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ON THE COMPARISON OF DENSITY TYPE TOPOLOGIES GENERATED BY FUNCTIONS

Abstract

In the paper there is presented a necessary and sufficient condition to compare f-density topologies.

1 Preliminaries

Throughout the paper we shall use standard notation: $\mathbb{R}$ will be the set of real numbers, $\mathbb{N}$ the set of positive integers, $\mathcal{L}$ the family of Lebesgue measurable subsets of $\mathbb{R}$ and $|E|$ the Lebesgue measure of a measurable set E. A point $x \in \mathbb{R}$ is a *right-hand density point* of a measurable set E if $\lim_{h\to 0^+} \frac{|(x,x+h)\cap E|}{h} = 1$, or equivalently if $\lim_{h\to 0^+} \frac{|(x,x+h)\setminus E|}{h} = 0$. In the same way we define a *left-hand density point* of E. We say that x is a *density point* of E if x is a right-hand density point and a left-hand density point of E. We will denote by $\Phi_d(E)$ the set of all density points of E (compare [8]).

From Lebesgue Density Theorem it follows that $|E \triangle \Phi_d(E)| = 0$ for any measurable set E. It is also well known ([8, Th. 22.5]) that the family $\mathcal{T}_d := \{E \in \mathcal{L} : E \subset \Phi_d(E)\}$ forms a topology on the real line, called the *density topology*.

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Replacing the denominator h in the definition of a density point by $f(h)$, where f is a nondecreasing function tending to zero at zero, we obtain general notions of density point, density operator, and density topology. To keep a connection with the "ordinary" density we should also assume that $\liminf_{x\to 0+} \frac{f(x)}{x} < \infty$. Otherwise, for any $x \in \mathbb{R}$ and $E \in \mathcal{L}$

$$\frac{|(x, x+h) \setminus E|}{f(h)} \leq \frac{|(x, x+h)|}{h} \cdot \frac{h}{f(h)} \underset{h\to 0+}{\longrightarrow} 0.$$

Let us denote by $\mathcal{A}$ the family of all nondecreasing functions $f : (0, \infty) \to (0, \infty)$ such that $\lim_{x\to 0+} f(x) = 0$ and $\liminf_{x\to 0+} \frac{f(x)}{x} < \infty$. Fix $f \in \mathcal{A}$, $E \in \mathcal{L}$ and $x \in \mathbb{R}$. We say that x is a *right-hand f-density point* of E if

$$\lim_{h\to 0+} \frac{|(x, x+h) \setminus E|}{f(h)} = 0.$$

By $\Phi_f^+(E)$ we denote the set of all right-hand f-density points of E. In the same way one may define *left-hand f-density points* of E and the set $\Phi_f^-(E)$. We say that x is an *f-density point* of E if it is a right and left-hand f-density point of E. By $\Phi_f(E)$ we denote the set of all f-density points of E, i.e. $\Phi_f(E) := \Phi_f^+(E) \cap \Phi_f^-(E)$.

It is easily seen that

- $\Phi_f^+(E + a) = \Phi_f^+(E) + a$,
- $\Phi_f^+(E \cap F) = \Phi_f^+(E) \cap \Phi_f^+(F)$,
- if $|E \triangle F| = 0$ then $\Phi_f^+(E) = \Phi_f^+(F)$,

and analogous properties hold for Φ_f^- and Φ_f. It is also clear that

- x is an f-density point of E if and only if

$$\lim_{\substack{h\to 0, k\to 0 \\ h\geq 0, k\geq 0, h+k>0}} \frac{|(x-h, x+k) \setminus E|}{f(h+k)} = 0.$$

In [2] it was proved that

- $\Phi_f^+(E)$ ($\Phi_f^-(E)$, $\Phi_f(E)$) are measurable,
- $\mathcal{T}_f := \{E \in \mathcal{L} : E \subset \Phi_f(E)\}$ is a topology stronger than the natural topology on the real line.

The topology $\mathcal{T}_f$ is called a *topology generated by a function* f or f*-density topology.* Among f-density topologies there are: the density topology, topologies generated by sequences (compare [3]), ψ-density topologies (compare [1]) and others. The purpose of this paper is to produce necessary and sufficient conditions for the inclusion $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$ for $f_1, f_2 \in \mathcal{A}$.

Different functions can generate the same operator and the same topology. The only important thing is the behavior of a function $f \in \mathcal{A}$ close to zero. Defining a function f on some interval $(0,\delta)$ we consider that it is specified in any permissible way on $[\delta,\infty)$. Obviously, even functions $f_1, f_2 \in \mathcal{A}$ such that $f_1(x) \neq f_2(x)$ for all $x > 0$ can generate the same operator (for example, this is true if for some $r > 0$, $\frac{1}{r} < \frac{f_1(x)}{f_2(x)} < r$ for all x sufficiently close to zero). Fortunately, different operators generate different topologies. To check if $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$, it is enough to examine right-hand f-density point at zero for all measurable sets. Indeed, from $\Phi_f(E+a) = \Phi_f(E) + a$, $\Phi_f^-(-E) = -\Phi_f^+(E)$ and [3, Prop. 4], it follows

Proposition 1. *For each $f_1, f_2 \in \mathcal{A}$ the following conditions are equivalent*

1. $\underset{E\in\mathcal{L}}{\forall}\left(0 \in \Phi_{f_1}^+(E) \Rightarrow 0 \in \Phi_{f_2}^+(E)\right)$,
2. $\underset{E\in\mathcal{L}}{\forall}\left(0 \in \Phi_{f_1}(E) \Rightarrow 0 \in \Phi_{f_2}(E)\right)$,
3. $\underset{E\in\mathcal{L}}{\forall}\Phi_{f_1}(E) \subset \Phi_{f_2}(E)$,
4. $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$.

2 Useful lemmas

In the paper we check whether, for certain functions $f_1, f_2 \in \mathcal{A}$, there exists a measurable set E, such that $0 \in \Phi_{f_2}(E) \setminus \Phi_{f_1}(E)$. To simplify proofs we will consider interval sets and functions "constant on intervals". We say that E is an *interval set* if $E = \cup_{n=1}^{\infty}[a_n, b_n]$ for some tending to zero sequences (a_n), (b_n) with $0 < b_{n+1} < a_n < b_n$, $n \in \mathbb{N}$. By $\mathcal{A}_s$ we denote the family of all functions $f \in \mathcal{A}$ of the form

$$f(x) = y_n \quad \text{for} \quad x \in (x_{n+1}, x_n]\,,$$

where sequences (x_n) and (y_n) are decreasing and tend to zero.

Fix an arbitrary function $f \in \mathcal{A}$. Let $a := f(1)$, (k_n) be an increasing sequence of all numbers n for which $f^{-1}\left(\left(\frac{a}{2^{k_n+1}}, \frac{a}{2^{k_n}}\right]\right)$ is a nondegenerated interval and let x_n be the right endpoint of this interval. In [3, Th. 1] it was demonstrated that the function $g(x) := \frac{a}{2^{k_n}}$ for $x \in (x_{n+1}, x_n]$ belongs to $\mathcal{A}_s$ and $\Phi_f = \Phi_g$. Thus we have

Lemma 1. *For any function $f \in \mathcal{A}$ there is a function $g \in \mathcal{A}_s$ such that $\Phi_f = \Phi_g$, $g \leq 2f$ and $f(x) \leq g(x)$ except for a countable set.*

Lemma 2. *Let $f \in \mathcal{A}_s$, t, h be positive numbers and E be a measurable set satisfying*

$$\limsup_{x\to 0+} \frac{|E \cap (0,x)|}{f(x)} > t.$$

There is an interval $[a,b] \subset (0,h)$ such that

$$\frac{|E \cap (a,b)|}{f(b)} = t \quad \textit{and} \quad \frac{|E \cap (a,x)|}{f(x)} \leq t \textit{ for } x \in (a,b].$$

PROOF. Let (x_n) and (y_n) be decreasing sequences tending to 0 such that $f(x) = y_n$ for $x \in (x_{n+1}, x_n]$. Since $\frac{|E\cap(0,x)|}{f(x)} \leq \frac{|E\cap(0,x_n)|}{f(x_n)}$ for $x \in (x_{n+1}, x_n]$,

$$\limsup_{n\to\infty} \frac{|E \cap (0,x_n)|}{f(x_n)} = \limsup_{x\to 0+} \frac{|E \cap (0,x)|}{f(x)} > t.$$

Fix a positive integer n_0 such that $x_{n_0} < h$ and $\frac{\left|E\cap\left(0,x_{n_0}\right)\right|}{f\left(x_{n_0}\right)} > t$. From the continuity of Lebesgue measure, it follows that $\frac{\left|E\cap\left(c,x_{n_0}\right)\right|}{f\left(x_{n_0}\right)} = t$ for some $c \in (0, x_{n_0})$. Let

$$p := \max\left\{n : n \geq n_0 \wedge x_n > c \wedge \frac{|E \cap (c,x_n)|}{f(x_n)} \geq t\right\}$$

and $b := x_p$. Using again the continuity of measure we find $a \in [c,b]$ with

$$\frac{|E \cap (a,b)|}{f(b)} = t.$$

Thus, for $x \in (a,b] \cap (x_{p+1}, x_p]$,

$$\frac{|E \cap (a,x)|}{f(x)} \leq \frac{|E \cap (a,x_p)|}{f(x_p)} = t.$$

On the other hand, if $x \in (a,b] \cap (a, x_{p+1}]$, then $x \in (x_{n+1}, x_n]$ for some $n > p$. Since $x_n > a \geq c$, we have $\frac{|E\cap(c,x_n)|}{f(x_n)} < t$ and

$$\frac{|E \cap (a,x)|}{f(x)} \leq \frac{|E \cap (c,x_n)|}{f(x_n)} < t.$$

$\square$

Lemma 3. *For any $f \in \mathcal{A}_s$ there exists an interval set C and a positive number t such that*

$$t \le \limsup_{x\to 0+} \frac{|C \cap (0,x)|}{f(x)} \le 2t.$$

Proof. Let (x_n) and (y_n) be decreasing sequences tending to 0 such that $f(x) = y_n$ for $x \in (x_{n+1}, x_n]$. Since $f \in \mathcal{A}$, there is $t > 0$ such that $\limsup_{x\to 0+} \frac{|\mathbb{R}\cap(0,x)|}{f(x)} = \limsup_{x\to 0+} \frac{x}{f(x)} > t$. Applying Lemma 2 we can choose sequences (a_n), (b_n) such that $a_n < b_n$, $b_{n+1} < \min\left\{\frac{1}{n}a_n, tf(a_n)\right\}$,

$$\frac{b_n - a_n}{f(b_n)} = t \text{ and } \frac{x - a_n}{f(x)} \le t \text{ for } x \in (a_n, b_n].$$

Put $C := \bigcup_{n=1}^{\infty} [a_n, b_n]$. For any $n \in \mathbb{N}$ we have $\frac{|C\cap(0,b_n)|}{f(b_n)} > \frac{b_n - a_n}{f(b_n)} = t$, and consequently $\limsup_{x\to 0+} \frac{|C\cap(0,x)|}{f(x)} \ge t$. On the other hand, if $x \in (a_n, b_n]$ then

$$\frac{|C\cap(0,x)|}{f(x)} \le \frac{x - a_n + b_{n+1}}{f(x)} \le \frac{x - a_n}{f(x)} + \frac{b_{n+1}}{f(a_n)} < t + t = 2t,$$

and if $x \in (b_n, a_{n-1}]$ then applying the previous result we obtain

$$\frac{|C\cap(0,x)|}{f(x)} = \frac{|C\cap(0,b_n)|}{f(x)} \le \frac{|C\cap(0,b_n)|}{f(b_n)} < 2t.$$

Hence $\limsup_{x\to 0+} \frac{|C\cap(0,x)|}{f(x)} \le 2t$. □

3 Comparison of topologies

By definition, from $\limsup_{x\to 0^+} \frac{f_1(x)}{f_2(x)} < \infty$ it follows that $\Phi_{f_1} \subset \Phi_{f_2}$ and $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$. Consequently, if $\liminf_{x\to 0^+} \frac{f_1(x)}{f_2(x)} > 0$ and $\limsup_{x\to 0^+} \frac{f_1(x)}{f_2(x)} < \infty$, then $\mathcal{T}_{f_1} = \mathcal{T}_{f_2}$. However, the functions

$$f_1(x) := \frac{1}{n!} \text{ for } x \in \left(\frac{1}{(n+1)!}, \frac{1}{n!}\right], n \in \mathbb{N}$$

and

$$f_2(x) := \frac{1}{n!} \text{ for } x \in \left[\frac{1}{(n+1)!}, \frac{1}{n!}\right), n \in \mathbb{N}$$

generate the same topology (see [3, Ex. 1]), although $\liminf_{x\to 0^+} \frac{f_1(x)}{f_2(x)} = 0$.

Theorem 1. *(compare [10, Th. 2.6]) For any functions $f_1, f_2 \in \mathcal{A}$ with $\lim_{x\to 0+} \frac{f_1(x)}{f_2(x)} = 0$ there is a measurable set D such that $0 \in \Phi_{f_2}(D) \setminus \Phi_{f_1}(D)$.*

Proof. By Lemma 1, there is a function $g_1 \in \mathcal{A}_s$ such that $g_1 \le 2f_1$ and $\Phi_{f_1} = \Phi_{g_1}$. Using Lemma 3 we can find an interval set C and a positive number t such that

$$t \le \limsup_{x\to 0+} \frac{|C \cap (0,x)|}{g_1(x)} \le 2t.$$

Let $D := \mathbb{R} \setminus C$. Since

$$\limsup_{x\to 0+} \frac{|(0,x)\setminus D|}{g_1(x)} = \limsup_{x\to 0+} \frac{|C\cap(0,x)|}{g_1(x)} \ge t > 0,$$

we have $0 \notin \Phi_{g_1}(D) = \Phi_{f_1}(D)$. On the other hand,

$$\begin{aligned}\limsup_{x\to 0+} \frac{|(0,x)\setminus D|}{f_2(x)} &\le \limsup_{x\to 0+} \frac{|(0,x)\setminus D|}{f_1(x)} \cdot \limsup_{x\to 0+} \frac{f_1(x)}{f_2(x)} \\ &\le 2\limsup_{x\to 0+} \frac{|C\cap(0,x)|}{g_1(x)} \cdot \limsup_{x\to 0+} \frac{f_1(x)}{f_2(x)} \le 4t\cdot 0 = 0,\end{aligned}$$

so $0 \in \Phi_{f_2}(D)$. □

Corollary 1. *If $\lim_{x\to 0+} \frac{f_1(x)}{f_2(x)} = 0$, then $\mathcal{T}_{f_1} \subsetneqq \mathcal{T}_{f_2}$.*

It is easily seen that the condition $\lim_{x\to 0+} \frac{f_1(x)}{f_2(x)} = 0$ is not necessary for $\mathcal{T}_{f_1} \subsetneqq \mathcal{T}_{f_2}$. Indeed, if $f_1(x) := x$ and $f_2(x) := \frac{1}{n!}$ for $x \in \left(\frac{1}{(n+1)!}, \frac{1}{n!}\right]$, then $\limsup_{x\to 0+} \frac{f_1(x)}{f_2(x)} = 1 > 0$ and $\mathcal{T}_{f_1} = \mathcal{T}_d \subsetneqq \mathcal{T}_{f_2}$ (see [3, Th. 4]).

S. J. Taylor in [9] investigated f-density points for a function $f(x) = x\psi(x)$, where ψ is a nondecreasing and continuous function with $\lim_{x\to 0+} \psi(x) = 0$. The symmetric version of Taylor's definition was used in [10] to introduce a notion of ψ-density point and ψ-density topology. A main result of [11] gave necessary and sufficient conditions for the the inclusion of one ψ-density topology in another. We will transfer it to f-density topologies. Fortunately, our general framework allows for a simplification of proofs given in [11].

Let $f_1, f_2 \in \mathcal{A}$. We define sequences

$$\begin{aligned} A_n &:= A_{nf_1f_2} := \left\{x \in (0,\infty) : f_1(x) < \frac{1}{n} f_2(x)\right\}, \\ \varepsilon_n &:= \varepsilon_{nf_1f_2} := \limsup_{x\to 0+} \frac{|A_n \cap (0,x)|}{f_1(x)}. \end{aligned}$$

Of course, these sequences are decreasing, and so (ε_n) is convergent.

Lemma 4. *If $0 \in \Phi^+_{f_2}(\mathbb{R} \setminus E)$, then for every positive integer n*

$$\limsup_{x \to 0+} \frac{|E \cap (0,x)|}{f_1(x)} = \limsup_{x \to 0+} \frac{|E \cap A_n \cap (0,x)|}{f_1(x)}.$$

Proof. By assumption,

$$\lim_{x \to 0+} \frac{|E \cap (0,x)|}{f_2(x)} = 0. \tag{1}$$

Let us fix a positive integer n and a positive x such that $f_1(x) < 1$. If $x \notin A_n$, then $f_1(x) \geq \frac{1}{n} f_2(x)$ and consequently

$$\frac{|(E \setminus A_n) \cap (0,x)|}{f_1(x)} \leq \frac{n\,|E \cap (0,x)|}{f_2(x)}. \tag{2}$$

If $(0,x] \subset A_n$, then

$$\frac{|(E \setminus A_n) \cap (0,x)|}{f_1(x)} = 0. \tag{3}$$

Finally, if $x \in A_n$ and $(0,x] \setminus A_n \neq \emptyset$, then for $b := \sup\left((0,x] \setminus A_n\right)$ and any a from $[b - bf_1(b), b] \setminus A_n$ we have $f_1(a) \geq \frac{1}{n} f_2(a)$, and so

$$\begin{aligned} \frac{|(E \setminus A_n) \cap (0,x)|}{f_1(x)} &= \frac{|(E \setminus A_n) \cap (0,b)|}{f_1(x)} \leq \frac{|(E \setminus A_n) \cap (0,a)|}{f_1(a)} + \frac{b-a}{f_1(b)} \\ &\leq \frac{n\,|E \cap (0,a)|}{f_2(a)} + b. \end{aligned} \tag{4}$$

Since $a \leq b \leq x$, conditions (1)-(4) imply

$$\lim_{x \to 0+} \frac{|(E \setminus A_n) \cap (0,x)|}{f_1(x)} = 0$$

which gives

$$\begin{aligned} \limsup_{x \to 0+} \frac{|E \cap (0,x)|}{f_1(x)} &\leq \limsup_{x \to 0+} \frac{|E \cap A_n \cap (0,x)|}{f_1(x)} + \limsup_{x \to 0+} \frac{|(E \setminus A_n) \cap (0,x)|}{f_1(x)} \\ &= \limsup_{x \to 0+} \frac{|E \cap A_n \cap (0,x)|}{f_1(x)} \leq \limsup_{x \to 0+} \frac{|E \cap (0,x)|}{f_1(x)}. \end{aligned}$$

$\square$

Corollary 2. *If* $0 \in \Phi^+_{f_2}(\mathbb{R} \setminus E)$*, then for every positive integer* n

$$\limsup_{x \to 0+} \frac{|E \cap (0,x)|}{f_1(x)} \le \limsup_{x \to 0+} \frac{|A_n \cap (0,x)|}{f_1(x)}.$$

Theorem 2. *Let* $f_1, f_2 \in \mathcal{A}$*. If* $\lim_{n\to\infty} \varepsilon_n = 0$*, then* $\mathcal{T}_{f_2} \subset \mathcal{T}_{f_1}$*.*

PROOF. Let us suppose that $0 \in \Phi^+_{f_2}(E)$. By Corollary 2,

$$0 \le \limsup_{x \to 0+} \frac{|(0,x) \setminus E|}{f_1(x)} \le \limsup_{x \to 0+} \frac{|A_n \cap (0,x)|}{f_1(x)} = \varepsilon_n$$

for every positive integer n. Thus $\limsup_{x\to 0+} \frac{|(0,x)\setminus E|}{f_1(x)} = 0$, and consequently $0 \in \Phi^+_{f_1}(E)$. From Proposition 1 we conclude that $\mathcal{T}_{f_2} \subset \mathcal{T}_{f_1}$. □

Theorem 3. *Let* $f_1, f_2 \in \mathcal{A}$*. If* $\lim_{n\to\infty} \varepsilon_n > 0$*, then there is a measurable set* D *such that* $0 \in \Phi_{f_2}(D) \setminus \Phi_{f_1}(D)$*.*

PROOF. By Lemma 1 there exists $g_1 \in \mathcal{A}_s$ such that $\Phi_{f_1} = \Phi_{g_1}$ and $g_1 \le 2f_1$. Let us define

$$\widetilde{A}_n := \left\{ x \in (0,\infty) : g_1(x) < \frac{1}{n} f_2(x) \right\} \quad \text{and} \quad \widetilde{\varepsilon}_n := \limsup_{x\to 0+} \frac{\left|\widetilde{A}_n \cap (0,x)\right|}{g_1(x)}.$$

Since $A_{2n} \subset \widetilde{A}_n$,

$$\widetilde{\varepsilon}_n = \limsup_{x\to 0+} \frac{\left|\widetilde{A}_n \cap (0,x)\right|}{g_1(x)} \ge \limsup_{x\to 0+} \frac{|A_{2n} \cap (0,x)|}{2f_1(x)} = \frac{1}{2}\varepsilon_{2n},$$

and consequently $\lim_{n\to\infty} \widetilde{\varepsilon}_n \ge \frac{1}{2}\lim_{n\to\infty} \varepsilon_{2n} > 0$. Set $t := \frac{1}{2}\lim_{n\to\infty}\widetilde{\varepsilon}_n$ and choose n_0 such that $\widetilde{\varepsilon}_n > t$ for $n \ge n_0$. Applying Lemma 2 we can define intervals $[a_n, b_n]$, $n \ge n_0$ such that $b_{n+1} < \min\left\{a_n, \frac{1}{n} f_2(a_n)\right\}$,

$$\frac{\left|\widetilde{A}_n \cap (a_n, b_n)\right|}{g_1(b_n)} = t \quad \text{and} \quad \frac{\left|\widetilde{A}_n \cap (a_n, x)\right|}{g_1(x)} \le t \text{ for } x \in (a_n, b_n]\,.$$

Let

$$E := \bigcup_{n=n_0}^{\infty} \left(\widetilde{A}_n \cap (a_n, b_n)\right) \quad \text{and} \quad D := \mathbb{R} \setminus E.$$

For $n \ge n_0$ we have $\frac{|E \cap (0,b_n)|}{g_1(b_n)} \ge \frac{\left|\widetilde{A}_n \cap (a_n,b_n)\right|}{g_1(b_n)} = t > 0$, and so $0 \notin \Phi_{g_1}(D) = \Phi_{f_1}(D)$.

It remains to prove that $0 \in \Phi_{f_2}(D)$. Consider $x \in (0, b_{n_0}]$. We first assume that $x \in (a_n, b_n]$ for some $n \geq n_0$ and define

$$x' := \inf\left\{y : \left|\widetilde{A}_n \cap [y,x]\right| = 0\right\}.$$

If $x' > a_n$, then one can find $x^* \in \widetilde{A}_n \cap (a_n, x']$ such that $x' - x^* < g_1(a_n)$. Thus

$$\begin{aligned} \frac{|E \cap (0,x)|}{f_2(x)} &\leq \frac{\left|\widetilde{A}_n \cap (a_n, x)\right| + b_{n+1}}{f_2(x)} \leq \frac{\left|\widetilde{A}_n \cap (a_n, x')\right|}{f_2(x^*)} + \frac{b_{n+1}}{f_2(a_n)} \\ &< \frac{\left|\widetilde{A}_n \cap (a_n, x^*)\right| + (x' - x^*)}{n g_1(x^*)} + \frac{1}{n} \leq \frac{t}{n} + \frac{1}{n} + \frac{1}{n} = \frac{t+2}{n}. \end{aligned} \tag{5}$$

If $x' \leq a_n$, since $\left|\widetilde{A}_{n+1} \cap (a_{n+1}, b_{n+1})\right| > 0$, we can apply (5) for $n+1$ and obtain

$$\frac{|E \cap (0,x)|}{f_2(x)} = \frac{|E \cap (0,b_{n+1})|}{f_2(x)} \leq \frac{|E \cap (0,b_{n+1})|}{f_2(b_{n+1})} < \frac{t+2}{n+1}. \tag{6}$$

Assume now that $x \in (b_{n+1}, a_n]$ for some $n \geq n_0$. As in the previous case, we get

$$\frac{|E \cap (0,x)|}{f_2(x)} \leq \frac{|E \cap (0,b_{n+1})|}{f_2(b_{n+1})} < \frac{t+2}{n+1}. \tag{7}$$

From (5)-(7) it follows that $\limsup_{x \to 0+} \frac{|E \cap (0,x)|}{f_2(x)} = 0$, which gives $0 \in \Phi_{f_2}(D)$. □

Corollary 3. *If* $\mathcal{T}_{f_2} \subset \mathcal{T}_{f_1}$, *then* $\lim_{n\to\infty} \varepsilon_n = 0$.

Theorem 2 and Corollary 3 lead to

Theorem 4. $\mathcal{T}_{f_2} \subset \mathcal{T}_{f_1}$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$.

We can use the above theorem to compare the density topology $\mathcal{T}_d$ with others f-density topologies. A topology $\mathcal{T}_f$ is stronger than $\mathcal{T}_d$ if and only if for any positive ε and sufficiently large n, the inequality

$$\frac{\left|\left\{x \in (0,h) : f(x) < \frac{x}{n}\right\}\right|}{f(h)} < \varepsilon$$

holds for sufficiently small h. Analogously, $\mathcal{T}_f \subset \mathcal{T}_d$ if and only if for any $\varepsilon > 0$ and sufficiently large n,

$$\frac{|\{x \in (0,h) : f(x) > nx\}|}{h} < \varepsilon$$

for sufficiently small h.

4 Applications

Using the condition formulated in Theorem 4 we have found answers for some questions and have simplified some difficult proofs. Let $\langle s \rangle = \{s_n\}_{n \in \mathbb{N}}$ be an unbounded, nondecreasing sequence of positive numbers. We say that $x \in \mathbb{R}$ is an $\langle s \rangle$-*density point* of a set $E \in \mathcal{L}$ if

$$\lim_{n\to\infty} \frac{\left|E \cap \left[x - \frac{1}{s_n}, x + \frac{1}{s_n}\right]\right|}{\frac{2}{s_n}} = 1.$$

For any $E \in \mathcal{L}$ we denote by $\Phi_{\langle s \rangle}(E)$ the set of all $\langle s \rangle$-density points E. The family $\mathcal{T}_{\langle s \rangle} := \left\{E \in \mathcal{L} : E \subset \Phi_{\langle s \rangle}(E\right\}$ is a topology called *topology generated by the sequence* $\langle s \rangle$ (see [6]). It is easy to check that $\mathcal{T}_{\langle s \rangle}$ is equal to the topology generated by the function $f_{\langle s \rangle}(x) := \frac{1}{s_n}$ for $x \in \left(\frac{1}{s_{n+1}}, \frac{1}{s_n}\right]$, and that $f_{\langle s \rangle}$ belongs to the family $\mathcal{A}^1 := \left\{f \in \mathcal{A} : \liminf_{x\to 0+} \frac{f(x)}{x} > 0\right\}$ (compare [3]). In [4] it is proved that there are continuum topologies $\mathcal{T}_f$ with $f \in \mathcal{A}_1$ such that $\mathcal{T}_f$ is different from all topologies generated by sequences. However, the construction presented in this paper is rather complicated. Theorem 4 lets us find simpler examples (compare [5]). In [6] a necessary and sufficient condition was given for comparability of density topologies generated by sequences. This condition can be easily obtained from Theorem 4. Finally, in [7] the condition from Theorem 4 is used to compare ψ-density topologies with topologies generated by functions from $\mathcal{A}^0 := \left\{f \in \mathcal{A} : \liminf_{x\to 0+} \frac{f(x)}{x} = 0\right\}$ more precisely than in [1].

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