RESEARCH

Real Analysis Exchange
Vol. 37(1), 2011/2012, pp. 83–116

Martial R. Hille, Humboldt University, Unter den Linden 6, D-10099 Berlin, Germany. email: Hille@math.hu-berlin.de

RESONANCES FOR GRAPH DIRECTED MARKOV SYSTEMS

Abstract

In this paper we introduce and study a certain zeta function and its zeros for conformal graph directed Markov systems (GDMS). These zeros are referred to as resonances.

We specify a list of geometric, combinatoric and analytic conditions on the GDMS under which this zeta function is indeed well defined and even holomorphic on the whole complex plane. In addition, we prove that there is a half-plane where there are no zeros. Finally, we transfer a result of Guillopé *et al.* in [15] on the zeros of the Selberg zeta function to our setting. More precisely, we give an upper bound for the number of resonances in a strip in terms of the Hausdorff dimension of the limit set of the GDMS.

We also briefly discuss relations to other zeta functions, in particular to the Selberg zeta function associated to a Kleinian group of Schottky type and to the geometric zeta function associated to a fractal string. Since the definition of the zeta function introduced in our paper is based on the transfer operator associated to the GDMS, these relations to other zeta functions indicate that our zeta function is a natural generalization of these zeta functions to conformal GDMSs.

1 Introduction

Graph directed Markov systems (GDMS) were introduced by Mauldin and Urbanski (see e.g. [20]). These systems form a significant generalization of the concept of an iterated function system (IFS) in fractal geometry. A large class of fractals can be described as limit sets obtained by iterating the maps

Mathematical Reviews subject classification: Primary: 37C30, 37F35; Secondary: 37C4540
Key words: graph directed Markov systems, Hausdorff dimension, zeta function, zeros
Received by the editors September 17, 2010
Communicated by: Lars Olsen

of such systems. Examples range from the well known middle third Cantor set to limit sets of certain types of Kleinian groups and Julia sets of a wide class of maps called parabolically semi hyperbolic generalized polynomial-like maps (c.f.g. [34]).

In this paper we consider a certain kind of zeta function associated to a finitely generated conformal GDMS which, in addition, satisfies a geometric condition referred to as nestedness condition (NC). This type of zeta function will be a kind of Ruelle zeta function, and hence, can be considered as a dynamical zeta function. The zeros of this function will be called resonances. We generalize a result in [15] on zeros of the Selberg zeta function to this type of Ruelle zeta function for the above mentioned GDMSs.

This zeta function will be defined via the determinant of the identity operator minus the complexified Frobenius-Perron-Ruelle-operator, which we shall refer to as FPR-Operator. The latter operator will act on a Hilbert space of complex valued functions defined in a complex neighborhood of the limit set of the GDMS. The main results of this paper are summarized in the following theorems (for the definitions see Section 2). Throughout, we write $a(w) \ll b(w)$ if there is a constant $c > 0$ (i.e. independent of w) with $a(w) \leq c \cdot b(w)$. We also write $a(w) \asymp b(w)$ if $a(w) \ll b(w)$ and $b(w) \ll a(w)$.

Main Theorem 1. *Let S be a finitely generated conformal GDMS acting on $\mathbb{R}^m$, which satisfies the strong separation condition (SSC) and the nestedness condition (NC). Then the complexification $\mathcal{L}_w$ of the Frobenius-Perron-Ruelle-operator associated to S as defined in Definition 12 is of trace class and the zeta function $\zeta : w \mapsto \det(1 - \mathcal{L}_w)$ is analytic on the whole complex plane. Moreover, there exists $c > 0$ such that ζ has no zeros in the half-plane $\{w \in \mathbb{C} \mid \operatorname{Re}(w) > c\}$.*

If the GDMS S is primitive, we investigate some relations between this zeta function and the Hausdorff dimension of the limit set of S, which we denote by $\delta(S)$.

Main Theorem 2. *Let S be as in Main Theorem 1. Further assume that S is primitive, then the following holds.*

1. *The zeta function ζ has no zeros in the half-plane*
$$\{w \in \mathbb{C} \mid \operatorname{Re}(w) > \delta(S)\}.$$
2. *Let $c > 0$ be fixed, then for every $w \in \{z \in \mathbb{C} \mid \operatorname{Re}(z) > -c$ and $|\operatorname{Im}(z)| > 1\}$ we have*
$$\log|\zeta(w)| \ll \mathrm{e}^{\delta(S)\cdot\log(|\operatorname{Im}(w)|)}.$$

Finally, for fixed $c > 0$ we consider the set of resonances in the half-plane $\{z \in \mathbb{C} \mid -c < \mathrm{Re}(z)\}$, which by the first statement of Main Theorem 2 equals the set of resonances in $\{z \in \mathbb{C} \mid -c < \mathrm{Re}(z) < \delta(S)\}$. Since in general there are infinitely many resonances in this half-plane, we rather consider the number of resonances in the area

$$A_c(k) := \{z \in \mathbb{C} \mid -c < \mathrm{Re}(z) \text{ and } |\operatorname{Im}(z)| \leq k\}.$$

Let $\mathrm{n}_\zeta(A_c(k))$ denote the number of zeros of ζ in $A_c(k)$ counted with multiplicity. By combining Main Theorem 2 with standard techniques from complex analysis, we give the following upper bound for the growth of the number of resonances.

Main Theorem 3. *Let S be as in Main Theorem 2. For fixed $c > 0$ and for all k sufficiently large the number of resonances in $A_c(k+1) \setminus A_c(k)$ counted by multiplicity is bounded by a constant multiple of $k^{\delta(S)}$, where the constant does not depend on k, namely*

$$\mathrm{n}_\zeta\left(A_c(k+1) \setminus A_c(k)\right) \ll k^{\delta(S)}.$$

In Section 2.1.1 we give a quick introduction to conformal GDMSs. In Section 2.1.3 we introduce the zeta function which we consider for finitely generated conformal GDMSs. In Sections 2.2.1 and 2.2.2 we recall all geometric and analytic preliminaries necessary for the proofs of our results. In Section 2.2.3 we prove Main Theorem 1 and the first statement of Main Theorem 2, while the proofs of the estimates stated in Main Theorem 2 and Main Theorem 3 are presented in Section 2.3. More precisely, in Section 2.3.2 we derive the upper bound for the modulus of this zeta function. This upper bound will be the main ingredient in the proof of the upper bound for the number of resonances in Main Theorem 3.

We conclude the paper by presenting a short comparison of ζ to other zeta functions. In Section 3.1 we mention some of the differences between ζ and the zeta function obtained if one does not consider the complexification. In 3.2 we give a short review of the geometric zeta function associated to a fractal string, while in 3.3 we recall that in the case of a GDMS coming from the action of a Kleinian group of Schottky type ζ actually equals the Selberg zeta function.

2 Resonances for GDMSs

Throughout this paper we study a finitely generated conformal GDMS S acting on $\mathbb{R}^m$. For the convenience of the reader we include a short introduction to such systems.

2.1 Basic notions

2.1.1 Basic notions of finitely generated GDMSs

In this section we collect some of the important basic geometric concepts necessary for the proof of our results. We begin by giving a detailed definition of a finitely generated GDMS. Note that each of these systems is based on a directed multigraph and *not* a graph. The finite multigraph consists of a finite set V of vertices and a finite set of directed edges E.

Definition 1. *A* finitely generated graph directed Markov system (fgGDMS) *S is defined by an octuple $(V, E, i, t, A, \{X_v\}_{v\in V}, \ell, \{\phi_e\}_{e\in E})$ given by the following list.*

- *A non-empty finite set V of vertices.*
- *A finite set E of directed edges.*
- *Two maps $i, t : E \to V$, which assign to each edge $e \in E$ its initial vertex $i(e)$ and terminal vertex $t(e)$.*
- *A $(\mathrm{card}\, E)\times(\mathrm{card}\, E)$-matrix A with entries in $\{0, 1\}$, which is also called transition matrix or edge incident matrix, since it determines which paths are to be admissible, that is, which edges may follow a given edge, and which satisfies that whenever $A_{e,f} = 1$ then $t(e) = i(f)$.*
- *A collection $\{X_v\}_{v\in V}$ of pairwise disjoint non-empty compact connected subsets $X_v \subset \mathbb{R}^m$ of a fixed $\mathbb{R}^m$, which are closures of open sets, that is $X_v = \overline{\mathrm{Int}(X_v)}$.*
- *Some constant $\ell \in (0, 1)$.*
- *Injective contractions $\phi_e : X_{i(e)} \to X_{t(e)}$ with Lipschitz constants less than $\ell \in (0, 1)$.*

We now recall some basic facts about fgGDMSs. For fgGDMSs these are well known, and we refer to the textbook [20] for the proofs and details.

For a fgGDMS S, we define the set of admissible words of length $n \in \mathbb{N}$ by

$$E^n := \left\{(e_1, \ldots, e_n) \mid e_i \in E \text{ such that } A_{e_i,e_{i+1}} = 1 \text{ for all } i \geq 1\right\}.$$

Also, let E^∞ denote the set of infinite (admissible) words, and define the set of finite (admissible) words by $E^* := \bigcup_{n\in\mathbb{N}} E^n$.

Definition 2. *We define the* limit set $L(S)$ *of a fgGDMS* S *by*

$$L(S) := \bigcap_{n\in\mathbb{N}} \bigcup_{(e_1,\ldots,e_n)\in E^n} \phi_{e_n} \circ \ldots \circ \phi_{e_1} \left(X_{i(e_1)}\right).$$

It is well known that the set $L(S)$ can be identified with the set of infinite words E^∞ (see e.g. [20, p.2]). We require the following properties.

Definition 3. *A fgGDMS* S *satisfies the* strong separation condition (SSC) *if for all* $e, f \in E$ *with* $e \neq f$,

$$\phi_e \left(X_{i(e)}\right) \cap \phi_f \left(X_{i(f)}\right) = \emptyset.$$

Definition 4. *A fgGDMS* S *satisfies the* bounded distortion property (BDP) *if there exists a constant* $c \geq 1$ *such that*

$$\frac{1}{c} \cdot \|\phi'_{\underline{e}}(y)\| \leq \|\phi'_{\underline{e}}(x)\| \leq c \cdot \|\phi'_{\underline{e}}(y)\|,$$

for all $\underline{e} \in E^*$ *and* $x, y \in X_{i(\underline{e})}$. *Here,* $\|\phi'_{\underline{e}}(x)\|$ *is any norm on the linear mappings on* $\mathbb{R}^m$ *(all such norms are equivalent).*

Definition 5. *A fgGDMS* S *satisfies the* nestedness condition (NC) *if for each* $e \in E$ *there is an open set* $U \subset X_{t(e)}$ *such that* $\phi_e \left(X_{i(e)}\right) \subset U$.

Definition 6. *A fgGDMS* S *is said to be* primitive *if there exists a* $p \geq 1$ *such that all entries of* A^p *are positive.*

Note that in order for a fgGDMS to be primitive it is necessary that the multigraph (V, E) is connected.

Finally, we adapt the definition of a conformal GDMS of [20] to our setting as follows.

Definition 7. *A fgGDMS* S *is said to be* conformal *if the following conditions are satisfied.*

- *For every vertex* $v \in V$ *there exists an open set* W_v *such that* $X_v \subset W_v$. *Moreover, for every* $e \in E$ *the map* ϕ_e *extends to a* C^1*-conformal diffeomorphism from* $W_{i(e)}$ *to* $W_{t(e)}$.
- *(Cone property) For every* $v \in V$ *there exist* $l > 0, \gamma \in (0, \pi/2)$, *such that for every* $x \in X_v$ *there exists an open cone* $\mathrm{Cone}(x, \gamma, l) \subset \mathrm{Int}(X_v)$ *with vertex* x, *central angle of measure* γ, *and altitude* l.

- *There exists a constant $c > 0$ such that for every $e \in E$ and all $x, y \in X_{i(e)}$ the following holds:*

$$| \, \|\phi'_e(x)\| - \|\phi'_e(y)\| \, | \leq c|x - y|.$$

- *The strong separation condition (SSC) is satisfied.*

REMARK: Definition 7 is a restricted version of the definition of a finitely generated conformal GDMS in [20, p.72]. The only difference is that we require SSC, while in [20] only the weaker open set condition was assumed. Hence, the conformal fgGDMSs as defined above are conformal GDMSs in the sense of [20]. It is well known that every conformal GDMS satisfies BDP (see [20, (4f),p73]).

Let us recall that a C^1 diffeomorphism $\phi : U \to \mathbb{R}^m$, where $m \geq 1$, from an open connected set $U \subset \mathbb{R}^m$ to $\mathbb{R}^m$ is conformal if its derivative at every point of U is a similarity map (cf. [20, p.62]). Note that for $m = 1$, C^1-conformality means that the maps ϕ_e are monotone C^1 diffeomorphisms, for each $e \in E$. For $m = 2$, C^1-conformal maps are holomorphic or antiholomorphic. For $m \geq 3$, conformal maps between domains in $\mathbb{R}^m$ are of the form $x \mapsto \lambda A i(x) + b$, where $\lambda > 0$, $b \in \mathbb{R}^m$, $A \in O(m)$ and i is either the identity or an inversion. Here, $O(m)$ denotes the orthogonal group. (This is Liouville's Theorem; a proof can be found for example in [6](Theorem A.3.7).) Recall that the inversion at the unit circle around zero is given by $x \mapsto \frac{x}{\|x\|^2}$, and this is C^∞ in $\mathbb{R}^m \setminus \{0\}$ (see [6, Proposition A.3.1]). Since $\|x\|^2 = x_1^2 + \ldots + x_m^2$, it immediately follows that this inversion is real analytic on $\mathbb{R}^m \setminus \{0\}$. Also, since it maps zero to ∞ and since in a fgGDMS the maps ϕ_e map compact sets to compact sets, it follows that the centre of the circle associated to the inversion is not included in the corresponding compact domain. This implies that for conformal fgGDMSs the maps ϕ_e are real analytic on the open sets $W_{i(e)}$, and that the $\|\phi'_e\|$ are non-zero.

2.1.2 Basic notions of functional analysis

In this section we recall the main definitions and facts from functional analysis which will be required later. (For a comprehensive introduction to functional analysis we refer to [24], [25] and [13].) In what follows, let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ denote Hilbert spaces. The inner product in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and we drop the subscript if it is clear from the context which Hilbert space is meant. The operator norm of $A : \mathcal{H}_1 \to \mathcal{H}_2$ is defined by $\|A\| := \sup \{ \|Af\|_{\mathcal{H}_2} \mid f \in \mathcal{H}_1, \ \|f\|_{\mathcal{H}_1} = 1 \}$. A linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is called compact if for any bounded subset $X \subset \mathcal{H}_1$ the image $A(X)$ is relatively compact in $\mathcal{H}_2$, that is, the closure $\overline{A(X)}$ is compact. Such an operator is necessarily a bounded operator, and it is therefore continuous. To each such A corresponds a unique $A^* : \mathcal{H}_2 \to \mathcal{H}_1$, called the adjoint of A, which is compact and satisfies $\langle Ax, y\rangle_{\mathcal{H}_2} = \langle x, A^* y\rangle_{\mathcal{H}_1}$, for all $y \in \mathcal{H}_2$ and all $x \in \mathcal{H}_1$. Furthermore, $\|A^*\| = \|A\|$ (see [26, 4.10]). Also, for a compact operator $A : \mathcal{H} \to \mathcal{H}$ there is an expansion

$$A = \sum_{n=0}^{N} \chi_n(A)\,\langle x_n, \cdot\rangle y_n,$$

where $N \in \mathbb{N} \cup \{0, \infty\}$, $\chi_n(A) \in \mathbb{R}$ and $\chi_n(A) \geq \chi_{n+1}(A) > 0$, for all $n \in \mathbb{N} \cup \{0\}$ (see [32, Theorem 1.4]). Moreover, $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ are orthonormal sets in $\mathcal{H}$. Furthermore, the $\chi_n(A)$ are uniquely determined, and they are referred to as *singular values* (see [32, Theorem 1.4]). For ease of exposition, let us only consider the case $N = \infty$. The resolvent set of A is defined by $\rho(A) := \{\mu \in \mathbb{C} \mid \exists\, (\mu - A)^{-1}\}$. The spectrum of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. If $A(f) = \lambda \cdot f$, then we call $\lambda = \lambda(A)$ an eigenvalue of A and $f \in \mathcal{H}$ its associated eigenvector. The dimension of $\{f \in \mathcal{H} \mid A(f) = \lambda f\}$ is called the geometric multiplicity of λ. Note that if λ is an eigenvalue of A, we necessarily have that $\lambda \in \sigma(A)$. By the well known spectral theorem for compact operators (see [32, Theorem 1.1]), we have that each non-zero $\lambda \in \sigma(A)$ is an eigenvalue of A of finite multiplicity, that $\sigma(A)$ is countable, that 0 is the only accumulation point of the non-zero eigenvalues, and hence, the function $z \mapsto (z - A)^{-1}$ has a pole at λ. The order of the pole is called the algebraic multiplicity. In what follows, we always refer to the algebraic multiplicity only, unless stated otherwise. Furthermore, $\{\lambda_n(A)\}_{n\in\mathbb{N}\cup\{0\}}$ refers to the collection of all non-zero eigenvalues of A repeated according to their algebraic multiplicity.

Let us make a few more comments about compact operators between two Hilbert spaces. For this let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator. Then $A^*A : \mathcal{H}_1 \to \mathcal{H}_1$ is self-adjoint, and hence has only real, non-negative eigenvalues which are equal to the eigenvalues of AA^*. Therefore, the eigenvectors of A^*A form an orthonormal basis (see [32, Theorem 1.1]). Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ be an orthonormal basis of eigenvectors of $(A^*A)^{1/2}$ and let $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ be an orthonormal basis of eigenvectors of $(AA^*)^{1/2}$. Then we have an expansion $A = \sum_{n=0}^{\infty} \sqrt{\lambda(A^*A)}\langle x_n, \cdot\rangle_{\mathcal{H}_2}\, y_n$ (cf. [32, Proof of Theorem 1.4]). From this one also sees that the singular values $\{\chi_n(A)\}_{n\in\mathbb{N}\cup\{0\}}$ are the non-zero eigenvalues of $(A^*A)^{1/2}$, counted according to their multiplicity.

We now recall the min-max principle for singular values (see [32, Theorem 1.5]), which follows from the fact that the singular values of A are exactly the non-zero eigenvalues of $(A^*A)^{1/2}$ and from the min-max-Theorem for eigenvalues (c.f.g. [25, Theorem XIII.1]).

Lemma 8. *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator. Then the singular values of A form a decreasing sequence with 0 being the only accumulation point. Also, the n-th singular value $\chi_n(A)$ of A is given by*

$$\chi_n(A) = \min_{\dim(\mathcal{H}_{1,n})=n} \max_{f\in\mathcal{H}_{1,n}^{\perp}} \frac{\|A(f)\|_{\mathcal{H}_2}}{\|f\|_{\mathcal{H}_1}}.$$

Here, the minimum is taken over all n-dimensional subspaces $\mathcal{H}_{1,n}$ of $\mathcal{H}_1$, while the maximum is taken over all elements in the orthogonal complement of $\mathcal{H}_{1,n}$.

From this it immediately follows that for any orthonormal basis $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of $\mathcal{H}_1$ we have

$$\chi_n(A) \leq \sum_{j=n}^{\infty} \|Ax_j\|. \tag{1}$$

REMARK: Note that in the literature one often finds $\dim(\mathcal{H}_{1,n}) = n+1$, rather than $\dim(\mathcal{H}_{1,n}) = n$. Consequently, one then has that χ_1 is the first singular value, while in our definition the first singular value is χ_0.

Recall that a compact operator $A : \mathcal{H} \to \mathcal{H}$ is said to be of trace-class if

$$\|A\|_1 := \sum_{n=0}^{\infty} \chi_n(A) < \infty.$$

Let $A = \sum_{n=0}^{\infty} \chi_n(A) \langle x_n, \cdot\rangle y_n$ be of trace-class. Then, for any choice of orthonormal basis $\{\eta_n\}_{n\in\mathbb{N}\cup\{0\}}$, the sum $\sum_{n=0}^{\infty} |\langle \eta_n, A\eta_n\rangle|$ converges. Moreover, the trace

$$\mathrm{Tr}(A) := \sum_{n=0}^{\infty} \langle \eta_n, A\eta_n\rangle = \sum_{n=0}^{\infty} \chi_n(A) \langle x_n, y_n\rangle$$

is independent of the basis (cf. [32, Theorem 3.1]). Furthermore, one can show that if A is of trace-class, then the series $\sum_{n\in\mathbb{N}\cup\{0\}} \lambda_n(A) \leq \sum_{n\in\mathbb{N}\cup\{0\}} \chi_n(A)$ is absolutely convergent, and the trace of A satisfies $\mathrm{Tr}(A) = \sum_{n\in\mathbb{N}\cup\{0\}} \lambda_n(A)$. This is Lidskii's equality (see e.g. [32, Theorem 3.7]). The following definition is adopted from [13].

Definition 9. *For an operator A of trace-class we define the* determinant $\det(1+A)$ *by*

$$\det(1+A) := \prod_{n=0}^{\infty} (1 + \lambda_n(A)).$$

Remark: There are several ways to define $\det(1+A)$ for a trace-class operator A. For example, in [9] one finds $\det(1+zA) := \exp(\operatorname{Tr}(\ln(1+zA)))$, for $z \in \mathbb{C}$ with $|z|$ small, and one then considers an analytic continuation of this locally holomorphic function.

We finish this section by recalling a well known fact which we need later. The following is an immediate implication of [32, (5.12)].

Lemma 10. *Let A be of trace-class and $\|A\|_1 < 1$, then we have that*

$$\det(1-A) = \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{Tr}\left(A^k\right)\right).$$

2.1.3 Definition of the zeta function

In this section we introduce the zeta function for a conformal fgGDMS used in the main theorems. This zeta function is a type of Artin-Mazur zeta-function (cf. [1]), which was generalized by Ruelle in [31], and hence is a type of dynamical zeta function. For a more comprehensive introduction to dynamical zeta-functions we refer to [31] and [5] (for further details also see [17]). Roughly speaking, the zeta function is defined as the Fredholm determinant of the difference of the identity and the complexified FPR-operator. In the definition of this function, the particular choice of the underlying function space will be essential. Here, the functions under consideration are holomorphic, square-integrable functions defined on a complex neighborhood of the limit set, rather than on the limit set only. One of the key facts in our investigation will be that the space of holomorphic L^2-functions is a Hilbert space. Furthermore, it turns out that this function space is somehow more natural, since the so obtained zeta-function coincides with the Selberg zeta function in the case in which S represents the action of a convex co-compact Schottky group (see Section 3.3).

Definition of the real valued FPR-operator

We start by recalling the definition of the usual (that is, not complexified) version of the FPR-operator. At this point we would like to remark that one can find many different names attached to this operator in the literature. Often it is referred to as the Ruelle transfer operator or just the Ruelle operator, since it was formally introduced by Ruelle in [27]. We refer to it as the Frobenius-Perron-Ruelle-operator (FPR-operator) in order to stress that one can prove a kind of Frobenius-Perron theorem for it.

Definition 11 (Frobenius-Perron-Ruelle-operator)**.** *The Frobenius-Perron-Ruelle operator (FPR-operator) $\mathcal{L}_s$ for a conformal fgGDMS S is defined for $s \in \mathbb{R}$, $x \in L(S)$ and $u : L(S) \to \mathbb{R}$ by*

$$\mathcal{L}_s\left(u\right)\left(x\right) := \sum_{e \in E} \|\phi_e'\left(x\right)\|^s u\left(\phi_e\left(x\right)\right).$$

Here, we use the convention that if $x \notin X_{i(e)}$, then $u\left(\phi_e\left(x\right)\right) := 0$.

Definition of the complexified FPR-operator

Recall that the compact sets X_v of a conformal fgGDMS S are subsets of $\mathbb{R}^m$. We now want to embed $\mathbb{R}^m$ into $\mathbb{C}^m$. For this, let $e \in E$ be fixed and recall that the maps ϕ_e and $\|\phi_e'\|$ are real analytic on $W_{i(e)}$ (see the discussion following Definition 7). Hence, we can complexify the real power series of ϕ_e and in this way we obtain a complex power series which converges in a complex neighborhood of its real domain (see [18, Proposition 2.3.15], see also the discussion at the beginning of [18, Section 2.3.1]). Choose a domain of convergence for the complexified power series of $\|\phi_e'\|$ and intersect it with a domain for the complexified power series of ϕ_e. This gives a complex domain, say $\left(X_e\right)_{\mathbb{C}}$, on which both, ϕ_e and $\|\phi_e'\|$, have holomorphic extensions. These holomorphic extensions will be denoted by $\left(\phi_e\right)_{\mathbb{C}}$, and $\left(\|\phi_e'\|\right)_{\mathbb{C}}$ respectively.

Since ϕ_e is a contraction on $X_{i(e)}$ with $\|\phi_e'\| < \ell < 1$, it follows that $\left(\phi_e\right)_{\mathbb{C}}$ is contracting on some sufficiently small complex domain containing $X_{i(e)}$ (with Lipschitz constant less than $\frac{\ell+1}{2}$). Therefore, we have that $|(\|\phi_e'\|)_{\mathbb{C}}(z)| < \frac{\ell+1}{2} < 1$ on some sufficiently small complex domain containing $X_{i(e)}$. Without loss of generality, we can assume that this domain is $\left(X_e\right)_{\mathbb{C}}$, since otherwise we can choose $\left(X_e\right)_{\mathbb{C}}$ to be the intersection of both domains. Moreover, recall that by Definition 7 we have $|\ \|\phi_e'(x)\| - \|\phi_e'(y)\|\ | \leq c|x-y|$. In other words, the maps $\|\phi_e'\|$ are Lipschitz continuous with a uniform Lipschitz constant $c > 0$. Hence, without loss of generality we can assume that the holomorphic extensions $(\|\phi_e'\|)_{\mathbb{C}}$ are also Lipschitz continuous with some uniform Lipschitz constant. Furthermore, since $\|\phi_e'\| > 0$, we can assume that $|\left(\|\phi_e'\|\right)_{\mathbb{C}}(z)|$ is bounded away from zero uniformly for all $e \in E$ and $z \in \left(X_e\right)_{\mathbb{C}}$.

For $v \in V$ let $\left(X_v\right)_{\mathbb{C}} := \bigcap_{e \in E, i(e)=v} \left(X_e\right)_{\mathbb{C}}$. Note that this intersection is again an open domain, since E is finite. For $\epsilon > 0$, let $B_\epsilon\left(X_{i(e)}\right) := \{z \in \mathbb{C}^m \mid \operatorname{dist}\left(z, X_{i(e)}\right) \leq \epsilon\}$. If the fgGDMS satisfies the nestedness condition (NC), then by combining the nestedness condition and the observation that $\left(\phi_e\right)_{\mathbb{C}}\left(B_\epsilon\left(X_{i(e)}\right) \cap \left(X_{i(e)}\right)_{\mathbb{C}}\right) \subset B_{\ell\cdot\epsilon}\left(X_{t(e)}\right)$ with the fact that E is finite, it follows that one can choose the elements of the sequence $\{(X_v)_{\mathbb{C}}\}_{v \in V}$ sufficiently small such that the sequence $\left\{\left(\phi_e\right)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right\}_{e \in E}$ is nested in

$\left(X_{t(e)}\right)_{\mathbb{C}}$. That is, we have that $(\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right) \subset \tilde{U} \subset \left(X_{t(e)}\right)_{\mathbb{C}}$ for some open set $\tilde{U} \subset \left(X_{t(e)}\right)_{\mathbb{C}}$.

Note that by choosing the sets $(X_v)_{\mathbb{C}}$ sufficiently small, if necessary, we can further assume that the sets $(X_v)_{\mathbb{C}}$ are pairwise disjoint. This can be done, since the finitely many sets X_v are pairwise disjoint and hence have a positive distance to each other.

Finally, let $X_{\mathbb{C}} := \bigcup_{v \in V} (X_v)_{\mathbb{C}} \subset \mathbb{C}$ be the union of the pairwise disjoint complex sets $(X_v)_{\mathbb{C}}$.

Definition 12 (Complexified FPR-operator)**.** *Let $\mathcal{H}(X_{\mathbb{C}})$ denote the Hilbert space of holomorphic L^2-functions on $X_{\mathbb{C}}$. Further let Φ_e denote the composition operator on $\mathcal{H}(X_{\mathbb{C}})$ given by $(\Phi_e(u))(z) := u((\phi_e)_{\mathbb{C}}(z))$, for each $u \in \mathcal{H}(X_{\mathbb{C}})$, and let $D_e : X_{\mathbb{C}} \to \mathbb{C}$ be given by $D_e(z) := (\|\phi_e'\|)_{\mathbb{C}}(z)$. Moreover, define $z^w := \left(|z| \cdot e^{\imath \operatorname{Arg}(z)}\right)^w$ with $-\pi \leq \operatorname{Arg}(z) < \pi$.*

Then for $w \in \mathbb{C}$, $z \in X_{\mathbb{C}}$ and $u \in \mathcal{H}(X_{\mathbb{C}})$, we define the complexified FPR-operator *$\mathcal{L}_w : \mathcal{H}(X_{\mathbb{C}}) \to \mathcal{H}(X_{\mathbb{C}})$ by*

$$\mathcal{L}_w(u)(z) := \sum_{e \in E} (D_e(z))^w (\Phi_e(u))(z).$$

Similarly to Definition 11, we use the convention that if $z \notin \left(X_{i(e)}\right)_{\mathbb{C}}$, then $(\Phi_e(u))(z) := 0$.

REMARK: Note that with $-\pi \leq \operatorname{Arg}(z) < \pi$ and the above mentioned facts regarding $(\|\phi_e'\|)_{\mathbb{C}}$, we can (by reducing $X_{\mathbb{C}}$ if necessary) assume that the argument $\operatorname{Arg}((\|\phi_e'\|)_{\mathbb{C}}(z))$ is bounded away from $\pm\pi$.

One can prove that $\mathcal{L}_w$ is a compact operator (see eg. [28, p.234]). In Proposition 20 we shall show that $\mathcal{L}_w$ is of trace-class. With this in mind, we can now define the zeta function as follows.

Definition 13. *The* zeta function *$\zeta : \mathbb{C} \to \mathbb{C}$ is given for $w \in \mathbb{C}$ by*

$$\zeta(w) := \det(1 - \mathcal{L}_w).$$

The zeros of ζ will be referred to as resonances*.*

2.2 Preparations for the proofs of the main results

2.2.1 Geometric preliminaries

An important tool for studying fractal sets is provided by fractal measures. Well studied examples of these measures are the Frostman measure for IFSs and the Patterson measure for Kleinian groups. As a consequence of the mass distribution principle, there is a direct connection between these measures and the Hausdorff dimension of the limit set. We use this connection when we apply the following theorem.

Theorem 14. *Let S be a finitely generated primitive conformal GDMS. Then there exists an Ahlfors-regular Borel probability measure μ supported on $L(S)$. Here, Ahlfors-regular means that the measure μ satisfies the following condition:*

$$\mu\left(B\left(x,r\right)\right) \asymp r^{\dim_{\mathrm{H}} L(S)} \text{ for all } x \in L(S)$$
$$\text{and } 0 < r < \frac{1}{2}\min\{\operatorname{diam} X_v \mid v \in V\}.$$

For the proof we refer to [20, proof of Theorem 4.2.11, p.79].

Using this theorem, we can prove the following lemma.

Lemma 15. *For a finitely generated primitive conformal GDMS S acting on $\mathbb{R}^m$, let $\phi_{\min} := \min_{e\in E} \|\phi_e'\|_\infty$, and for $r > 0$ define*

$$E\left(r\right) := \{\underline{e} \in E^* \mid r \geq \operatorname{diam}\left(\phi_{\underline{e}}\left(X_{i(\underline{e})}\right)\right) \geq r \cdot \phi_{\min}\}.$$

We then have that $\operatorname{card} E\left(r\right) \asymp r^{-\delta(S)}$, *where $\delta\left(S\right) := \dim_{\mathrm{H}} L(S)$ refers to the Hausdorff dimension of the limit set $L(S)$ of S.*

PROOF. Clearly, $\bigcup_{e\in E(r)} \phi_e\left(X_{i(e)}\right)$ is a cover of $L_{dyn}(S)$ and hence of $L(S)$, since the $\phi_e(X_{i(e)})$ are compact and E is finite. For $1 > \phi_{\max} := \max_{e\in E} \|\phi_e'\|_\infty$ note that $\phi_{\max}\cdot\operatorname{diam}(A) \leq \operatorname{diam}\phi_f(A) \leq \phi_{\min}\cdot\operatorname{diam}(A)$, for all sets $A \subset X_{i(f)}$ and all $f \in E$. Hence, a straight forward calculation shows that the multiplicity of this cover is at most $\frac{\log(|\phi_{\max}|)}{\log(|\phi_{\min}|)}$. By Theorem 14, we have that there exists an Ahlfors-regular Borel probability measure μ on $L(S)$. Hence, we have

$$\begin{aligned}1 = \mu\left(L(S)\right) = \mu\left(\bigcup_{e\in E(r)} \phi_e\left(X_{i(e)}\right)\right) &\asymp \sum_{e\in E(r)} \mu\left(\phi_e\left(X_{i(e)}\right)\right)\\ &\asymp \operatorname{card} E\left(r\right)\cdot r^{\delta(S)}.\end{aligned}$$

□

2.2.2 Functional analytic preliminaries

In this section we present some important facts from functional analysis which will be required later. The main aim is to show the following inequality, which will be crucial in the proofs of our main results. Namely, for non-empty finite index sets I and J, and for a family of trace-class operators $\{A_{i,j}\}_{(i,j)\in I\times J}$, we have that

$$\left|\det\left(1-\sum_{i\in I}\bigoplus_{j\in J}A_{i,j}\right)\right|\leq\prod_{l=0}^{\infty}\left(1+\sharp I\max_{i\in I,j\in J}\chi_{\left[\frac{l}{\sharp I\cdot\sharp J}\right]}\left(A_{i,j}\right)\right). \qquad (2)$$

Here, $[x]$ denotes the Gauss bracket (or floor function), and $\sharp I$ refers to the cardinality of I. In what follows we require the following lemma.

Lemma 16. *Let $A:\mathcal{H}_A\to\mathcal{H}_A$ and $B:\mathcal{H}_B\to\mathcal{H}_B$ be compact operators. For all $l\in\mathbb{N}\cup\{0\}$, we have for the l-th singular value $\chi_l\left(A\oplus B\right)$ of the direct sum of A and B that*

$$\chi_l\left(A\oplus B\right)=\min\left\{\max\left\{\chi_j\left(A\right);\chi_n\left(B\right)\right\}\mid j+n=l\right\}. \qquad (3)$$

Proof. One easily verifies that the set of singular values of $A\oplus B$ is equal to the union of the singular values of A and B. The assertion in (3) now follows by a straight forward combinatoric argument. □

Proposition 17. *For a compact operator $A:\mathcal{H}\to\mathcal{H}$, let $\{\chi_n\left(A\right)\}_{n\in\mathbb{N}\cup\{0\}}$ denote the decreasing set of singular values. Moreover, let $\{A_j\}_{j\in\{1,\ldots,k\}}$ refer to some finite family of compact operators. Then the following inequalities hold for all $l\in\mathbb{N}\cup\{0\}$.*

1. $\left|\det\left(1+A\right)\right|\leq\prod_{n=0}^{\infty}\left(1+\chi_n\left(A\right)\right).$
2. $\chi_l\left(\sum_{j=1}^{k}A_j\right)\leq k\cdot\max\left\{\chi_{\left[\frac{l}{k}\right]}\left(A_j\right)\mid j\in\{1,\ldots,k\}\right\}.$
3. $\chi_l\left(\bigoplus_{j=1}^{k}A_j\right)\leq\max\left\{\chi_{\left[\frac{l}{k}\right]}\left(A_j\right)\mid j\in\{1,\ldots,k\}\right\}.$

Proof. The assertion in *1* is well known and can be found in the literature, for example in [32] (see there the equation following (3.8), where one has to set $z=1$). In contrast, the assertions of *2* and *3* are less well known, and we therefore include their proofs.

To prove the assertion in *2*, we use Fan's inequality [32, Theorem 1.7] (see also [10] and [11]), which states that for compact operators A and B we have for all $l,j\in\mathbb{N}\cup\{0\}$ that

$$\chi_{l+j}\left(A+B\right)\leq\chi_l\left(A\right)+\chi_j\left(B\right). \qquad (4)$$

Now, let $\{A_j\}_{j\in\{1,\ldots,k\}}$ be some family of compact, normal operators of trace-class. For all $l \in \mathbb{N}\cup\{0\}$, we then have that

$$\begin{aligned}\chi_l\left(\sum_{j=1}^{k} A_j\right) &\leq \min_{j_1+\ldots+j_k=l}\left\{\chi_{j_1}(A_1)+\ldots+\chi_{j_k}(A_k)\right\}\\ &\leq k\cdot\max_{j=1,\ldots,k}\chi_{\left[\frac{l}{k}\right]}(A_j).\end{aligned}$$

This completes the proof of the assertion in point *2*.

In order to prove the assertion in point *3*, observe that it is implied by (3), since for all $l\in\mathbb{N}\cup\{0\}$ we have

$$\begin{aligned}\chi_l\left(\bigoplus_{j=1}^{k} A_j\right) &= \min\left\{\max\left\{\chi_{j_1}(A_1),\ldots,\chi_{j_k}(A_k)\right\}\mid j_1+\ldots+j_k=l\right\}\\ &\leq \max_{j=1,\ldots,k}\chi_{\left[\frac{l}{k}\right]}(A_j).\end{aligned}$$

This completes the proof of the proposition. □

Note that we can now use Proposition 17 to obtain the statement in (2) as follows. By applying first part *1*, then part *2*, and finally part *3* of Proposition 17, we derive for the family $\{A_{i,j}\}$ of bounded normal operators

$$\begin{aligned}\left|\det\left(1-\sum_{i\in I}\bigoplus_{j\in J}A_{i,j}\right)\right| &\leq \prod_{l=0}^{\infty}\left(1+\chi_l\left(\sum_{i\in I}\bigoplus_{j\in J}A_{i,j}\right)\right)\\ &\leq \prod_{l=0}^{\infty}\left(1+\sharp I\max\left\{\chi_{\left[\frac{l}{\sharp I}\right]}\left(\bigoplus_{j\in J}A_{i,j}\right)\mid i\in I\right\}\right)\\ &\leq \prod_{l=0}^{\infty}\left(1+\sharp I\max\left\{\chi_{\left[\frac{l}{\sharp I\cdot\sharp J}\right]}(A_{i,j})\mid i\in I, j\in J\right\}\right).\end{aligned}$$

In the proof of Main Theorem 2 we also need the following lemma.

Lemma 18. *Let $\{A_j\}_{j\in J}$ be a finite family of trace-class operators. Then, for each $c_0>0$, there exists a constant $c_1>0$ such that for all $j\in J$ and $k\in\mathbb{N}$, we have that*

$$\sum_{l=0}^{\infty}\log\left(1+c_0\cdot\max_{j\in J}\chi_{\left[\frac{l}{k}\right]}(A_j)\right)\leq c_1\cdot k\cdot\max_{j\in J}\sum_{l=0}^{\infty}\chi_l(A_j)\ll k.$$

PROOF. Recall that the singular values are positive and bounded from above. This implies that for each $j \in J$ we have

$$c_0 \cdot \chi_{\left[\frac{l}{k}\right]}(A_j) \asymp \log\left(1 + c_0 \cdot \chi_{\left[\frac{l}{k}\right]}(A_j)\right).$$

Since J is finite, we obtain that

$$\begin{aligned}\sum_{l=0}^{\infty} \log\left(1 + c \cdot \max_{j\in J} \chi_{\left[\frac{l}{k}\right]}(A_j)\right) &\asymp \sum_{l=0}^{\infty} c_0 \cdot \max_{j\in J} \chi_{\left[\frac{l}{k}\right]}(A_j) \\ &= c_0 \sum_{l=0}^{\infty} k \cdot \max_{j\in J} \chi_l(A_j) \\ &= c_0 \cdot k \sum_{l=0}^{\infty} \max_{j\in J} \chi_l(A_j).\end{aligned}$$

Now we have $\sum_{l=0}^{\infty} \max_{j\in J} \chi_l(A_j) \leq \sum_{l=0}^{\infty} \sum_{j\in J} \chi_l(A_j)$, which is finite, since J is finite and the operators A_j are of trace-class. This completes the proof. $\square$

We end this section by giving an estimate for the determinant of a particular type of finite dimensional matrix, which we shall use in Section 2.2.3.

Lemma 19. *Let $U \subset \mathbb{R}^m$ be open, and let $g : U \to U$ be differentiable and Lipschitz with Lipschitz constant less than $0 < \ell < 1$. Let g' denote the Jacobian of g. We then have for all $x \in U$ that*

$$|\det(1 - g'(x))| \geq (1 - \ell)^m. \tag{5}$$

PROOF. Since $\ell > 0$ is the Lipschitz-constant of g, we have that each eigenvalue $\lambda(g')$ of g' satisfies the inequality $|\lambda(g')| < \ell$. Therefore, $|1-\lambda(g')| \geq 1-\ell$. Since the Jacobian g' is an $m\times m$-matrix, it has exactly m (complex) eigenvalues. Hence, $\det(1 - g') = \prod_{j=1}^{m}(1 - \lambda_j(g'))$. Combining these observations, we obtain $|\det(1 - g')| \geq (1 - \ell)^m$. $\square$

2.2.3 Combining geometric and analytic facts

In this section we show how the nestedness condition (NC) of the conformal fgGDMS S comes into play. Namely, we show that $\mathcal{L}_w$ is of trace-class, and the proof mainly relies on the nestedness condition (NC). Furthermore, we show that there is some half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > c\}$ on which ζ has no zeros. The proof of this fact will mainly rely on the nestedness condition (NC) and on an Atiyah-Bott-type fixed point theorem of Ruelle from [28].

Proposition 20. *Let S be a conformal fgGDMS satisfying the nestedness condition (NC). Then $\mathcal{L}_w$ is of trace-class for all $w \in \mathbb{C}$.*

PROOF. The proof will be given in several steps. First fix $w \in \mathbb{C}$ and note that it is enough to show that the sum $\sum_{l=1}^{\infty} \chi_l \left(\mathcal{L}_w\right)$ is finite. Hence, it is enough to find appropriate bounds for $\chi_l \left(\mathcal{L}_w\right)$. Recall that $\mathcal{L}_w = \sum_{e \in E} \left(D_e\right)^w \Phi_e$. Combining this with (4) (Fan's inequality), we have that

$$\chi_l \left(\mathcal{L}_w\right) \leq \operatorname{card}(E) \cdot \max\{\chi_l \left(\left(D_e\right)^w \Phi_e\right) \mid e \in E\}.$$

Clearly, we have that $\chi_l \left(\left(D_e\right)^w \Phi_e\right) \leq \|\left(D_e\right)^w\|_\infty \cdot \chi_l \left(\Phi_e\right)$ (see [32, Theorem 1.6]). Also, one immediately verifies that for every $w \in \mathbb{C}$ we have $\|\left(D_e\right)^w\|_\infty = \sup_{z \in X_{\mathbb{C}}} |\left(D_e(z)\right)^w|$. Moreover, the latter supremum is bounded from above by some finite constant, since $\left(D_e\right)^w$ is a continuous map defined on a compact set. Therefore, we only have to find bounds for $\chi_i \left(\Phi_e\right)$.

For this, we study $\Phi_e : \mathcal{H}\left(\left(X_{t(e)}\right)_{\mathbb{C}}\right) \to \mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)$. Without loss of generality, we can assume that for each $e \in E$ there exists a family of open sets $\{B_k(e)\}_{k=1}^N$, for some $N \in \mathbb{N}$, which does not depend on $e \in E$, such that the following holds. Let $\overline{B_k(e)}$ denote the closure of $B_k(e)$, then $\left(X_{t(e)}\right)_{\mathbb{C}} = \bigcup_{k=1}^N \overline{B_k(e)}$ and $\left(X_{t(e)}\right)_{\mathbb{C}} \setminus \partial\left(X_{t(e)}\right)_{\mathbb{C}} = \bigcup_{k=1}^N B_k(e)$, and each $\overline{B_k(e)}$ is biholomorphic to $\overline{B_1(0)}$, the closed unit ball in $\mathbb{C}^m$. Let $b_{e,k}$ denote this biholomorphic map, so that $b_{e,k} : \overline{B_k(e)} \to \overline{B_1(0)}$. Recall from the discussion preceding Definition 12 that the nestedness condition (NC) implies that $\left(\phi_e\right)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)$ is nested in $\left(X_{t(e)}\right)_{\mathbb{C}}$ and note that this implies that

$$\left(\phi_e\right)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right) \subset \bigcup_{k=1}^{N} {b_{e,k}}^{-1}\left(\overline{B_\rho(0)}\right)$$

for some $\rho \in (0,1)$. We can take ρ to be independent of $e \in E$ (by taking the maximum of the ρ's), since E is finite.

We can express Φ_e as a composition of maps in the following way.

$$\begin{tikzcd}
\mathcal{H}\left(\bigcup_{k=1}^N \overline{B_k(e)}\right) \arrow[r] & \bigoplus_{k=1}^N \mathcal{H}\left(\overline{B_k(e)}\right) \arrow[r] & \bigoplus_{k=1}^N \mathcal{H}\left(\overline{B_1(0)}\right) \arrow[d, "\bigoplus_{k=1}^N R_\rho"] \\
\mathcal{H}\left(\left(\phi_e\right)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \arrow[d, "\widetilde{\Phi_e}"] & \mathcal{H}\left(\bigcup_{k=1}^N b_{e,k}\left(\overline{B_\rho(0)}\right)\right) \arrow[l] & \bigoplus_{k=1}^N \mathcal{H}\left(\overline{B_\rho(0)}\right) \arrow[l] \\
\mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right) & &
\end{tikzcd}$$

Here, the restriction operator $R_\rho : \mathcal{H}\left(\overline{B_1(0)}\right) \to \mathcal{H}\left(\overline{B_\rho(0)}\right)$ is given by $R_\rho(f) := f_{|\overline{B_\rho(0)}}$.

Note that the norms of the bi-holomorphic maps $b_{e,k}$ are uniformly bounded, as are the norms of the natural restrictions (that is, all maps corresponding to horizontal arrows in the above diagram are uniformly bounded). In order to see that $\widetilde{\Phi_e}$ is bounded, note that by substitution one has

$$\begin{aligned}
\sup_{\|u\|=1} \|u \circ (\phi_e)_{\mathbb{C}}\| &= \sup_{\|u\|=1} \int_{X_{\mathbb{C}}} u^2((\phi_e)_{\mathbb{C}}(z))dz \\
&= \sup_{\|u\|=1} \int_{(\phi_e)_{\mathbb{C}}(X_{\mathbb{C}})} \frac{1}{\|(\phi_e)'_{\mathbb{C}}(y)\|} u^2(y)dy \\
&\leq \sup_{\|u\|=1} \frac{1}{\|(\phi_e)'_{\mathbb{C}}\|_\infty} \|u\| = \frac{1}{\|(\phi_e)'_{\mathbb{C}}\|_\infty}.
\end{aligned}$$

Note that the latter expression is uniformly bounded for each $e \in E$. Therefore, it is enough to find bounds for the singular values of the operator R_ρ. We now show that there exists $\gamma \in (0,1)$ such that $\chi_l(R_\rho) \ll \gamma^{l^{1/m}}$, for all $l \in \mathbb{N} \cup \{0\}$. This is sufficient, since $\sum_{l=0}^{\infty} \gamma^{l^{\frac{1}{m}}}$ is a convergent series, which then implies that R_ρ is of trace-class, and hence that Φ_e is of trace-class as well.

Since the polynomials form a basis of $\mathcal{H}$, it follows from (1) that it is sufficient to consider, for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m$, the normalized polynomials u_α, given by $u_\alpha(z) := c_\alpha \prod_{i=1}^m z_i^{\alpha_i}$ for $c_\alpha \in \mathbb{C}$, $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$. With $|\alpha| := \sum_{i=1}^m \alpha_i$, we have that

$$\begin{aligned}
\|R_\rho(u_\alpha)\|_{L^2}^2 &= \int_{B(0,\rho)} \left| c_\alpha \prod_{i=1}^m z_i^{\alpha_i} \right|^2 dz \\
&\leq \int_{B(0,\rho)} |c_\alpha|^2 \left| \prod_{i=1}^m |z|^{\alpha_i} \right|^2 dz \\
&= \int_{B(0,\rho)} |c_\alpha|^2 |z|^{2\sum_{i=1}^m \alpha_i} dz \\
&= |c_\alpha|^2 \int_{\mathbb{S}^{2m-1}} \int_0^\rho r^{2|\alpha|} \cdot r^{2m-1}\, dr\, d\omega \\
&= |c_\alpha|^2 \cdot vol\left(\mathbb{S}^{2m-1}\right) \cdot \frac{1}{2(|\alpha|+m)} \rho^{2(|\alpha|+m)}.
\end{aligned}$$

Clearly, we can assume that the basis of polynomials is ordered such that $\{\hat{u}_j\}_{j\in\mathbb{N}} := \{u_\alpha\}_\alpha$ and $\deg(\hat{u}_j) \leq \deg(\hat{u}_{j+1})$, where deg refers to the degree of a polynomial. Combining this with the estimate above and with (1), we have for all $l \in \mathbb{N} \cup \{0\}$ that

$$\chi_l\left(R_\rho\right) \leq \sum_{j=l}^{\infty} \|R_\rho \hat{u}_j\|_{L^2} \ll \sum_{|\alpha| \geq l^{1/m}} \rho^{|\alpha|+m}.$$

Also, observe that $\operatorname{card}\{|\alpha| = k\} \ll k^{m-1}$. Hence, we have

$$\sum_{|\alpha| \geq l^{1/m}} \rho^{|\alpha|+m} \ll \sum_{k \geq l^{1/m}} k^{m-1} \rho^{k+m}.$$

We require the following estimate:

$$\int_{l^{1/m}}^{\infty} x^{m-1} \rho^x dx \ll l \cdot \rho^{\left(l^{1/m}\right)}, \text{ for all } l \in \mathbb{N} \cup \{0\}. \tag{6}$$

This estimate is well known, since the left hand side of the latter inequality is equal to the well known upper incomplete gamma function. (For an introduction to the incomplete gamma function we refer to [35, Chapter 11.2].) However, for sake of completeness, we include an elementary proof of this inequality. Indeed, this estimate can be obtained by integration by parts, as follows,

$$\int_{l^{1/m}}^{\infty} x^{m-1} \rho^x dx = \left[\sum_{i=1}^{m} x^{m-i} \ln\left(\rho\right)^{-i} \rho^x \frac{(m-1)!}{(m-i)!} \left(-1\right)^{i-1}\right]_{l^{1/m}}^{\infty} =: \left[I\left(x\right)\right]_{l^{1/m}}^{\infty}.$$

Since $\lim_{x \to \infty} x^{m-i} \rho^x = 0$ for all $i \in \{1, \ldots, m\}$, it follows that $\lim_{x \to \infty} I\left(x\right) = 0$. Hence, the integral in the equation above is equal to $-I\left(l^{1/m}\right)$. Finally, observe that

$$\begin{aligned} -I\left(l^{1/m}\right) &= -\sum_{i=1}^{m} \left(l^{1/m}\right)^{m-i} \left(\ln\left(\rho\right)\right)^{-i} \rho^{\left(l^{1/m}\right)} \frac{(m-1)!}{(m-i)!} \left(-1\right)^{i-1} \\ &\leq l \cdot \rho^{\left(l^{1/m}\right)} \cdot (m-1)! \cdot \sum_{i=1}^{m} |\ln\left(\rho\right)|^{-i} \\ &\ll l \cdot \rho^{\left(l^{1/m}\right)}. \end{aligned}$$

This verifies (6).

To finish the proof of the proposition, note that for $\tilde{\rho} \in (\rho, 1)$ there exists some $\tilde{c}$ such that

$$l \cdot \rho^{\left(l^{1/m}\right)} \leq \tilde{c} \cdot \tilde{\rho}^{\left(l^{1/m}\right)}.$$

Hence, for all $l \in \mathbb{N} \cup \{0\}$ we have that

$$\chi_l(R_\rho) \ll \sum_{k \geq l^{1/m}} k^{m-1} \rho^{k+m} \ll \int_{l^{1/m}}^{\infty} x^{m-1} \rho^x dx \ll l \cdot \rho^{\left(l^{1/m}\right)} \ll \tilde{\rho}^{(l^{1/m})}.$$

Since $\sum_l \tilde{\rho}^{(l^{1/m})}$ is a convergent series, this implies that R_ρ is of trace-class. Recall that we expressed Φ_e as a composition of several operators, and these were all bounded. Hence, we can use the fact that $\chi_l(AB) \leq \|A\| \cdot \chi_l(B)$ to bound the singular values of Φ_e by $\chi_l(R_\rho)$. Hence, Φ_e is of trace-class, since R_ρ is of trace-class. From the discussion at the beginning of this proof it now follows that $\sum_{l=0}^{\infty} \chi_l(\mathcal{L}_w) < \infty$, and hence, that $\mathcal{L}_w$ is of trace-class. □

For the proof of the next proposition we need the following theorem, which is due to Ruelle ([28]). Its proof is based on a fixed point theorem by Atiyah and Bott [2](see also [3],[4] and [16, Theorem 4.1]).

Theorem 21. *Let $U \subset \mathbb{C}^m$ be a non-empty open bounded complex domain. Let $\psi : U \to \mathbb{C}$ and $\phi : U \to U$ be holomorphic functions with continuous extensions to $\overline{U}$, and assume that $\phi\left(\overline{U}\right) \subset U$. Then ϕ has a unique fixed point $z^* \in U$, and the weighted composition operator $T : \mathcal{H}(U) \to \mathcal{H}(U)$, given by $(Tu)(z) := \psi(z)(u \circ \phi)(z)$, is of trace-class with trace given by the Atiyah-Bott type fixed point formula*

$$\mathrm{Tr}(T) = \frac{\psi(z^*)}{\det(1 - \phi'(z^*))}.$$

We use this theorem to prove the following proposition.

Proposition 22. *Let S be a conformal fgGDMS satisfying the nestedness condition (NC). With the notion as above we then have that there exists a constant $c \in \mathbb{R}$ such that for all w with $\Re(w) > c$, we have $|\zeta(w)| > \widetilde{c}$, for some $\widetilde{c} > 0$. If, in addition, S is primitive then there are no zeros of ζ in the half-plane $\{w \in \mathbb{C} \mid \Re(w) > \dim_{\mathrm{H}} L(S)\}$.*

PROOF. Combining Lemma 10 and the fact that $\det(1 - \mathcal{L}_w)$ is analytic (cf. [14], see also [28]), one easily verifies that

$$\det(1 - \mathcal{L}_w) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}(\mathcal{L}_w^n)\right).$$

In order to evaluate the traces, we use the notation from Definition 12 and write

$$\mathcal{L}_w(u)(z) = \sum_{e \in E} (D_e(z))^w \cdot \Phi_e(u)(z),$$

where $\Phi_e : \mathcal{H}\left(\left(X_{t(e)}\right)_{\mathbb{C}}\right) \to \mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)$. We use an idea similar to the one of Ruelle in [28]. For ease of notation, we define the operator L_e by $L_e(u)(z) := \left(D_e\left(z\right)\right)^w \cdot \Phi_e(u(z))$. With this notation we then have that (see [28, p. 235])

$$\operatorname{Tr}\left(\mathcal{L}_w^n\right) = \sum_{\substack{(e_1,\ldots,e_n)\in E^n \\ i(e_1)=t(e_n)}} \operatorname{Tr}\left(L_{e_1}\ldots L_{e_n}\right).$$

For each $\underline{e} := (e_1, \ldots, e_n) \in E^n$ we have that

$$L_{e_1}\ldots L_{e_n}(u)\left(z\right) = \left(\left(D_{\underline{e}}\right)\left(z\right)\right)^w \cdot \left(u \circ \left(\phi_{e_n}\right)_{\mathbb{C}} \circ \ldots \circ \left(\phi_{e_1}\right)_{\mathbb{C}}\right)\left(z\right),$$

where $D_{\underline{e}}(z) = D_{e_1}(z) \cdot D_{e_2}\left(\left(\phi_{e_1}\right)_{\mathbb{C}}\left(z\right)\right) \cdot \ldots \cdot D_{e_n}\left(\left(\phi_{e_{n-1}}\right)_{\mathbb{C}} \circ \ldots \circ \left(\phi_{e_1}\right)_{\mathbb{C}}\left(z\right)\right)$. Note that $L_{\underline{e}} := L_{e_1}\ldots L_{e_n}$ has the form of a weighted composition operator T given by $T\left(u\right)\left(z\right) = h\left(z\right) \cdot \left(u \circ g_{\mathbb{C}}\right)\left(z\right)$, for functions $h : U \to \mathbb{C}$ and $g_{\mathbb{C}} : U \to U$. Further, note that $g_{\mathbb{C}}(\overline{U}) \subset U$, since S satisfies the nestedness condition (NC). Hence, by Theorem 21, we have that

$$\operatorname{Tr}\left(L_{e_1}\ldots L_{e_n}\right) = \frac{\left(D_{\underline{e}}\left(z_{\underline{e}}^*\right)\right)^w}{\det\left(1 - \left(g_{\mathbb{C}}\right)'\left(z_{\underline{e}}^*\right)\right)},$$

where $z_{\underline{e}}^*$ is the unique fixed point of $g_{\mathbb{C}} = (\phi_{\underline{e}})_{\mathbb{C}} = \left(\phi_{e_n}\right)_{\mathbb{C}} \circ \ldots \circ \left(\phi_{e_1}\right)_{\mathbb{C}}$ and $\left(g_{\mathbb{C}}\right)'\left(z_{\underline{e}}^*\right)$ denotes the Jacobian of $g_{\mathbb{C}}$ at $z_{\underline{e}}^*$. Recall that the maps ϕ_e are real analytic and that $\left(\phi_e\right)_{\mathbb{C}}$ are holomorphic maps defined via exactly the same power series. Therefore, the fixed point $z_{\underline{e}}^*$ of $g_{\mathbb{C}}$ is equal to the fixed point of $g := \phi_{e_n} \circ \ldots \circ \phi_{e_1}$, and thus $z_{\underline{e}}^*$ belongs to $\mathbb{R}^m$ (in particular, $z_{\underline{e}}^* \in L(S)$). Let us now evaluate the determinant $\det\left(1 - \left(g_{\mathbb{C}}\right)'\left(z_{\underline{e}}^*\right)\right)$. Since $\left(\phi_e\right)_{\mathbb{C}}$ is defined via the power series of ϕ_e, it follows that the entries of their Jacobian coincide (in the sense that each entry of the Jacobian is an analytic function and so it is a power series; for g' and $g'_{\mathbb{C}}$ the coefficients of these power series coincide). Hence, the Jacobian of $g_{\mathbb{C}}$ evaluated at $z_{\underline{e}}^* \in \mathbb{R}^m$ equals the Jacobian of g at $z_{\underline{e}}^*$. It therefore suffices to evaluate $\det\left(1 - g'\left(z_{\underline{e}}^*\right)\right)$. By Lemma 19, we now have

$$\left|\det\left(1 - \left(\left(\phi_{\underline{e}}\right)_{\mathbb{C}}\right)'\left(z_{\underline{e}}^*\right)\right)\right| = \left|\det\left(1 - \left(g_{\mathbb{C}}\right)'\left(z_{\underline{e}}^*\right)\right)\right| \geq \left(1 - \ell\right)^m. \tag{7}$$

This follows, since all maps of the fgGDMS S are contracting at least by some factor $\ell < 1$.

Recall that $D_e\left(z\right) = \left(\|(\phi_e')\|\right)_{\mathbb{C}}\left(z\right)$. Since $\left(\|(\phi_e')\|\right)_{\mathbb{C}}$ is a holomorphic extension of $\|(\phi_e')\|$, we have that $\left(\|(\phi_e')\|\right)_{\mathbb{C}}$ evaluated at a real $z_{\underline{e}}^*$ is the same as $\|(\phi_e')\|$ evaluated at $z_{\underline{e}}^*$, that is $\|(\phi_e')\|(z_{\underline{e}}^*) = (\|(\phi_e')\|)_{\mathbb{C}}(z_{\underline{e}}^*)$. Therefore, we have that $D_{\underline{e}}\left(z_{\underline{e}}^*\right) = \left\|\left(\phi_{\underline{e}}'\right)\left(z_{\underline{e}}^*\right)\right\| \in \mathbb{R}$. Hence,

$$\left|D_{\underline{e}}\left(z_{\underline{e}}^*\right)^w\right| = \left\|\left(\phi_{\underline{e}}'\right)\left(z_{\underline{e}}^*\right)\right\|^{\Re(w)} \leq \ell^{\Re(w)}.$$

We are now ready to complete the proof. First recall that

$$\zeta(w) = \det(1-\mathcal{L}_w) \quad = \quad \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left(\mathcal{L}_w^n\right)\right).$$

Now we can bound the exponent of the right hand side of the above equation in the following way:

$$\begin{aligned}
\left|\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left(\mathcal{L}_w^n\right)\right| &= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{\substack{\underline{e}=(e_1,\dots,e_n)\in E^n\\ i(e_1)=t(e_n)}}\left|\frac{\left(D_{\underline{e}}\left(z_{\underline{e}}^*\right)\right)^w}{\det\left(1-\left(g_{\mathbb{C}}\right)'\left(z_{\underline{e}}^*\right)\right)}\right| \\
&\leq \sum_{n=1}^{\infty}\frac{1}{n}\sum_{\substack{\underline{e}\in E^n\\ i(e_1)=t(e_n)}}\left|\frac{\left\|\left(\phi_{\underline{e}}'\right)\left(z_{\underline{e}}^*\right)\right\|^{\mathrm{Re}(w)}}{(1-\ell)^m}\right| \\
&\leq \frac{1}{(1-\ell)^m}\sum_{\underline{e}\in E^\star}\frac{1}{|\underline{e}|}\left\|\left(\phi_{\underline{e}}'\right)\left(z_{\underline{e}}^*\right)\right\|^{\mathrm{Re}(w)}.
\end{aligned}$$

If the fgGDMS is primitive, then it follows from [20, Proposition 4.2.8] (combined with [20, Theorem 4.2.13]) that the latter sum converges for all $\mathrm{Re}(w) > \dim_{\mathrm{H}} L(S)$. If the fgGDMS is not primitive, note that

$$\sum_{\underline{e}\in E^\star}\frac{1}{|\underline{e}|}\left\|\left(\phi_{\underline{e}}'\right)\left(z_{\underline{e}}^*\right)\right\|^{\mathrm{Re}(w)} \leq \sum_{n=1}^{\infty}\left(\mathrm{card}\,(E)\cdot\ell^{\Re(w)}\right)^n.$$

The series in the latter expression is a geometric series, so it converges for $\Re(w)$ large enough. Hence, there exists a positive constant $\tilde{c}$ such that for $\Re(w)$ sufficiently large, we have $|\zeta(w)| > \tilde{c} > 0$. $\square$

2.3 Proofs of main results

2.3.1 Refinement of the FPR-operator

For the following lemma, recall that in Definition 12 we defined $\mathcal{H}(Y_{\mathbb{C}})$ to be the Hilbert space of holomorphic L^2-functions on a complex neighborhood $Y_{\mathbb{C}} \subset \mathbb{C}^m$.

Lemma 23. *Let S be a conformal fgGDMS which satisfies (SSC) and (NC). Then for each $e \in E$ and $r > 0$ sufficiently small, there exists a refinement $\tilde{E}_r(i(e)) \subset E^*$ and a refined FPR-operator $\widetilde{\mathcal{L}_w}$ which is of the form*

$$\widetilde{\mathcal{L}_w} = \sum_{e \in E} \bigoplus_{\underline{f} \in \tilde{E}_r(i(e))} \left(D_{e,\underline{f}}\right)^w \cdot \Phi_{e,\underline{f}}.$$

Here, $w \in \mathbb{C}$ and $D_{e,\underline{f}}$ is given for $z \in \left(X_{i(\underline{f})}\right)_{\mathbb{C}}$ by $D_{e,\underline{f}}(z) := (\|(\phi_e')\|)_{\mathbb{C}}(z)$, and $\Phi_{e,\underline{f}} : \mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \to \mathcal{H}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)\right)$ is given by $u(z) \mapsto u\left((\phi_e)_{\mathbb{C}}(z)\right)$. Furthermore, we have

$$\det(1 - \mathcal{L}_w) = \det(1 - \widetilde{\mathcal{L}_w}).$$

PROOF. Recall that the FPR-operator was defined by

$$\mathcal{L}_w(u)(z) = \sum_{e \in E} \left((\|(\phi_e')\|)_{\mathbb{C}}(z)\right)^w u\left((\phi_e)_{\mathbb{C}}(z)\right) = \sum_{e \in E} \left(D_e(z)\right)^w \cdot \Phi_e(u)(z).$$

Let us concentrate on the composition operator Φ_e in one of these summands for the moment. For this we have that

$$\Phi_e : \mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \to \mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right).$$

Recall that we defined $\mathcal{L}_w$ and all associated operators on functions which are defined on small neighborhoods of the limit set. We now refine these neighborhoods. For this, let $r > 0$ and define

$$E_r(i(e)) := \left\{\underline{f} \in E^* \mid \phi_{\underline{f}}\left(X_{i(\underline{f})}\right) \subset X_{i(e)} \text{ and } \operatorname{diam}\left(\phi_{\underline{f}}\left(X_{i(\underline{f})}\right)\right) \asymp r\right\}.$$

Then $\bigcup_{\underline{f} \in E_r(i(e))} \phi_{\underline{f}}\left(X_{i(\underline{f})}\right)$ is a cover of $L(S) \cap X_{i(e)}$. Therefore, we can restrict $\mathcal{L}_w$ to the space $\mathcal{H}\left(\bigcup_{e \in E} \bigcup_{\underline{f} \in E_r(i(e))} \left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)\right)$, which corresponds to restricting Φ_e to $\mathcal{H}\left(\bigcup_{\underline{f} \in E_r(i(e))} \left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)\right)$. Since S satisfies SSC, we have for $f, g \in E_r(i(e))$ that either $\phi_f\left(X_{i(f)}\right)$ and $\phi_g\left(X_{i(g)}\right)$ are disjoint, or one is a subset of the other. In the latter case, if $\phi_f\left(X_{i(f)}\right) \subset \phi_g\left(X_{i(g)}\right)$, we write $f < g$. Since the cardinality of $E_r(i(e))$ is finite, this partial ordering allows us to determine maximal elements in $E_r(i(e))$, and this allows us to define the set $\tilde{E}_r(i(e))$ of maximal elements in $E_r(i(e))$. We then have that $\bigcup_{\underline{f} \in \tilde{E}_r(i(e))} \phi_{\underline{f}}\left(X_{i(\underline{f})}\right)$ is a cover of $L(S) \cap X_{i(e)}$ consisting of pairwise disjoint sets. Therefore, instead of considering the function space $\mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)$, we consider the direct sum

$$\bigoplus_{\underline{f}\in\tilde{E}_r(i(e))} \mathcal{H}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)\right).$$

For this we have

$$\widetilde{\Phi_e} : \mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \to \bigoplus_{\underline{f}\in\tilde{E}_r(i(e))} \mathcal{H}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)\right),$$

given by $u(z) \mapsto u\left((\phi_e)_{\mathbb{C}}(z)\right)$. Hence, we have for each $e \in E$ that

$$\begin{aligned}
\left(\left(\|(\phi_e')\|\right)_{\mathbb{C}}(z)\right)^w \cdot \widetilde{\Phi_e}(u)(z) &= \bigoplus_{\underline{f}\in\tilde{E}_r(i(e))} \left(\left(\|(\phi_e')\|\right)_{\mathbb{C}}(z)\right)^w \cdot \Phi_{e,\underline{f}}(u)(z) \\
&= \bigoplus_{\underline{f}\in\tilde{E}_r(i(e))} \left(D_{e,\underline{f}}(z)\right)^w \cdot \Phi_{e,\underline{f}}(u)(z),
\end{aligned}$$

with $\Phi_{e,\underline{f}} : \mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \to \mathcal{H}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)\right)$ given by $u(z) \mapsto u\left((\phi_e)_{\mathbb{C}}(z)\right)$.

In order to prove that $\det(1-\mathcal{L}_w) = \det(1-\widetilde{\mathcal{L}_w})$, recall that with the notation as in the proof of Proposition 22, we have that

$$\det(1-\mathcal{L}_w) = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{\substack{\underline{e}=(e_1,\ldots,e_n)\in E^n\\ i(e_1)=t(e_n)}} \frac{\left(\left(D_{\underline{e}}\right)\left(z^*_{\underline{e}}\right)\right)^w}{\det\left(1-(g_{\mathbb{C}})'\left(z^*_{\underline{e}}\right)\right)}.$$

Note that here the inner sum is actually taken over the unique fixed points of $\phi_{\underline{e}}$, $\underline{e} \in E^n$ and the summands are traces of weighted composition operators. Recall that these traces are given by evaluating certain expressions at the fixed points. Let us now compare these expressions with the corresponding expressions for $\widetilde{\mathcal{L}_w}$. Let $\underline{e} \in E^n$ be given and let $z^*_{\underline{e}}$ be the unique fixed point of $(\phi_{\underline{e}})_{\mathbb{C}}$. Recall that we have $z^*_{\underline{e}} \in L(S)$. Recall that we have $\widetilde{\Phi_e} = \bigoplus_{f\in\tilde{E}_r(i(e))}\Phi_{e,f}$. Note that there is a unique $f_1 \in \bigcup_{e\in E}\tilde{E}_r(i(e))$ such that $z^*_{\underline{e}} \in (\phi_f)_{\mathbb{C}}(X_{i(f)})$, since $\{\phi_{\underline{f}}(X_{i(\underline{f})})\}_{\underline{f}\in\tilde{E}_r(i(e))}$ is a cover of $L(S)\cap X_{i(e)}$ (and $\{X_{i(e)}\}_{e\in E}$ is a cover of $L(S)$) by pairwise disjoint sets. From this it is easy to see that there is a unique operator $\Phi_{e_n,f_n}\circ\ldots\circ\Phi_{e_1,f_1}$ with associated contraction having the unique fixed point $z^*_{\underline{e}}$ and $\underline{e} = (e_1,\ldots,e_n)$. Note that $\Phi_{e_n,f_n}\circ\ldots\circ\Phi_{e_1,f_1}$ is a composition operator. Applying the formula from

Theorem 21, it is clear that its trace coincides with the trace of $\Phi_{\underline{e}}$. From this it follows that $\mathrm{Tr}(\mathcal{L}_w^n) = \mathrm{Tr}(\widetilde{\mathcal{L}_w}^n)$, for all $n \in \mathbb{N}$. Hence, we have that $\det(1 - \mathcal{L}_w) = \det(1 - \widetilde{\mathcal{L}_w})$. This completes the proof. $\square$

2.3.2 An upper bound for the zeta function

The following lemma gives the key observation of this section. It will allow us to show that if the refinements in Lemma 23 are chosen appropriately, then $|D^w_{e,\underline{f}}(z)|$ can be bounded from above by some uniform constant.

Lemma 24. *Let S be a conformal fgGDMS which satisfies (SSC) and (NC). Let $e \in E$ be fixed and let $c > 0$ be given. Let $D_{e,\underline{f}}$ be as in Lemma 23, with $\underline{f} \in E_r(i(e))$ for some $r > 0$ sufficiently small. If for $w \in \mathbb{C}$ we have $\Re(w) \geq -c$ and $|\mathrm{Im}(w)| \asymp r^{-1}$, then*

$$|(D_{e,\underline{f}}(z))^w| \ll 1, \text{ for all } z \in \left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right).$$

In particular, $\|(D_{e,\underline{f}}(\cdot))^w\|_\infty \ll 1$.

PROOF. Although the following calculation is relatively straightforward, we present the details here for the sake of completeness. Here, $\mathrm{Arg}(z)$ denotes the number $-\pi \leq \mathrm{Arg}(z) < \pi$ with $z = |z| \cdot e^{\imath \mathrm{Arg}(z)}$. For two complex numbers $z, w \in \mathbb{C}$ we have that

$$\begin{aligned} |z^w| &= \left| \left(|z| \cdot \mathrm{e}^{(\imath \cdot \mathrm{Arg}(z))}\right)^w \right| \\ &\leq |z|^{\Re(w)} \cdot \left|\mathrm{e}^{(\imath \cdot w \cdot \mathrm{Arg}(z))}\right| \\ &\leq |z|^{\Re(w)} \cdot \left|\mathrm{e}^{(-\mathrm{Im}(w) \cdot \mathrm{Arg}(z))}\right| \\ &\leq |z|^{\Re(w)} \cdot \mathrm{e}^{(|\mathrm{Im}(w)| \cdot |\mathrm{Arg}(z)|)}. \end{aligned}$$

Applying this inequality to the operator $D_{e,\underline{f}}$ given in Lemma 23, for e, $\underline{f}$, w and z as stated in the lemma, we obtain that

$$|(D_{e,\underline{f}}(z))^w| \leq \left|D_{e,\underline{f}}(z)\right|^{\mathrm{Re}(w)} \cdot \mathrm{e}^{(|\mathrm{Im}(w)| \cdot |\mathrm{Arg}(D^w_{e,\underline{f}}(z))|)}. \tag{8}$$

Recall from the discussion preceding Definition 12 that $(\|\phi'_e\|)_{\mathbb{C}}$ was a holomorphic extension of $\|\phi'_e\|$ on a small neighborhood $\left(X_{i(e)}\right)_{\mathbb{C}}$ and that $|(\|\phi'_e\|)_{\mathbb{C}}(z)| < 1$, for all $z \in \left(X_{i(e)}\right)_{\mathbb{C}}$. Therefore, for $\Re(w) \geq -c$ we have

$$\left|\left|D_{e,\underline{f}}(z)\right|^{\Re(w)}\right| = \left| \,|(\|\phi'_e\|)_{\mathbb{C}}(z)|^{\Re(w)}\right| \leq \left|\,|(\|\phi'_e\|)_{\mathbb{C}}(z)|^{-c}\right| \ll 1. \tag{9}$$

Note that for $\mathrm{Im}(w)=0$ this already proves the assertion in the theorem. Hence, assume $|\mathrm{Im}(w)|>0$. Recall that by Definition 7 we have $|\,\|\phi_e'(x)\|-\|\phi_e'(y)\|\,|\le c|x-y|$. In other words, the maps $\|\phi_e'\|$ are Lipschitz continuous with a uniform Lipschitz constant $c>0$. Hence, we assumed that the holomorphic extensions $(\|\phi_e'\|)_{\mathbb{C}}$ are also Lipschitz continuous with some uniform Lipschitz constant (see 2.1.3). Therefore, for all $z_1,z_2\in\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)$, we have that

$$|(\|\phi_e'\|)_{\mathbb{C}}(z_1)-(\|\phi_e'\|)_{\mathbb{C}}(z_2)|\ll|z_1-z_2|\le\operatorname{diam}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)\right)\asymp r.$$

Further recall from the discussion following Definition 7 that $\|\phi_e'\|$ is non-zero. In particular, by combining the bounded distortion property and the fact the S is finitely generated, we have that $\|\phi_e'(x)\|$ is uniformly bounded away from zero, for all $e\in E$ and all $x\in X_{i(e)}$. Hence, we assumed that $|(\|\phi_e'\|)_{\mathbb{C}}(z)|$ is bounded away from zero uniformly, say by some constant $c_0>0$, for all $e\in E$ and $z\in X_{\mathbb{C}}$ (see 2.1.3). Combining these observations, we conclude that for all $z\in\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)$ we have that

$$\left|\operatorname{Arg}\left(D_{e,\underline{f}}(z)\right)\right| \quad\le\quad \arctan\left(\operatorname{diam}\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(X_{i(\underline{f})}\right)/c_0\right)\asymp r.$$

Finally, for $|\mathrm{Im}(w)|>0$ we can choose $r\asymp|\mathrm{Im}(w)|^{-1}$ and then the previous estimate implies that $|\mathrm{Im}(w)|\cdot\left|\operatorname{Arg}\left(D_{e,\underline{f}}(z)\right)\right|\ll 1$. Combining this with the inequalities (8) and (9), the lemma follows. $\square$

2.3.3 Proof of the main theorems

We are now ready to prove the main theorems. First note that Main Theorem 1 and the first statement of Main Theorem 2 follow directly from Proposition 22. Recall that the remaining statement of Main Theorem 2 is as follows.

> *Let S be a primitive conformal fgGDMS acting on $\mathbb{R}^m$ and satisfying the strong separation condition (SSC) and the nestedness condition (NC). For each $c>0$ and $w\in\{z\in\mathbb{C}\mid\Re(z)>-c,\ |\mathrm{Im}(z)|>1\}$, we then have*
>
> $$\log|\zeta(w)|\ll\mathrm{e}^{\delta(S)\cdot\log(|\mathrm{Im}(w)|)}.$$

Let S be as stated in Main Theorem 2. For the estimate in Main Theorem 2, let $w\in\{z\in\mathbb{C}\mid\Re(z)>-c,\ |\mathrm{Im}(z)|>1\}$, for some $c>0$. Combining

Lemma 23 and equation (2) with the definition of the zeta function, we have for all $r > 0$ that

$$|\zeta(w)| \leq \prod_{l=0}^{\infty} \left(1 + \sharp E \cdot \max_{\substack{e \in E, \\ \underline{f} \in \tilde{E}_r(i(e))}} \chi_{\left[\frac{l}{\sharp E \cdot \sharp \tilde{E}_r(i(e))}\right]} \left(\left(D_{e,\underline{f}} \right)^w \cdot \Phi_{e,\underline{f}} \right) \right).$$

By Lemma 24, we have that if $r^{-1} \asymp |\operatorname{Im}(w)|$ then there is some $c_2 > 0$ such that for every $\underline{f} \in E_r(i(e))$ we have $\|(D_{e,\underline{f}})^w\|_\infty < c_2$. Hence, let us choose r in this way, that is, let $r^{-1} \asymp |\operatorname{Im}(w)|$. This can be done, since we have seen before that ζ is independent of the choice of r. Combining this with the fact that $\chi_l(AB) \leq \|A\| \chi_l(B)$, we have that

$$|\zeta(w)| \leq \prod_{l=0}^{\infty} \left(1 + c_2 \cdot \sharp E \cdot \max_{\substack{e \in E, \\ \underline{f} \in \tilde{E}_r(i(e))}} \chi_{\left[\frac{l}{\sharp E \cdot \sharp \tilde{E}_r(i(e))}\right]} \left(\Phi_{e,\underline{f}} \right) \right).$$

Note that since $\Phi_{e,\underline{f}}$ is a restriction of Φ_e, we have that $\chi_l\left(\Phi_{e,\underline{f}}\right) \leq \chi_l(\Phi_e)$, for all $l \in \mathbb{N} \cup \{0\}$. Hence, we have

$$|\zeta(w)| \leq \prod_{l=0}^{\infty} \left(1 + c_2 \cdot \sharp E \cdot \max_{\substack{e \in E, \\ \underline{f} \in \tilde{E}_r(i(e))}} \chi_{\left[\frac{l}{\sharp E \cdot \sharp \tilde{E}_r(i(e))}\right]} (\Phi_e) \right).$$

Applying Lemma 18, we obtain the estimate

$$|\zeta(w)| \ll \mathrm{e}^{\sharp E \cdot \sharp \tilde{E}_r(i(e))}.$$

Taking the logarithm on both sides of the above inequality and recalling that we have $\sharp \tilde{E}_r(i(e)) \leq \sharp E_r(i(e))$, it follows that

$$\log(|\zeta(w)|) \ll \sharp E \cdot \sharp E_r(i(e)) + \mathit{const.} \asymp \mathit{const.} + \sharp E_r(i(e)).$$

Applying Lemma 15, we then have

$$\log(|\zeta(w)|) \ll \mathit{const.} + r^{-\delta(S)}.$$

Since $r^{-1} \asymp |\operatorname{Im}(w)|$ we hence have $\log(|\zeta(w)|) \ll \mathit{const.} + |\operatorname{Im}(w)|^{\delta(S)}$. Note that $|\operatorname{Im}(w)|^{\delta(S)} + \mathit{const.} \ll |\operatorname{Im}(w)|^{\delta(S)}$, since $|\operatorname{Im}(w)| > 1$. From this the inequality in Main Theorem 2 follows. This completes the proof of Main Theorem 2.

For the proof of Main Theorem 3, let us define the rectangle $Q_{c,d}^{a,b} \subset \mathbb{C}$ for $a, b, c, d \in \mathbb{R}$ by

$$Q_{c,d}^{a,b} := \{z \in \mathbb{C} \mid a \leq \Re(z) \leq b,\ c \leq \operatorname{Im}(z) \leq d\}.$$

Moreover, let c be a fixed positive constant. We claim that for all sufficiently large $k \in \mathbb{R}$, the following upper bound for the growth of the number of resonances within the strip $Q_{k,k+1}^{-c,\infty}$ (counted with multiplicity) holds:

$$\mathrm{n}_\zeta\left(Q_{k,k+1}^{-c,\infty}\right) \ll k^{\delta(S)}.$$

To show this, first recall that by Proposition 22, there exist two real constants $c_4, \widetilde{c_4}$ such that $|\zeta(z)| > \widetilde{c_4} > 0$ on the half-plane $\{z \in \mathbb{C} : \Re(z) \geq c_4\}$. Consequently, the strip $Q_{k,k+1}^{-c,\infty}$ can be replaced by the rectangle $Q_{k,k+1}^{-c,c_4}$. Let us now consider the ball $B_{c_5}(\imath k + c_4)$ such that $Q_{k,k+1}^{-c,c_4} \subset B_{c_5}(\imath k + c_4)$. In order to be able to apply certain standard techniques from complex analysis, let us normalize the situation as follows. With $\mathcal{Z} : \mathbb{C} \to \mathbb{C}$ given by $\mathcal{Z}(w) := \zeta(w + \imath k + c_4)$, we have that

$$\mathrm{n}_\zeta\left(Q_{k,k+1}^{-c,c_4}\right) \leq \mathrm{n}_{\mathcal{Z}}\left(B_{c_5}(0)\right).$$

Let $n_{\mathcal{Z}}(t)$ denote the number of zeros of $\mathcal{Z}$ inside the ball $B_t(0)$, for t positive, counted with multiplicity. Applying Jensen's Formula (see for example [36, 3.62(2)]), we obtain

$$\int_0^t \frac{n_{\mathcal{Z}}(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log\left|\mathcal{Z}\left(t \cdot \mathrm{e}^{\imath\theta}\right)\right| d\theta - \log|\mathcal{Z}(0)|.$$

Note that $|\mathcal{Z}(0)| = |\zeta(\imath k + c_4)| > \widetilde{c_4} > 0$. Therefore, we have $-\log|\mathcal{Z}(0)| < -\log(\widetilde{c_4})$, and hence $-\log|\mathcal{Z}(0)|$ is finite. However, if $-\log|\mathcal{Z}(0)| > 0$, then there is at least one zero. Hence, for t sufficiently large, we have that

$$\int_0^t \frac{n_{\mathcal{Z}}(x)}{x} dx \ll \frac{1}{2\pi} \int_0^{2\pi} \log\left|\mathcal{Z}\left(t \cdot \mathrm{e}^{\imath\theta}\right)\right| d\theta.$$

Moreover, one immediately verifies that

$$n_{\mathcal{Z}}(t) \leq \frac{1}{\log 2} \int_t^{2t} \frac{n_{\mathcal{Z}}(x)}{x} dx \leq \frac{1}{\log 2} \int_0^{2t} \frac{n_{\mathcal{Z}}(x)}{x} dx.$$

Combining these two observations, it follows that

$$n_{\mathcal{Z}}(t) \ll \frac{1}{2\pi} \int_0^{2\pi} \log\left|\mathcal{Z}\left(2t \cdot \mathrm{e}^{\imath\theta}\right)\right| d\theta \ll \max_{\theta \in [0,2\pi]} \log\left|\mathcal{Z}\left(2t \cdot \mathrm{e}^{\imath\theta}\right)\right|.$$

This implies for $c_6 \geq c_5$ sufficiently large, that

$$\mathrm{n}_{\mathcal{Z}}\left(B_{c_6}(0)\right) = n_{\mathcal{Z}}(c_6) \ll \max_{\theta \in [0,2\pi]} \log\left|\mathcal{Z}\left(2c_6 \cdot \mathrm{e}^{\imath\theta}\right)\right|.$$

Also, by the definition of $\mathcal{Z}$, we have that

$$\max_{\theta \in [0,2\pi]} \log\left|\mathcal{Z}\left(2c_6 \cdot \mathrm{e}^{\imath\theta}\right)\right| \leq \max_{w \in B_{2c_6}(\imath k + c_4)} \log|\zeta(w)|.$$

Furthermore, note that if $w \in B_{2c_6}(\imath k + c_4)$ then $\Re(w) \geq -2c_6 + c_4$, and hence

$$B_{2c_6}(\imath k + c_4) \subset Q_{k-2c_6,\, k+2c_6}^{c_4-2c_6,\, c_4+2c_6}.$$

This shows that

$$\max\left\{\log|\zeta(w)| : w \in B_{2c_6}(\imath k + c_4)\right\} \leq \max\left\{\log|\zeta(w)| : w \in Q_{c_4-2c_6,\, c_4+2c_6}^{k-2c_6,\, k+2c_6}\right\}.$$

Combining these observations with the estimate in Main Theorem 2, it now follows that for k sufficiently large, we have

$$\begin{aligned} \mathrm{n}_\zeta\left(Q_{k,k+1}^{-c_4,\infty}\right) &\ll \max_{w \in B_{2c_6}(\imath k + c_4)} \log|\zeta(w)| \\ &\leq \max_{\substack{\Re(w) > c_4 - 2c_6, \\ \mathrm{Im}(w) \in [k-2c_6, k+2c_6]}} \log|\zeta(w)| \ll k^{\delta(S)}, \end{aligned}$$

where, as always in this paper, the resonances are counted with multiplicities. Exactly the same calculation can be done for $Q_{-k-1,-k}^{-c_4,\infty}$. This completes the proof of Main Theorem 3.

Remark: Note that the setting we have used in this paper is a special case of the setting used by Ruelle in [28] (see also [29] and [30]). More precisely, in [28] Ruelle considered more general transfer operators on exterior forms and has used arbitrary holomorphic functions instead of the very special functions $(\|\phi_e'\|)_{\mathbb{C}}^w$ which we have used in this paper. However, Ruelle studied transfer operators on the Banach-space of holomorphic functions with continuous extensions to the boundary of the domain of definition equipped with the uniform norm. In contrast, we have followed the approach of Guillopé *et al.* to study $\mathcal{L}_w$ on the Hilbert space of holomorphic L^2-functions. Note that the techniques we have used here are generalizations of the techniques used in [15] to conformal fgGDMSs.

3 Comparison to some other zeta functions

3.1 Comparison to the (not-complexified) Ruelle zeta function

One might wonder why we went to such length in Section 2.1.3 to complexify the transfer operator and then used this complexified operator in the definition of the zeta function. There are of course several reasons for this, one of which will become clear in Section 3.3. However, in principle there is no problem defining the zeta function using the non-complexified transfer operator. Note however, that for the analytic part to work one would prefer to consider a Hilbert space of functions on $L(S)$, and hence the canonical choice would be $L^2(L(S), \mu)$, where μ is the unique measure mentioned in Theorem 14.

In this case Proposition 20 would still hold and the zeta function wold still be well defined. As for Proposition 22, our proof of Proposition 22 relies in particular on the Athiya-Bott type Theorem 21 and hence on the special choice of function space. An analog of Theorem 21 for functions on $L^2(L(S), \mu)$ seems yet unknown. Since the proofs of Section 2.3 including the main theorems rely on the Athiya-Bott type Theorem 21, these results might not hold for the non-complexified zeta function.

Finally, let us remark that in the special case of a fgGDMS which consists of only a single contraction, one easily calculates that the set of zeros of the non-complexified zeta function equals $\left\{\imath k \frac{2\pi}{\|\phi'\|}, k \in \mathbb{Z}\right\}$, while the set of zeros of ζ equals $\left\{\imath k \frac{2\pi}{\|\phi'\|} - m, k \in \mathbb{Z}, m \in \mathbb{N} \cup \{0\}\right\}$. Note that a similar discrepancy turns up in the next section.

3.2 Comparison to the geometric zeta function on fractal strings

Given a fractal in the unit interval the complement of the fractal is a disjoint union of (open) intervals. The set of length $\mathcal{L} := l_1, l_2, \dots$ of these intervals is called a fractal string. Associated to a fractal string is the so called *geometric zeta function* which is the meromorphic continuation of the map $\zeta_{\mathcal{L}} : s \mapsto \sum_j l_j^s$. The poles of $\zeta_{\mathcal{L}}$ are referred to as *complex dimensions* (c.f.g [19] and references therein).

If C is the fractal string associated to the 1/3-Cantor set, then the set of complex dimensions is given by

$$\left\{\frac{\log 2}{\log 3} + \imath k \frac{2\pi}{\log 3} \mid k \in \mathbb{Z}\right\}.$$

If $S := \{S_1, \dots, S_n\}$ is a self-similar IFS and if all S_i share the same contraction ratio $1/S_i' =: r \in (1, \infty)$, then the set of resonances of the zeta function ζ defined in this paper can be easily calculated. Namely, it is given by

$$\left\{ \frac{\log n}{\log r} + \imath k \frac{2\pi}{\log r} - m \mid k \in \mathbb{Z}, m \in \mathbb{N} \cup \{0\} \right\}.$$

Hence, at least in these situations, the set of resonances of the zeta function studied in this paper contains all information given by the complex dimensions of a fractal string.

3.3 Comparison to the Selberg zeta function for convex co-compact Schottky groups

Finally, we like to remark that if a fgGDMS comes from the action of a Kleinian group of Schottky type Γ (c.f.g. [20, Example 5.1.5]), then the zeta function ζ defined above coincides with the Selberg zeta function associated to the hyperbolic manifold $\mathbb{H}^{n+1}/\Gamma$. (This is essentially proven in [15, Proposition 3.4], where the transfer-operator is considered with respect to the Bowen-Series map [7] associated to the action of Γ. For further details on the Bowen-Series map we refer to [33] and references given therein. Also note that a Kleinian group of Schottky type is necessarily free. For non-free groups, the associated zeta functions are slightly more complicated, c.f.g. [17].)

For the Selberg zeta function there is an extensive literature on the the distribution of the poles and zeros, (c.f.g. [21] and references therein). In particular note, that in these cases the distribution of resonances in the direction of the negative real axis seems (in most cases) not to exhibit the simple lattice-like behavior observed above, where the dependence on $m \in \mathbb{N} \cup \{0\}$ was always given by a shift by $-m$, with $m \in \mathbb{N} \cup \{0\}$ (c.f.g. [15]).

REMARK: It seems possible to extend the results of this paper at least to (Gromov)-hyperbolic convex co-compact groups which satisfy analogs of (SSC) and (NC) or to subshifts of finite type. On the one hand, one would need a suspension of the shift in order to get the geodesic flow (c.f.g. [22], [23] and references given therein). On the other hand, one would need to ensure the existence of an Ahlfors regular measure (as in Theorem 14).

Acknowledgements I am grateful to the unknown referee for valuable constructive comments on the first draft of this paper.

References

[1] M. Artin, B. Mazur, *On periodic points*, Ann. of Math. **81(2)** (1965) 82–99.

[2] M. F. Atiyah, R. Bott, *A Lefschetz fixed point formula for elliptic differential operators*, Bull. Amer. Math. Soc. **72** (1966) 245–250.

[3] M. F. Atiyah, R. Bott, *A Lefschetz fixed point formula for elliptic complexes I*, Ann. of Math. **86** (1967) 374–407.

[4] M. F. Atiyah, R. Bott, *A Lefschetz fixed point formula for elliptic complexes II*, Ann. of Math. **88** (1968) 451–491.

[5] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, Advanced Series in Nonlinear Dynamics, 16. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[6] R. Bennedetti, C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag, Berlin, 1992.

[7] R. Bowen, C. Series, *Markov maps associated to Fuchsian groups*, Publ. Math., Inst. Hautes Études Sci. **50** (1979) 153–170.

[8] L. Carleson, T. W. Gamelin, *Complex Dynamics*, Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[9] N. Dunford, J. T. Schwarz, *Linear Operators, Part 2*, John Wiley, New York, 1988.

[10] Ky Fan, *On a theorem of Weyl concerning the eigenvalues of linear transformations II*, Proc. Natl. Acad. Sci. USA **36** (1950) 31–35.

[11] Ky Fan, *Some inequalities concerning positive-definite Hermitian matrices*, Math. Proc. Cambridge Philos. Soc. **51** (1955) 414–421.

[12] I. C. Gohberg, S. Goldberg, N. Krupnik, *Traces and Determinants of linear Operators*, Operator Theory 116, 1998.

[13] I. C. Gohberg, M. C. Krein, *Introduction to the Theory of linear non-selfadjoint Operators*, Trans. Math. Monographs 18, AMS, 1969.

[14] A. Grothendieck, *La théorie de Fredholm*, Sém. Bourbaki No. 91 (also in Bull. Soc. Math. France, **84** (1956) 319–384).

[15] L. Guillopé, K. Lin, M. Zworski, *The Selberg zeta function for convex co-compact Schottky groups*, Comm. Math. Phys. **245** (2004) 149–176.

[16] J. Hilgert, F. Rilke, *Meromorphic continuation of dynamical zeta functions via transfer operators*, J. Funct. Anal. **254** (2008) 476–505.

[17] A. Juhl, *Cohomological Theory of Dynamical Zeta Functions*, Progress in Mathematics, 194. Birkhäuser Verlag, Basel, 2001.

[18] S. G. Krantz, *Function Theory of several complex Variables*, 2nd edition, Oxford University Press, 2001.

[19] M. L. Lapidus, M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions: geometry and spectra of fractal strings*, Springer Monographs in Mathematics, (Springer-Verlag, New York, 2006).

[20] R. D. Mauldin, M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge Tracts in Mathematics 148, Cambridge University Press, Cambridge, 2003.

[21] P. Perry, *The spectral geometry of geometrically finite hyperbolic manifolds*, Festschrift in Honour of Barry Simon's Sixtieth Birthday, in Proc. Sympos. Pure Math. (2007) 289–328.

[22] W. Parry, M. Pollicott, *An Analogue of the Prime Number Theorem for Closed Orbits of Axiom A Flows*, Ann. of Math. **118** (1983) 573–591.

[23] M. Pollicott, *Notes on thermodynamic formalism for Anosov flows*, Rencontres de theorie spectrale et geometrie, Grenoble, 1991, 123–128.

[24] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. I, Functional Analysis*, Academic Press, 1972.

[25] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. IV, Analysis of Operators*, Academic Press, 1978.

[26] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[27] D. Ruelle, *Statistical mechanics of a one-dimensional lattice gas*, Comm. Math. Phys. **9** (1968) 267–278.

[28] D. Ruelle, *Zeta-functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976) 231–242.

[29] D. Ruelle, *Repellers for expanding analytic maps*, Ergodic Theory Dynam. Systems **2** (1982) 99–107.

[30] D. Ruelle, *The thermodynamic formalism for expanding maps*, Comm. Math. Phys. **125** (1989) 239–262.

[31] D. Ruelle, *Dynamical zeta functions and transfer operators*, Notices Amer. Math. Soc. **49** (2002), 887–895.

[32] B. Simon, *Trace Ideals and their Applications*, 2nd edition, Math. Surveys Monogr. **120**, AMS, 2002.

[33] M. Stadlbauer, *The Bowen-Series map for some free groups*, Preprint series: Mathematica Gottingensis **5** (2002). [webdoc.sub.gwdg.de/ebook/e/2003/mathgoe/]

[34] B. O. Stratmann, M. Urbański, *Real analyticity of topological pressure for parabolically semihyperbolic generalized polynomial-like maps*, Indag. Math. (2003) 119–134.

[35] Nico M. Temme, *Special Functions: An Introduction to the classical Functions of Mathamatical Physics*, Wiley-Interscience, New York, 1996.

[36] E. C. Titchmarsh, *The Theory of Functions*', 2nd edition, Oxford University Press, 1939.