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BV_p-FUNCTIONS AND CHANGE OF VARIABLE

Abstract

In this note we discuss some interconnections between the space $BV_p[a,b]$ $(1 \leq p < \infty)$ of functions of bounded p-variation (in Wiener's sense) and the space $Lip_\alpha[a,b]$ $(0 < \alpha \leq 1)$ of Hölder continuous functions. In particular, we show that $f \in BV_p[a,b]$ if and only if $f = g \circ \tau$, with $g \in Lip_{1/p}[a,b]$ and τ being monotone, and that $f \in BV_p[a,b] \cap C[a,b]$ if and only if $f = g \circ \tau$, with $g \in Lip_{1/p}[a,b]$ and τ being a homeomorphism.

1 Introduction

In this note we will discuss some interconnections between functions of bounded p-variation for $p \in [1,\infty)$ (in Wiener's sense), on the one hand, and Hölder continuous functions with Hölder exponent $\alpha \in (0,1]$, on the other. Roughly speaking, classical functions of bounded variation (i.e., $p = 1$) under these interconnections correspond to Lipschitz continuous functions (i.e., $\alpha = 1$). Passing from Lipschitz to Hölder continuity, however, is often highly nontrivial and by no means "automatic". For instance, a function $f \in Lip[a,b]$ is always differentiable a.e. on $[a,b]$, but this is not true for $f \in Lip_\alpha[a,b]$ in case $\alpha < 1$. Similarly, every Lipschitz continuous function has bounded variation, but this fails for Hölder continuous functions of order $\alpha < 1$. Finally,

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every function in $Lip[a,b]$ has the Luzin (N)-property of preserving Lebesgue nullsets, while this is not true for functions from $Lip_\alpha[a,b]$.

The main purpose of this note is to find out which results for functions $f \in BV[a,b]$ (respectively, $f \in Lip[a,b]$) carry over to $f \in BV_p[a,b]$ (respectively, $f \in Lip_\alpha[a,b]$), and which do not. Examples of the "asymmetry" between the cases $p = 1$ and $p > 1$ are given in Theorem 1 and Theorem 4 below.

2 Main Results

Before we begin our discussion, we briefly recall some definitions and notation. Throughout this note, by $\mathcal{P}[a,b]$ we denote the family of all partitions $P = \{t_0, t_1, \dots, t_m\}$ $(m \in \mathbb{N})$ of the interval $[a,b]$, and $p \geq 1$ is a real number. Given a function $f : [a,b] \to \mathbb{R}$ we put

$$\mathrm{Var}_p(f, P; [a,b]) := \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^p \qquad (P = \{t_0, t_1, \dots, t_m\})$$

and

$$\mathrm{Var}_p(f; [a,b]) := \sup\{\mathrm{Var}_p(f, P; [a,b]) : P \in \mathcal{P}[a,b]\}, \tag{1}$$

where the supremum in (1) is taken over all partitions of $[a,b]$, and call (1) the (total) *p-variation* of f over $[a,b]$. It is not hard to show that the linear space $BV_p[a,b]$ of all functions with finite p-variation over $[a,b]$, equipped with the norm

$$\|f\|_{BV_p} = |f(a)| + \mathrm{Var}_p(f; [a,b])^{1/p}, \tag{2}$$

is a Banach space. For $f \in BV_p[a,b]$ and $a \leq x \leq b$ we further put

$$V_{f,p}(x) := \mathrm{Var}_p(f; [a,x]) \qquad (a \leq x \leq b). \tag{3}$$

Thus, the map $x \mapsto V_{f,p}(x)$ is increasing with $V_{f,p}(a) = 0$ and $V_{f,p}(b) = \mathrm{Var}_p(f; [a,b])$. A detailed study of the properties of functions $f \in BV_p[a,b]$ may be found in [5]. Apart from the space $BV_p[a,b]$, in what follows we will also consider the Banach space $Lip_\alpha[a,b]$ $(0 < \alpha \leq 1)$ of all Hölder continuous (or Lipschitz continuous, for $\alpha = 1$) functions $f : [a,b] \to \mathbb{R}$ endowed with the norm

$$\|f\|_{Lip_\alpha} := |f(a)| + lip_\alpha(f),$$

where

$$lip_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

In case $p = 1$ or $\alpha = 1$ we will drop the subscript 1, so we write $\mathrm{Var}(f, P; [a, b])$, $\mathrm{Var}(f; [a, b])$, $BV[a, b]$, $V_f(x)$, $lip(f)$, and $Lip[a, b]$ instead of $\mathrm{Var}_1(f, P; [a, b])$, $\mathrm{Var}_1(f; [a, b])$, $BV_1[a, b]$, $V_{f,1}(x)$, $lip_1(f)$, and $Lip_1[a, b]$, respectively. A straightforward calculation shows that

$$Lip_\alpha[a, b] \subseteq BV_{1/\alpha}[a, b] \qquad (0 < \alpha \leq 1); \tag{4}$$

in particular, $Lip[a, b] \subseteq BV[a, b]$. The following example shows that the inclusion (4) is actually strict for any $\alpha \in (0, 1]$.

Example 1. For $\gamma > 0$, let $g_\gamma : [0, 1] \to \mathbb{R}$ be the "zigzag function" defined by

$$g_\gamma(x) := \begin{cases} 0 & \text{for } x = 0, \\ \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^\gamma} & \text{for } x = a_n, \\ \text{linear} & \text{otherwise,} \end{cases} \tag{5}$$

where $a_n := 1 - 2^{-n}$. Geometrically, the graph of g_γ starts at the origin and increases linearly by 1 on the interval $[0, 1/2]$ so that $g_\gamma(1/2) = 1$. Then we let g_γ decrease linearly by $2^{-\gamma}$ on $[1/2, 3/4]$, increase linearly by $3^{-\gamma}$ on $[3/4, 7/8]$, decrease linearly by $4^{-\gamma}$ on $[7/8, 15/16]$, and so on. It follows from the construction and continuity of this zigzag function that

$$g_\gamma(1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^\gamma}, \qquad \mathrm{Var}_p(g_\gamma; [0, 1]) = \sum_{k=1}^{\infty} \frac{1}{k^{p\gamma}}. \tag{6}$$

In particular, $g_\gamma \in BV_p([0, 1])$ if and only if $p\gamma > 1$. On the other hand, the function g_γ does not belong to *any* Hölder space $Lip_\alpha([0, 1])$. In fact, a simple geometric reasoning shows that

$$lip_\alpha(g_\gamma) \geq \sup\,\{2^{n\alpha} n^{-\gamma} : n = 1, 2, 3, \ldots\}$$

for $0 < \alpha \leq 1$ and $\gamma > 0$, and the exponential growth of $2^{n\alpha}$ always dominates the power type growth of n^γ.

Of course, the zigzag function (5) may also be used to show that the inclusion $BV_p[a, b] \subseteq BV_q[a, b]$ is strict for $1 \leq p < q$.

We point out that the inclusion $Lip[a, b] \subseteq BV[a, b]$ is in a certain sense sharp, inasmuch as one may construct, for fixed $\alpha \in (0, 1)$, a function which belongs to $Lip_\alpha[0, 1]$ but not to $BV[0, 1]$, see [2, Exercise 14.28], or even a function which belongs to $Lip_\alpha[0,1]$ for *every* $\alpha \in (0,1)$ but not to $BV[0,1]$, see [2, Exercise 14.29]. Such examples, however, are somewhat more complicated than our Example 1. Since the Russian reference [2] is not easily accessible, for the reader's ease we briefly recall these examples.

Example 2. The first function constructed in [2, Exercise 14.28] looks very much like a "mirror reversed version" of our zigzag function (5). Define a constant γ and a sequence $(t_n)_n$ in $[0,1]$ by

$$\gamma := \sum_{k=1}^{\infty} \frac{1}{k^{1/\alpha}}, \qquad t_n := \frac{1}{\gamma} \sum_{k=n}^{\infty} \frac{1}{k^{1/\alpha}}.$$

Then we define $f : [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & \text{for } x = 0, \\ \dfrac{(-1)^n}{n} & \text{for } x = t_n, \\ \text{linear} & \text{otherwise.} \end{cases}$$

By choosing partitions containing $t_1, t_2, \ldots, t_n$ and using the divergence of the harmonic series, it is easy to see that $f \notin BV[0,1]$. On the other hand, distinguishing several cases for x and y, one may prove that $|f(x) - f(y)| \leq 4|x-y|^\alpha$, and so $f \in Lip_\alpha[0,1]$.

In [2, Exercise 14.29] the authors replace γ and $(t_n)_n$ in this example by

$$\gamma := \sum_{k=1}^{\infty} \frac{1}{k \log^2(k+1)}, \qquad t_n := \frac{1}{\gamma} \sum_{k=n}^{\infty} \frac{1}{k \log^2(k+1)},$$

and define $f : [0,1] \to \mathbb{R}$ precisely as before. Again, one may show, by considering partitions containing $t_1, t_2, \ldots, t_n$, that $f \notin BV[0,1]$. On the other hand, a somewhat cumbersome calculation shows that f belongs to $Lip_\alpha[0,1]$ for *any* $\alpha < 1$.

Our first theorem is concerned with the "interaction" between the variation function $V_{f,p}$ given in (3) and its parent function f. A detailed discussion of such interactions may be found in the survey paper [7]; for example, it is well-known that $V_{f,p}$ is (absolutely) continuous if f is (absolutely) continuous, and vice versa. Here we prove a special result related to Hölder continuity (in particular, Lipschitz continuity) of the function (3).

Theorem 1. *For $f \in BV_p[a,b]$ and $V_{f,p}$ as in (3), the following statements are true. (a) The function f is Hölder continuous of order $\alpha = 1/p$ if and* *only if the function $V_{f,p}$ is Lipschitz continuous; moreover, in this case we have $lip_{1/p}(f) = lip(V_{f,p})^{1/p}$. (b) The function f is Hölder continuous of order $\alpha/p \in (0,1)$ if the function $V_{f,p}$ is Hölder continuous order α; moreover, in this case we have $lip_{\alpha/p}(f) \le lip_\alpha(V_{f,p})^{1/p}$.*

PROOF. Suppose that $f \in Lip_{1/p}[a,b]$, $L > lip_{1/p}(f)$, and $a \le x < y \le b$, and let $P = \{t_0, t_1, \dots, t_m\} \in \mathcal{P}[x,y]$ be any partition of the interval $[x,y]$. Then

$$\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^p \le L^p \sum_{j=1}^{m} (t_j - t_{j-1}) = L^p(y-x)$$

and so

$$V_{f,p}(y) - V_{f,p}(x) = \mathrm{Var}_p(f;[x,y]) \le L^p(y-x),$$

which shows that $V_{f,p} \in Lip[a,b]$ with $lip(V_{f,p}) \le lip_{1/p}(f)^p$. Conversely, suppose that $V_{f,p} \in Lip[a,b]$ and $a \le x < y \le b$. Then

$$|f(x) - f(y)|^p \le \mathrm{Var}_p(f;[x,y]) = V_{f,p}(y) - V_{f,p}(x) \le lip(V_{f,p})|x-y| \tag{7}$$

which shows that $f \in Lip_{1/p}[a,b]$ with $lip_{1/p}(f) \le lip(V_{f,p})^{1/p}$ and proves (a). To prove (b) observe that (7) in case $V_{f,p} \in Lip_\alpha[a,b]$ reads

$$|f(x) - f(y)|^p \le \mathrm{Var}_p(f;[x,y]) = V_{f,p}(y) - V_{f,p}(x) \le lip_\alpha(V_{f,p})|x-y|^\alpha$$

which shows that $f \in Lip_{\alpha/p}[a,b]$ with $lip_{\alpha/p}(f) \le lip_\alpha(V_{f,p})^{1/p}$. □

The proof of (a) shows that $\|V_{f,p}\|_{Lip} = \|f\|^p_{Lip_{1/p}}$ (in particular, $\|V_f\|_{Lip} = \|f\|_{Lip}$) for all functions $f \in Lip_{1/p}[a,b]$ satisfying $f(a) = 0$. Observe that there is an asymmetry in statement (b) of Theorem 1: we did *not* claim that $f \in Lip_{\alpha/p}$ (hence $f \in BV_{p/\alpha}[a,b]$) implies $V_{f,p} \in Lip_\alpha$. In fact, to the best of our knowledge this is an open problem even in case $p = 1$, i.e., for functions $f \in BV[a,b] \cap Lip_\alpha[a,b]$ for $0 < \alpha < 1$. Of course, if one merely requires $f \in Lip_\alpha[a,b]$, Example 2 shows that the answer is negative, because in this case the function $x \mapsto V_f(x)$ jumps from 0 to ∞ as soon as x gets positive.

Our next theorem gives a simple sufficient condition under which a "change of variables" preserves bounded p-variation.

Theorem 2. *Let $g : [c,d] \to \mathbb{R}$ a bounded map and $\tau : [a,b] \to [c,d]$ strictly increasing and onto. Then $f := g \circ \tau \in BV_p[a,b]$ if and only if $g \in BV_p[c,d]$.*

PROOF. First of all, note that τ is continuous, by the intermediate value theorem, and so a homeomorphism. Moreover, our assumptions on τ imply that

$$\tau(\{t_0, t_1, \dots, t_m\}) = \{\tau(t_0), \tau(t_1), \dots, \tau(t_m)\}$$

is a bijection between $\mathcal{P}[a,b]$ and $\mathcal{P}[c,d]$. Therefore, for every function $g \in BV[c,d]$ we have $\mathrm{Var}_p(f,P;[a,b]) = \mathrm{Var}_p(g,\tau(P);[c,d])$, hence

$$\mathrm{Var}_p(f;[a,b]) \leq \mathrm{Var}_p(g;[c,d]).$$

Applying this reasoning to the function τ^{-1} we conclude that also

$$\mathrm{Var}_p(g;[c,d]) = \mathrm{Var}_p(f \circ \tau^{-1};[c,d]) \leq \mathrm{Var}_p(f;[a,b]).$$

This shows that g and $f = g \circ \tau$ have the same total p-variation on their domain of definition, and so the assertion follows. □

Our proof shows even more: by definition of the norm (2), the map $g \mapsto f = g \circ \tau$ is an *isometry* between the spaces $(BV_p[a,b], \|\cdot\|_{BV_p})$ and $(BV_p[c,d], \|\cdot\|_{BV_p})$, since $f(a) = g(\tau(a)) = g(c)$ and $f(b) = g(\tau(b)) = g(d)$. The following two examples show that we cannot drop the continuity or monotonicity assumption on τ in Theorem 2.

Example 3. Define $\tau : [0,4] \to [0,4]$ by $\tau(0) := 0$ and $\tau(t) := 3 + t/4$ for $0 < t \leq 4$. Then τ is strictly increasing with $\tau(0) = 0$ and $\tau(4) = 4$, but discontinuous at $t = 0$. The function $g : [0,4] \to \mathbb{R}$ defined by

$$g(x) := \begin{cases} 0 & \text{for} \quad 0 \leq x \leq 1, \\ \tan \frac{\pi}{2}(x-1) & \text{for} \quad 1 < x < 2, \\ 0 & \text{for} \quad 2 \leq x \leq 4, \end{cases}$$

does not belong to $BV_p[0,4]$ for any p, since it is unbounded near $x = 2$. On the other hand, the function $f(t) = (g \circ \tau)(t) \equiv 0$ trivially belongs to $BV_p[0,4]$ for all p.

Example 4. For $p \geq 1$, define $\tau : [0,1] \to [0,1]$ by

$$\tau(t) := \begin{cases} t\left|\sin \dfrac{1}{t}\right|^p & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0. \end{cases}$$

Then τ is continuous, but of course far from being monotone. The function $g : [0,1] \to \mathbb{R}$ defined by $g(x) := x^{1/p}$ belongs to $Lip_{1/p}[0,1]$, hence also to $BV_p[0,1]$, by (4). On the other hand, the function $f = g \circ \tau$ does not belong to $BV_p[0,1]$, which can be seen as follows. For $n \in \mathbb{N}$, consider the partition

$$P_n := \{0,1\} \cup \{s_1, \ldots, s_n\} \cup \{t_1, \ldots, t_n\},$$

where

$$s_j := \frac{1}{4j\pi}, \quad t_j := \frac{1}{(4j+1)\pi} \qquad (j = 1, 2, \dots, n).$$

Since $f(s_j) = 0$ and $f(t_j) = t_j$, the partition P_n gives the contribution

$$\mathrm{Var}_p(f, P_n; [0,1]) \geq \left(\frac{2}{\pi}\right)^{1/p} \sum_{k=1}^{n} \frac{1}{(4k+1)^{1/p}}, \tag{8}$$

and the sum in (8) is unbounded as $n \to \infty$, because $p \geq 1$.

Theorem 2 shows that, roughly speaking, monotone surjective maps are the only suitable changes of variables which preserve bounded p-variation (in particular, bounded variation).

In the historical paper [8] in which Camille Jordan introduced the class $BV[a, b]$ he also proved that the function $f - V_f$ is increasing for $f \in BV[a, b]$, and so every function of bounded variation may be represented as difference of two increasing functions. Now we discuss another type of decomposition of a function $f \in BV_p[a, b]$ (in particular, $f \in BV[a, b]$) into a Hölder (in particular, Lipschitz) continuous function and a monotone change of variables. The following result may be found in [4] without proof.

Theorem 3. *A function f belongs to $BV_p[a, b]$ if and only if it may be represented as composition $f = g \circ \tau$, where $\tau : [a, b] \to [c, d]$ is increasing and $g \in Lip_{1/p}[c, d]$ with Hölder constant $lip_{1/p}(g) = 1$.*

PROOF. Suppose that $f = g \circ \tau$, where g and τ have the mentioned properties. Given any partition $P = \{t_0, t_1, \dots, t_m\} \in \mathcal{P}[a, b]$, we get

$$\begin{aligned} \mathrm{Var}_p(f, P; [a, b]) &= \sum_{j=1}^{m} |g(\tau(t_j)) - g(\tau(t_{j-1}))|^p \\ &\leq \sum_{j=1}^{m} |\tau(t_j) - \tau(t_{j-1})| \\ &= |\tau(b) - \tau(a)|, \end{aligned}$$

hence $f \in BV_p[a, b]$. Conversely, let $f \in BV_p[a, b]$, and put $\tau(x) = V_{f,p}(x)$, see (4). Then τ maps $[a, b]$ into $[c, d]$, where $c = 0$ and $d = \mathrm{Var}_p(f; [a, b])$. If we define the function g on the range $\tau([a, b]) \subseteq [c, d]$ by putting $g(\tau(x)) := f(x)$, then the decomposition $f = g \circ \tau$ holds trivially by construction and

$$|g(\tau(s)) - g(\tau(t))| = |f(s) - f(t)| \leq \mathrm{Var}_p(f; [s, t])^{1/p} \leq |\tau(s) - \tau(t)|^{1/p}$$

for $a \leq s < t \leq b$. Consequently, g is in fact Hölder continuous with Hölder exponent $\alpha = 1/p$ and Hölder constant 1, but only on $\tau([a, b])$.

It remains to extend g as a Hölder continuous function with the same Hölder exponent to the whole interval $[c, d]$. Here we may use a general result by McShane [10] which reads as follows. If $M \subset \mathbb{R}$ and $g : M \to \mathbb{R}$ is Hölder continuous with Hölder exponent $\alpha \in (0, 1]$, then the map $\overline{g} : \mathbb{R} \to \mathbb{R}$ defined by

$$\overline{g}(x) := \sup \{f(z) - lip_\alpha(f)|x - z|^\alpha : z \in M\} \tag{9}$$

is Hölder continuous on $\mathbb{R}$ with $lip_\alpha(\overline{g}) = lip_\alpha(g)$ and satisfies $\overline{g}(x) = g(x)$ for $x \in M$. Applying this to g as above on $M = \tau([a, b])$ we obtain the desired map. $\square$

We illustrate Theorem 3 by means of the following simple

Example 5. Let $[a, b] = [0, 2]$ and $f = \chi_{\{1\}}$ be the characteristic function of the singleton $\{1\}$. The variation function $\tau : [0, 2] \to [0, 2]$ from (1) in this case has the form

$$\tau(x) = 1 + \operatorname{sgn}(x - 1) = \begin{cases} 0 & \text{for} \quad 0 \leq x < 1, \\ 1 & \text{for} \quad x = 1, \\ 2 & \text{for} \quad 1 < x \leq 2. \end{cases}$$

Observe that $\tau([0, 2]) = \{0, 1, 2\}$, $g(0) = g(2) = 0$, and $g(1) = 1$, hence $lip_\alpha(g) = 1$ in this example. Applying the McShane extension (9) to g we end up with the function

$$\overline{g}(x) = \max \{-|x|^\alpha, 1 - |x - 1|^\alpha, -|x - 2|^\alpha\} = 1 - |x - 1|^\alpha \quad (0 \leq x \leq 2)$$

which is easily seen to be Hölder continuous with Hölder exponent α on the whole interval $[0, 2]$.

The following result may be considered as a refinement of Theorem 2: it shows that a *continuous* functions of bounded p variation may be "made" Hölder continuous with Hölder exponent $1/p$, and even differentiable with bounded derivative, after a suitable homeomorphic change of variables. In case $p = 1$ this result has been proved in [3].

Theorem 4. *For a function $g : [a, b] \to \mathbb{R}$, the following are equivalent.*

(a) The function g is continuous and has bounded p-variation.

(b) There exists a homeomorphism $\tau : [a, b] \to [a, b]$ such that $f = g \circ \tau : [a, b] \to \mathbb{R}$ is Hölder continuous on $[a, b]$ with Hölder exponent $1/p$.

Proof. Without loss of generality we take $[a,b]=[0,1]$. Suppose first that $g \in C[0,1] \cap BV_p[0,1]$ and put $V_{g,p}(1) =: \omega$, see (1). To prove (b) we define $\sigma : [0,1] \to [0,1+\omega]$ by

$$\sigma(x) := x + V_{g,p}(x) \qquad (0 \le x \le 1). \tag{10}$$

Clearly, σ is strictly increasing and surjective and satisfies

$$|g(x)-g(y)|^p \le |V_{g,p}(x)-V_{g,p}(y)| \le |V_{g,p}(x)+x-V_{g,p}(y)-y| = |\sigma(x)-\sigma(y)| \tag{11}$$

for all $x,y \in [0,1]$. So the map $\tau : [0,1] \to [0,1]$ defined by

$$\tau(t) := \sigma^{-1}(t+\omega t) \qquad (0 \le t \le 1) \tag{12}$$

is strictly increasing with $\tau(0)=0$ and $\tau(1)=1$, hence an homeomorphism. Moreover, from (11) it follows that the map $f = g \circ \tau$ satisfies

$$|f(s)-f(t)| \le |g(\tau(s))-g(\tau(t))| \le |\sigma(\tau(s))-\sigma(\tau(t))|^{1/p} \le (1+\omega)^{1/p}|s-t|^{1/p}$$

for all $s,t \in [0,1]$. This shows that $f \in Lip_{1/p}[0,1]$ with $lip_{1/p}(f) \le (1+\omega)^{1/p}$, and so we have proved (b).

The fact that (b) implies (a) follows from Theorem 2. Indeed, $g \circ \tau \in Lip_{1/p}[a,b] \subset BV_p[a,b]$ implies $g = g \circ \tau \circ \tau^{-1} \in BV_p[a,b]$, since every homeomorphism of an interval onto itself is strictly monotone. $\square$

Observe the subtle difference between Theorems 2 and 4: While a generic function $g \in BV_p[a,b]$ in general remains in $BV_p[a,b]$ (hence discontinuous) after a homeomorphic change of variables, a function $g \in BV_p[a,b] \cap C[a,b]$ becomes even *Hölder continuous* of order $1/p$. So adding continuity bridges the gap (which is essential, as Example 1 shows) between $Lip_{1/p}[a,b]$ and $BV_p[a,b]$.

We illustrate Theorem 4 by means of two examples. The function f in the first example belongs to $BV_p[0,1]$, but does not belong to $Lip_\alpha[0,1]$ for any $\alpha \in (0,1]$.

Example 6. For $\gamma > 0$, let $g_\gamma : [0,1] \to \mathbb{R}$ be defined as in Example 1. Theorem 4 gives a constructive recipe how to transform the function g_γ into a function $f = g_\gamma \circ \tau \in Lip_\alpha([0,1])$ with arbitrary $\alpha < \gamma$. Putting $a_n = 1-2^{-n}$ as in Example 1, we have

$$P_n := \{0, \frac{1}{2}, \frac{3}{4}, \ldots, 1-2^{-n}\} \in \mathcal{P}[0,a_n], \quad Var_p(g_\gamma, P_n; [0,a_n]) = \sum_{k=1}^{n} \frac{1}{k^{p\gamma}}.$$

Therefore, in case $p\gamma > 1$ the function (10) has the form

$$\sigma(x) = \begin{cases} x + \displaystyle\sum_{k=1}^{n(x)} \frac{1}{k^{p\gamma}} & \text{for} \quad 0 \le x < 1, \\ 1 + \mathrm{Var}_p(g_\gamma; [0,1]) & \text{for} \quad x = 1, \end{cases}$$

where $n(x)$ denotes the largest natural number n such that $x \ge a_n$, i.e., $2^{-n} \ge 1 - x$. Since ω is given by the value of the second series in (6), we may use (12), at least theoretically, to calculate the homeomorphism τ piecewise in this example.

Example 7. Let $g : [0,1] \to [0,1]$ be the Cantor function associated to the classical perfect Cantor nullset $C \subset [0,1]$. It is well known [6] that g is a continuous increasing surjective map from $[0,1]$ onto itself. Moreover, g cannot be absolutely continuous, by the Vitali-Banach-Zaretskij theorem [9], since the image $g(C)$ of the nullset C has positive measure, and so g does not have the Luzin property. However, one may show [1] that g is Hölder continuous with best possible Hölder exponent $\alpha = \log 2/\log 3$ which precisely coincides with the Hausdorff dimension of the Cantor set C. By (4), we conclude that $g \in BV_p[0,1]$ for $p = \log 3/\log 2$.

However, we can do better. Indeed, since the Cantor function is monotone, it belongs to $BV[0,1]$, so we may choose $p = 1$ in Theorem 4 and find a homeomorphism $\tau : [0,1] \to [0,1]$ such that $f = g \circ \tau$ is even *Lipschitz continuous* on $[0,1]$. Moreover, the proof of Theorem 4 shows how to do this. Since $V_g = g$, we see that $\sigma(x) = x + g(x)$ and therefore

$$f(t) = g(\sigma^{-1}(2t)) \qquad (0 \le t \le 1). \tag{13}$$

To make this more explicit, we consider this function at the endpoints of the deleted intervals in the construction of the Cantor set C. Clearly.

$$g(1 \cdot 3^{-n}) = g(2 \cdot 3^{-n}) = 1 \cdot 2^{-n}, \quad g(7 \cdot 3^{-n}) = g(8 \cdot 3^{-n}) = 3 \cdot 2^{-n},$$
$$g(19 \cdot 3^{-n}) = g(20 \cdot 3^{-n}) = 5 \cdot 2^{-n}, \ldots$$

and, more generally,

$$g(1 - 2 \cdot 3^{-n}) = g(1 - 1 \cdot 3^{-n}) = 1 - 1 \cdot 2^{-n} \quad (n = 1, 2, 3, \ldots).$$

A straightforward, but somewhat cumbersome calculation gives then the values of the function f in (13) at the points $a_n := 2 - (2 \cdot 3^{-n} + 1 \cdot 2^{-n}) \in [0,1]$, and we only have to extend f linearly to a Lipschitz continuous function on the whole interval $[0,1]$.

Although the explicit computation of the function $f = g \circ \tau$ in Example 7 is rather messy, this example has a certain theoretical interest. In [3, Theorem 1] it was shown that, in case of a function $g \in C[a, b] \cap BV[a, b]$ one may even find a homeomorphism $\tau : [a, b] \to [a, b]$ such that $f = g \circ \tau : [a, b] \to \mathbb{R}$ is differentiable with bounded derivative on $[a, b]$. The proof is based on the fact that in this case we may assume that g is Lipschitz continuous, and so differentiable a.e. on $[a, b]$. By Zahorski's theorem [11,12] one may then find a homeomorphism $\tau : [0, 1] \to [0, 1]$ which is differentiable with bounded derivative τ' on $[0, 1]$ and satisfies $\tau'(t) = 0$ precisely for $t \in \tau^{-1}(G)$, where G is an appropriate G_δ nullset which contains all points of non-differentiability of g. This homeomorphism has then the desired properties. Unfortunately, a Hölder continuous function need not be differentiable a.e., and so this proof does not work for $g \in BV_p[a, b]$ in case $p > 1$.

The question arises whether or not one may choose, in case $p = 1$, the homeomorphism τ in such a way that $f = g \circ \tau$ is even differentiable with *continuous* derivative. Example 7 shows that the answer is negative. In fact, suppose that $f = g \circ \tau \in C^1[0, 1]$ for some homeomorphism $\tau : [0, 1] \to [0, 1]$. The derivative f' of f is equal to 0 at each point of $[0, 1] \setminus \tau^{-1}(C)$. But $\tau^{-1}(C)$ cannot be a nullset, since f, being Lipschitz continuous, has the Luzin property, and so $g(C) = (f \circ \tau^{-1})(C)$ would be a nullset as well, a contradiction. Therefore the derivative of $f = g \circ \tau$ cannot be 0 a.e. on $[0, 1]$. So Theorem 1 in [3] is in a certain sense optimal in case $p = 1$. To show that our Theorem 4 is optimal in case $p > 1$, one should find a function $g \in BV_p[a, b] \cap C[a, b]$ such that no homeomorphism $\tau : [a, b] \to [a, b]$ makes $f = g \circ \tau$ differentiable; this seems to be an open problem.

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