But $F_1, F_2 \in \underline{M}$ on P; so $F_1, F_2 \in \underline{AC}$ on Q. Hence $F \in \underline{AC}$ on Q.

Let $Z \subset P$, $m(Z) = 0$. For $\epsilon/2 > 0$, let $\delta_\epsilon > 0$ be given by the fact that $F \in \underline{AC}$ on Q. By Proposition 1 there exists $\{(a_i, b_i)\}_i$, a sequence of nonoverlapping open intervals, such that $Z \cap Q \subset \cup_{i=1}^{\infty}(a_i, b_i)$, $\sum_{i=1}^{\infty}(b_i - a_i) < \delta_\epsilon$ and $\sum_{i=1}^{\infty} \mathcal{O}_-\big(F; Z \cap Q \cap (a_i, b_i)\big) > -\epsilon$. Hence

$$\mathcal{O}_-^{\infty}(F; Z) \geq -\epsilon - \Big(\sum_{n=1}^{\infty}(\mathcal{O}(F; Z \cap I_n) + \mathcal{O}(F; Z \cap J_n)\Big) > -2\epsilon.$$

Since $\mathcal{O}_-^{\infty}(F; Z) \leq 0$ and ϵ is arbitrary, it follows that $\mathcal{O}_-^{\infty}(F; Z) = 0$. Hence $F \in (\underline{N})$ on P.

(ii) The first two inclusions are evident, since $(N) \subset (M)$ (see the Banach-Zarecki Theorem). We prove the last inclusion. Let F_1, F_2, A_1, A_2, I_n, J_n and Q be defined as in the proof of (i). Suppose that $F_1, F_2 \in VB \cap (M)$ on P. From the definition of (M) it follows that $F \in AC \subset (N)$ on Q. Let $Z \subset P$, $m(Z) = 0$. Then

$$m^*(F(Z)) \leq m^*\big(F(Z \cap Q)\big) + \sum_{n=1}^{\infty} m^*\big(F(Z \cap I_n)\big) + \sum_{n=1}^{\infty} m^*\big(F(Z \cap J_n)\big) < \epsilon.$$

Since ϵ is arbitrary, we obtain that $m^*(F(Z)) = 0$. Hence $F \in (N)$ on P. $\square$

Lemma 2. *Let P be a closed subset of $[a,b]$. Then we have:*

(i) $VB \cap (\underline{N}) = VB \cap \underline{M}$ *is an upper real linear space on* P.

(ii) $VB \cap (N) = VB \cap (M)$ *is a real linear space on* P.

(iii) $VB \cap (M) = VB \cap \underline{M} \cap \overline{M} = VB \cap (\underline{N}) \cap (\overline{N}) = VB \cap (N)$ *on* P.

PROOF. (i) This follows by Lemma 1, (i).
(ii) This follows by Lemma 1, (ii).
(iii) We have $VB \cap (N) \subseteq VB \cap (\underline{N}) \cap (\overline{N}) = VB \cap \underline{M} \cap \overline{M} = VB \cap (M) = VB \cap (N)$. (The equalities follow by (i), (ii) and the fact that we always have $(M) = \underline{M} \cap \overline{M}$.) $\square$

Lemma 3. *Let $F : [a,b] \to \mathbb{R}$, $E_k \subset [a,b]$, $k = 1, 2, \ldots$, and $E = \cup_{i=1}^{\infty} E_k$.*

(i) F is (N) (respectively $(\underline{N})$) on E if and only if F is (N) (respectively $(\underline{N})$) on each E_k.

(ii) If in addition each E_k is a closed set, then F is (M) (respectively $\underline{M}$) on E if and only if $F \in (M)$ (respectively $\underline{M}$) on each E_k.

PROOF. (i) For (N) the proof is evident. For $(\underline{N})$ the necessity is also obvious, and the sufficiency follows by definitions and Lemma 2.20.1 of [1].

(ii) The "$\Rightarrow$" part is evident. We show the converse. Let Q be a closed subset of E such that $F \in VB \cap \mathcal{C}$ on Q. Clearly $F \in VB \cap \mathcal{C}$ on each closed set $Q \cap E_k$. Since F is (M) (respectively $\underline{M}$) on each E_k, it follows that F is AC (respectively $\underline{AC}$) on each $Q \cap E_k$. Therefore $F \in VB \cap \mathcal{C} \cap ACG = AC$ (respectively $F \in VB \cap \mathcal{C} \cap \underline{AC}G = \underline{AC}$) on Q (see Corollary 2.21.1, (iv), (iii) of [1]). Therefore F is (M) (respectively $\underline{M}$) on E. $\square$

Theorem 1. *Let P be a closed subset of $[a,b]$. Then we have:*

(i) $[VBG] \cap (\underline{N}) = [VBG] \cap \underline{M}$ *is an upper real linear space on* P.

(ii) $[VBG] \cap (N) = [VBG] \cap (M)$ *is a real linear space on* P.

(iii) $[VBG] \cap (M) = [VBG] \cap \underline{M} \cap \overline{M} = [VBG] \cap (\underline{N}) \cap (\overline{N}) = [VBG] \cap (N)$ *on* P.

PROOF. (i) Since $(\underline{N}) \subset \underline{M}$, we have $[VBG] \cap (\underline{N}) \subset [VBG] \cap \underline{M}$ on P. Let $F \in [VBG] \cap \underline{M}$. Then there exists a sequence of closed sets $\{P_n\}_n$ such that $P = \cup_{n=1}^{\infty} P_n$ and $F \in VB \cap \underline{M} = VB \cap (\underline{N})$ on each P_n (see Lemma 2, (i)). By Lemma 3, (i) it follows that $F \in (\underline{N})$ on P; so $[VBG] \cap \underline{M} \subset [VBG] \cap (\underline{N})$. We show that $[VBG] \cap (\underline{N})$ is an upper linear space. Let $F_1, F_2 : P \to \mathbb{R}$,

$F_1, F_2 \in [VBG] \cap (\underline{N})$. Then there exists $\{Q_n\}_n$, a sequence of closed sets, such that $P = \cup_{n=1}^{\infty} Q_n$ and $F_1, F_2 \in VB \cap (\underline{N})$ on each Q_n. By Lemma 2, (i), $F_1 + F_2 \in VB \cap (\underline{N})$ on each Q_n. Now by Lemma 3, (i) it follows that $F_1 + F_2 \in [VBG] \cap (\underline{N})$ on P.

(ii) The proof is similar to that of (i), using Lemma 2, (ii) and Lemma 3, (i).

(iii) By (i), (ii) and because we always have $(M) = \underline{M} \cap \overline{M}$, it follows that

$$[VBG] \cap (N) \subseteq [VBG] \cap (\overline{N}) \cap (\underline{N}) = [VBG] \cap \overline{M} \cap \underline{M} =$$

$$= [VBG] \cap (M) = [VBG] \cap (N)\,. \qquad \square$$

3 Conditions (N), ($\underline{\mathbf{N}}$), (M), $\underline{\mathbf{M}}$ and VB on Borel Sets

Lemma 4. *Let $F : P \to \mathbb{R}$ be an increasing function, $P \subset [a, b]$. Then $F \in (\overline{N})$ if and only if $F \in (N)$ on P.*

PROOF. "$\Rightarrow$" Suppose that $F \in (\overline{N})$ on P, and let $Z \subset P$ such that $m(Z) = 0$ Then $\mathcal{O}_+^{\infty}(F; Z) = 0$; i.e., for every $\epsilon > 0$, there is a sequence $\{Z_i\}_i$ of sets such that $Z = \cup_{i=1}^{\infty} Z_i$ and $0 \le \sum_{i=1}^{\infty} \mathcal{O}_+(F; Z_i) < \epsilon$. Since F is increasing, it follows that $\mathcal{O}_+(F; Z_i) = \mathcal{O}(F; Z_i)$. Therefore

$$m^*(F(Z)) \le \sum_{i=1}^{\infty} m^*(F(Z_i)) \le \sum_{i=1}^{\infty} \mathcal{O}(F; Z_i) < \epsilon\,.$$

Since ϵ is arbitrary, we obtain that $m^*(F(Z)) = 0$. Hence $F \in (N)$ on P.
"$\Leftarrow$" $(N) \subseteq (\overline{N})$ is always true (see Theorem 2.20.1 of [1]). $\square$

Lemma 5 (Fundamental Lemma)**.** *Let $P \subset [a, b]$ be a Borel set and let $G : P \to \mathbb{R}$, $G \in VB$.*

- *(i) If $G \notin (\overline{N})$ on P, then there exists a compact set $K \subset P$ with $m(K) = 0$ such that $G_{|K}$ is strictly increasing and $G(K)$ is a compact set of positive measure.*
- *(ii) If $G \notin (N)$ on P, then there exists a compact set $K \subset P$ with $m(K) = 0$ such that $G_{|K}$ is strictly monotone and $G(K)$ is a compact set of positive measure.*

PROOF. (i) By Lemma 4.1 of [8] (p. 221), there exists $F : [a, b] \to \mathbb{R}$ such that $F \in VB$ and $F_{|P} = G$. Let $E = \{x \in [a, b] : F'(x)$ does not exist, finite or infinite$\}$. By Theorem 7.2 of [8] (p. 230), we have $m(F(E)) = 0$. Since

$F \notin (\overline{N})$ on P, it follows that there exists a set $Z \subset P$ with $m(Z) = 0$ and $\mathcal{O}_+^\infty(F; Z) > 0$. Hence

$$F \notin (\overline{N}) \text{ on } Z. \tag{4}$$

Let $A = Z \cap E$. Then

$$m(F(A)) = 0\,. \tag{5}$$

Let $A_1 = \{x \in Z : |F^{'}(x)| < 1\}$. Then

$$F \in (N) \text{ on } A_1 \tag{6}$$

(see Theorem 10.5, p. 235 or Theorem 4.6, p. 271 of [8]). Let $B = \{x \in Z : |F^{'}(x)| \geq 1\}$, $B_+ = \{x \in Z : F^{'}(x) \geq 1\}$ and $B_- = \{x \in Z : F^{'}(x) \leq -1\}$. Using the proof of Theorem 10.1 of [8] (pp. 234-235), it follows that the set B_- can be written as the union of a finite or countable family of sets $\{B^{'}_n\}_n$, such that F is strictly decreasing on each $B^{'}_n$. Clearly $\mathcal{O}_+(F; B^{'}_n) = 0$; so $\mathcal{O}_+^\infty(F; B_-) = 0$. Hence

$$F \in (\overline{N}) \text{ on } B_-\,. \tag{7}$$

The set B_+ can also be written as the union of a finite or countable family of sets $\{B_n\}_n$, such that $F - I$ is increasing on each of them (here $I(x) = x$ for each $x \in [a, b]$). By (5), (6), (7) and Lemma 3, (i), it follows that

$$F \in (\overline{N}) \text{ on } A \cup A_1 \cup B_-\,. \tag{8}$$

Since $Z = A \cup A_1 \cup B_- \cup (\cup_n B_n)$, by (4), (8), Lemma 3 and Lemma 4, it follows that there exists at least a positive integer n such that $F \notin (N)$ on B_n. Fix such a positive integer n. Since $F \in VB$ on $[a, b]$, $F - I$ is bounded on B_n. By Lemma 4.1 of [8] (p. 221), it follows that there exists $\widetilde{F - I} : [a, b] \to \mathbb{R}$ such that $\widetilde{F - I}_{|B_n} = F - I$ and $\widetilde{F - I}$ is increasing on $[a, b]$. Let B_0 be a G_δ-set of measure zero that contains B_n. Let

$$\tilde{B} = P \cap B_0 \cap \{x \in [a, b] : (\widetilde{F - I})(x) = (F - I)(x)\}\,.$$

Since $\widetilde{F - I}$, $F - I \in VB \subset$ Borel functions on P, it follows that $\tilde{B}$ is a Borel set of measure zero, $m^*(F(\tilde{B})) > 0$ (because $\tilde{B} \subseteq B_n$) and $F = (F - I) + I$ is strictly increasing on $\tilde{B}$. From [4] (pp. 391, 387, 365), we obtain that $F(\tilde{B})$ is a Lebesgue measurable set (because the image of a Borel set under a Borel function is an analytic set, and an analytic set is Lebesgue measurable). Therefore $F(\tilde{B})$ contains a compact set Q of positive measure.

Let $E = \tilde{B} \cap F^{-1}(Q)$. Then $F_{|E}$ is a strictly increasing function and $F(E) = Q$. So $F_{|E}$ admits an inverse on E, namely $(F_{|E})^{-1} : Q \to E$, that is strictly increasing. Let $Q_1 \subset Q$ be a compact set of positive measure such that Q_1 does not contain the countable set of discontinuity points of $(F_{|E})^{-1}$. Let $K = (F_{|E})^{-1}(Q_1)$. Then K is a compact set (because any continuous function maps a compact set into a compact set). Clearly $K \subset \tilde{B}$. It follows that $m(K) = 0$, $F_{|K} = G_{|K}$ is strictly increasing and $G(K) = Q_1$.

(ii) Since $F \notin (N)$ on P, there exists $Z \subset P$ such that $m(Z) = 0$ and $m^*(F(Z)) > 0$. Hence $F \notin (N)$ on Z. Let A, A_1, B, B_+ and B_- be defined as in the proof of (i). Since $Z = A \cup A_1 \cup B_+ \cup B_-$ and $F \in (N)$ on $A \cup A_1$, by Lemma 3, (i) it follows that $F \notin (N)$ either on B_+ or on B_-. We may suppose without loss of generality that $F \notin (N)$ on B_+. Then there exists at least one positive integer n such that $F \notin (N)$ on B_n. Fix such a positive integer n and continue as in the proof of (i). $\square$

Lemma 6. *Let P be a Borel subset of $[a, b]$. Then we have:*

(i) $VB \cap (\overline{N}) \subseteq VB \cap \overline{M} \subseteq (VB \cap \overline{M}) \boxplus (VB \cap \overline{M}) \subseteq VB \cap (\overline{N})$ *on P.*

(ii) $VB \cap (N) \subseteq VB \cap (M) \subseteq (VB \cap (M)) \oplus (VB \cap (M)) \subseteq VB \cap (N)$ *on P.*

PROOF. (i) The first two inclusions are evident. We show the last one. Let $F_1, F_2 : P \to \mathbb{R}$, $F_1, F_2 \in VB \cap \overline{M}$. Clearly $F = F_1 + F_2 \in VB$ on P. Suppose to the contrary that $F \notin (\overline{N})$ on P. By Lemma 5, (i) it follows that P contains a compact set K of measure zero such that $F_{|K}$ is strictly increasing and $F(K)$ is a compact set of positive measure. By Lemma 2, (i) we obtain that $F \in (\overline{N})$ on K. Since F is increasing on K, by Lemma 4, it follows that $F \in (N)$ on K. Therefore $m(F(K)) = 0$, a contradiction.

(ii) Let $F_1, F_2 : P \to \mathbb{R}$, $F_1, F_2 \in VB \cap (M)$. Clearly $F = F_1 + F_2 \in VB$ on P. Suppose to the contrary that $F \notin (N)$ on P. Then P contains a compact set K of measure zero such that $F_{|K}$ is strictly monotone and $F(K)$ is a compact set of positive measure (see Lemma 5, (ii)). By Lemma 2, (ii) we obtain that $F \in (N)$ on K. Therefore $m(F(K)) = 0$, a contradiction. $\square$

Lemma 7. *Let P be a Borel subset of $[a, b]$. Then we have:*

(i) $VB \cap (\underline{N}) = VB \cap \underline{M}$ *is a real upper linear space on P.*

(ii) $VB \cap (N) = VB \cap (M)$ *is a real algebra on P.*

(iii) $VB \cap (M) = VB \cap \underline{M} \cap \overline{M} = VB \cap (\underline{N}) \cap (\overline{N}) = VB \cap (N)$ *on P.*

PROOF. (i) This follows by Lemma 6, (i).

(ii) That $VB \cap (N) = VB \cap (M)$ is a real linear space on P follows by Lemma 6, (ii). Let $F_1, F_2 \in VB \cap (N)$. Clearly F_1 and F_2 are bounded on P. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$G_1(x) := F_1(x) + \alpha_1 \geq 1 \ \text{ and } \ G_2(x) := F_2(x) + \alpha_2 \geq 1 \ \text{ on } P\,.$$

Then $F_1 \cdot F_2 = G_1 \cdot G_2 - \alpha_1 F_2 - \alpha_2 F_1 - \alpha_1 \alpha_2$. Since ln is a Lipschitz function on $[1, +\infty)$, it follows that $\ln \circ G_1$ and $\ln \circ G_2$ are $VB \cap (N)$ on P. But

$$\ln(G_1 \cdot G_2) = \ln(G_1) + \ln(G_2) \in VB \cap (N)$$

(because $VB \cap (N)$ is a real linear space). Then

$$G_1 \cdot G_2 = \exp(\ln(G_1 \cdot G_2)) \in VB \cap (N) \ \text{ on } P$$

(since the exponential function is Lipschitz on each compact interval).

(iii) This follows by (i), (ii) and the fact that always $(M) = \underline{M} \cap \overline{M}$. □

Remark 2. If $F_1, F_2 : [a, b] \to \mathbb{R}$ are $VB \cap (\underline{N})$, then it is possible that $F_1 \cdot F_2 \notin (\underline{N})$. Indeed, let F_1 be the Cantor function on $[0, 1]$ and $F_2(x) = -1$ for $x \in [0, 1]$. Then $F_1 \cdot F_2 = -F_1 \notin (\underline{N})$ (see Lemma 4).

Theorem 2. *We denote by $\mathcal{B}or$ the collection of all real Borel measurable functions. Let P be a Borel subset of $[a, b]$. Then we have:*

(i) $VBG \cap (\underline{N}) \cap \mathcal{B}or = VBG \cap \underline{M} \cap \mathcal{B}or$ *is a real upper linear space on P.*

(ii) $VBG \cap (N) \cap \mathcal{B}or = VBG \cap (M) \cap \mathcal{B}or$ *is a real algebra on P.*

(iii) $VBG \cap (N) \cap \mathcal{B}or = VBG \cap \underline{M} \cap \overline{M} \cap \mathcal{B}or = VBG \cap (\underline{N}) \cap (\overline{N}) \cap \mathcal{B}or = VBG \cap (N) \cap \mathcal{B}or$ *on P.*

PROOF. (i) Clearly $VBG \cap (\underline{N}) \subset VBG \cap \underline{M}$ on any set $E \subset [a, b]$ (E not necessarily a Borel set). Let $F : P \to \mathbb{R}$, $F \in VBG \cap \underline{M}$. Then there exists a sequence $\{P_n\}_n$ of sets such that $P = \cup_n P_n$ and F is VB on each P_n. By Lemma 4.1 of [8] (p. 221), there exists a function $F_n : [a, b] \to \mathbb{R}$, $F_n \in VB$ on $[a, b]$, such that $(F_n)_{|P_n} = F$. Let $Q_n = \{x \in P : F(x) = F_n(x)\}$. Since F and F_n are Borel functions, it follows that Q_n is a Borel set, that obviously contains the set P_n. Thus $F \in VB \cap \underline{M} = VB \cap (\underline{N})$ on Q_n (see Lemma 7, (i)). Since $P = \cup_n Q_n$, by Lemma 3, (i) we obtain that $F \in (\underline{N})$ on P. Hence

$$VBG \cap \underline{M} \cap \mathcal{B}or \subset VBG \cap (\underline{N}) \cap \mathcal{B}or \ \text{ on } P.$$

Let $F_1, F_2 : P \to \mathbb{R}$, $F_1, F_2 \in VBG \cap (\underline{N}) \cap \mathcal{B}or$ on P. Then there exists a sequence $\{E_n\}_n$ of sets such that $P = \cup_n E_n$ and $F_1, F_2 \in VB$ on each E_n.

Arguing as above, we may suppose without loss of generality that each E_n is a Borel set. By Lemma 7, (i), $VB \cap (\underline{N})$ is a real upper linear space on each E_n. Hence $F_1 + F_2 \in VB \cap (\underline{N})$ on each E_n. By Lemma 3, (i) it follows that $F \in (\underline{N})$ on P; so $F_1 + F_2 \in VBG \cap (\underline{N}) \cap \mathcal{B}or$ on P.

(ii) The proof is as that of (i), using Lemma 7, (ii) and Lemma 3, (i).

(iii) Clearly, we always have

$$VBG \cap (N) \subset VBG \cap (\underline{N}) \cap (\overline{N}) \subset VBG \cap \underline{M} \cap \overline{M} = VBG \cap (M)\,.$$

By (ii), we obtain that $VBG \cap (M) \cap \mathcal{B}or = VBG \cap (N) \cap \mathcal{B}or$. □

Remark 3. That $VBG \cap (N) \cap \mathcal{B}or$ is a real linear space on a Borel set was shown first (in a different manner) in [2].

4 Characterizations of $\underline{\mathbf{M}}$ and (M)

Theorem 3. *Let $P \subset [a, b]$ and $F : P \to \mathbb{R}$.*

(i) The following assertions are equivalent.

1) $F \in \underline{M}$ on P.

2) If $F \in VB$ on a Borel set $Q \subset P$, then $F \in (\underline{N})$ on Q.

3) If $F \in VB$ on a closed set $Q \subset P$, then $F \in (\underline{N})$ on Q.

4) If $F \in VB \cap \mathcal{C}_i$ on a closed set $Q \subset P$, then $F \in \underline{AC}$ on Q (see also [1], p. 84).

5) If F is decreasing and bounded on a Borel set $Q \subset P$, then $F \in (N)$ on Q.

6) If F is decreasing on a closed set $Q \subset P$, then $F \in (N)$ on Q.

7) If F is strictly decreasing and continuous on a closed set $Q \subset P$, then $F \in AC$ on Q.

(ii) If P is a Borel set and F is a Borel function, then $F \in \underline{M}$ on P if and only if $F \in (\underline{N})$ on any Borel subset Q of P on which F is VBG.

PROOF. (i) 1) ⇒ 2) Let $Q \subset P$ be a Borel set such that $F \in VB$ on Q. By 1) it follows that $F \in VB \cap \underline{M} = VB \cap (\underline{N})$ on Q (see Lemma 7, (i)). Hence $F \in (\underline{N})$ on Q.

2) ⇒ 3) This is obvious.

3) ⇒ 4) Let Q be a closed subset of P such that $F \in VB \cap \mathcal{C}_i$ on Q. By 3), $F \in VB \cap \mathcal{C}_i \cap (\underline{N}) = \underline{AC}$ on Q (see Corollary 2.21.1, (iii) of [1]).

4) $\Rightarrow$ 1) Let Q be a closed subset of P such that $F \in VB \cap \mathcal{C}$ on Q. Then $F \in VB \cap \mathcal{C}_i$ on Q, and by 4), F is $\underline{AC}$ on Q. Therefore $F \in \underline{M}$.

1) $\Rightarrow$ 5) Let Q be a Borel subset of P such that F is decreasing and bounded on Q. Clearly F is VB on Q, and by 1), $F \in VB \cap \underline{M} = VB \cap (\underline{N})$ on Q (see Lemma 7, (i)). Thus $F \in (\underline{N})$ on Q. By Lemma 4, $F \in (N)$ on Q.

5) $\Rightarrow$ 6) A real valued function that is decreasing on a bounded closed set is bounded on that set. Now the assertion is obvious.

6) $\Rightarrow$ 7) Let Q be a closed subset of P such that F is continuous and decreasing on Q. By 6), $F \in (N)$ on Q. Clearly $F \in VB \cap \mathcal{C} \cap (N) = AC$ (see the Banach-Zarecki Theorem).

7) $\Rightarrow$ 1) By Corollary 2.21.1, (iii) of [1], we have that $VB \cap \mathcal{C} \cap (\underline{N}) \subseteq \underline{AC}$ on a closed set. Suppose that 7) is true and 1) isn't. Since $F \notin \underline{M}$ on P, there exists a closed set $Q \subset P$ such that $F \in VB \cap \mathcal{C}$ but $f \notin \underline{AC}$ on Q. It follows that $F \notin (\underline{N})$ on Q. By Lemma 5, (i), there exists a compact set $K \subset Q$ of measure zero such that $m(F(K)) > 0$ and F is strictly decreasing on K. By 7), F is AC on K. Since $AC \subset (N)$, we obtain a contradiction.

(ii) "$\Rightarrow$" Let $Q \subset P$ be a Borel set such that $F_{|Q}$ is VBG. By hypotheses, $F \in VBG \cap \underline{M} = VBG \cap (\underline{N})$ (see Theorem 2, (i)). Therefore $F \in (\underline{N})$ on Q.

"$\Leftarrow$" Let Q be a closed subset of P such that $F_{|Q} \in VB \cap \mathcal{C}$. By hypotheses, $F \in VB \cap \mathcal{C} \cap (\underline{N}) \subset \underline{AC}$ on Q (see Corollary 2.21.1 (iii) of [1]). $\square$

Theorem 4. *Let $P \subset [a, b]$ and $F : P \to \mathbb{R}$.*

(i) The following assertions are equivalent.

1) $F \in (M)$ on P.

2) If $F \in VB$ on a Borel set $Q \subset P$, then $F \in (N)$ on Q.

3) If $F \in VB$ on a closed set $Q \subset P$, then $F \in (N)$ on Q.

4) If F is monotone and bounded on a Borel set $Q \subset P$, then $F \in (N)$ on Q.

5) If F is monotone on a closed subset Q of P, then $F \in (N)$ on Q.

6) If F is strictly monotone and continuous on a closed set $Q \subset P$, then $F \in AC$ on Q.

(ii) If P is a Borel set and F is a Borel function, then $F \in (M)$ if and only if $F \in (N)$ on any Borel set $Q \subset P$ on which F is VBG.

PROOF. (i) 1) $\Rightarrow$ 2) Let $Q \subset P$ be a Borel set such that $F \in VB$ on Q. By 1), $F \in VB \cap (M) = VB \cap (N)$ on Q (see Lemma 7, (ii)).

2) $\Rightarrow$ 3) This is obvious.

3) $\Rightarrow$ 1) Let Q be a closed subset of P such that $F_{|Q}$ is $VB \cap \mathcal{C}$. By 3), $F_{|Q} \in VB \cap \mathcal{C} \cap (N) = AC$ (see the Banach-Zarecki Theorem). Therefore $F \in (M)$ on P.

1) $\Rightarrow$ 4) Let Q be a Borel subset of P such that F is monotone and bounded on Q. Then $F \in VB$ on Q. By 1), $F \in VB \cap (M) = VB \cap (N)$ on Q (see Lemma 7, (ii)).

4) $\Rightarrow$ 5) This is obvious.

5) $\Rightarrow$ 6) Let Q be a closed subset of P such that F is strictly monotone and continuous on Q. By 5), $F \in (N)$ on Q. Clearly $F \in VB \cap \mathcal{C} \cap (N) = AC$ on Q (see the Banach-Zarecki Theorem).

6) $\Rightarrow$ 1) Suppose that 6) is true and 1) isn't. Since $F \notin (M)$ on P, it follows that there exists a closed set $Q \subset P$ such that $F \in VB \cap \mathcal{C}$ on Q, but $F \notin AC$ on Q. Since $VB \cap \mathcal{C} \cap (N) = AC$ on a closed set (see the Banach-Zarecki Theorem), we obtain that $F \notin (N)$ on Q. By Lemma 5, (ii), there exists a compact set $K \subset Q$ of measure zero such that $m(F(K)) > 0$ and F is strictly monotone on K. By 6), $F \in AC$ on K, a contradiction.

(ii) "$\Rightarrow$" Let $Q \subset P$ be a Borel set such that $F_{|Q} \in VBG$. By hypothesis, $F \in VBG \cap (M) = VBG \cap (N)$ (see Theorem 2, (ii)). Hence $F \in (N)$ on Q.

"$\Leftarrow$" This follows by the Banach-Zarecki Theorem. $\square$

Corollary 1. *Let $P \subset [a, b]$ be a Borel set. Then we have:*

(i) $(VBG \cap \underline{M} \cap \mathcal{B}or) \boxplus (\underline{M} \cap \mathcal{B}or) = (\underline{M} \cap \mathcal{B}or)$ *on* P.

(ii) $(VBG \cap (M) \cap \mathcal{B}or) \oplus ((M) \cap \mathcal{B}or) = (M) \cap \mathcal{B}or$ *on* P.

PROOF. Let $F_1, F_2, F : P \to \mathbb{R}$, $F = F_1 + F_2$.

(i) Suppose that $F_1 \in VBG \cap \underline{M} \cap \mathcal{B}or$ and $F_2 \in \underline{M} \cap \mathcal{B}or$ on P. Let Q be a Borel subset of P such that $F_{|Q}$ is VB. Clearly $F_2 = F - F_1$ is $VBG \cap \underline{M} \cap \mathcal{B}or$ on Q. By Theorem 2, (i), it follows that $F \in (\underline{N})$ on Q, and by Theorem 3, 1), 2) we obtain that $F \in \underline{M}$ on P.

(ii) Suppose that $F_1 \in VBG \cap (M) \cap \mathcal{B}or$ and $F_2 \in (M) \cap \mathcal{B}or$ on P. Let $Q \subset P$ be a Borel set such that $F_{|Q}$ is VB. Clearly $F_2 = F - F_1$ is $VBG \cap (M) \cap \mathcal{B}or$ on Q. By Theorem 2, (ii), it follows that $F \in (N)$ on Q, and by Theorem 4, 1), 2), we obtain that $F \in (M)$ on P. $\square$

Remark 4. In Corollary 1, (ii), Foran's condition (M) cannot be replaced by Lusin's condition (N), although $VBG \cap (N) \cap \mathcal{B}or = VBG \cap (M) \cap \mathcal{B}or$ (see Theorem 2, (ii)). This follows from an example of Mazurkiewicz ([5] or [1], p. 226). He constructed a continuous function $f(x)$ on $[0, 1]$, such that $f \in (N)$, but for $b \neq 0$ the function $f(x) + bx \notin (N)$.

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