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A TALE OF TWO (s)-ITIES

Abstract

In the product $X \times Y$ of two uncountable complete separable metric spaces, not every (s)-set belongs to the σ-algebra generated by the products of (s)-sets in X with (s)-sets in Y. The construction makes use of the fact that the Boolean algebra $(s)/(s_0)$ is complete.

It was the best of $\times$, it was the worst of $\times$.

Let X and Y be complete, separable metric spaces. Jack Brown posed the following question: Does every (s)-set in $X \times Y$ belong to the σ-algebra generated by all sets of the form $A \times B$, where $A \subset X$ and $B \subset Y$ are (s)-sets? This question is particularly natural in the light of the recent result of Elalaoui-Talibi [1] that the graph of an (s)-measurable function $f : X \to Y$ does belong to this σ-algebra.

However, the answer to Brown's question is "no." The construction and proof are motivated by John Morgan's theory of Category Bases, as we discuss below.

Thanks to Jack Brown for suggesting the problem and drastically simplifying the solution.

A set M in a complete separable metric space is said to be have property (s) if every perfect set has a perfect subset which is either a subset of M or is disjoint from M. The set M has property (s_0) if every perfect set has a perfect subset which is disjoint from M. These notions were introduced by Szpilrajn-Marczewski in [7], where it is proved, among many other things, that the class of sets having property (s) is a σ-algebra, and the class of sets having property (s_0) is a σ-ideal.

Now let Z be a complete, separable metric space. The Boolean algebra $\mathcal{S}(Z)$ of (s)-sets modulo the (s_0)-sets is defined in the usual manner; for each set $M \subset Z$ with property (s), $[M]$, the equivalence class of M 'mod (s_0),'

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is an element of $\mathcal{S}(Z)$. It is clear from the facts cited above that $\mathcal{S}(Z)$ is a countably complete Boolean algebra. In fact, more is true;

$$\mathcal{S}(Z) \text{ is a complete Boolean algebra.} \tag{cBa}$$

There are two proofs of (cBa) in the literature. In [6], it is proved that there is a topology $\mathcal{T}$ on Z such that the $\mathcal{T}$-Baire property on Z coincides with property (s), and the class of $\mathcal{T}$-meager (= first category) sets in Z coincides with the class of sets with property (s_0). Thus (cBa) follows from the classical Birkhoff-Ulam theorem (see [5, p. 75]): In any topological space, "Baire property modulo meager" is a complete Boolean algebra. A second proof, using the general notion of 'category bases,' is found in [4, p. 37].

We now use this result to show that the Fubini property fails spectacularly for the properties (s) and (s_0). Let X and Y be complete separable metric spaces, and consider the space $X \times Y$. To fix notation, for every $M \subset X \times Y$, $x \in X$ and $y \in Y$, the cross-sections of M are defined by $M_x = M \cap (\{x\} \times Y)$ and $M^y = M \cap (X \times \{y\})$.

Proposition 1. *Let* $\left[\hat{M}\right] = \Sigma_{y \in Y} [X \times \{y\}]$. *(This infinite join is computed in the complete Boolean algebra* $\mathcal{S}(X \times Y)$*). Then* $\hat{M}$ *has property* (s)*, and*

(i) for all $y \in Y$*, the set* $(X \times \{y\}) \setminus \hat{M}^y$ *has property* (s_0)*, and*

(ii) for all $x \in X$*,* $\hat{M}_x$ *has property* (s_0)*.*

In other words, in the sense of properties (s) and (s_0), the cross-sections $\hat{M}^y$ are all nearly full subsets of $(X \times \{y\})$, while the cross-sections $\hat{M}_x$ are all negligible subsets of $\{x\} \times Y$. (The best of $\times$, the worst of $\times$??)

PROOF. To prove 1(i), fix $y \in Y$. Note that by definition of join, $\left[\hat{M}\right] \geq [X \times \{y\}]$, so $\left[\hat{M}^y\right] = \left[\hat{M} \cap (X \times \{y\})\right] = [X \times \{y\}]$. To prove (ii), fix $x \in X$. Note that for all $y \in Y$, the set $(\{x\} \times Y) \cap (X \times \{y\})$ is a singleton and therefore has property (s_0). Now $\left[\hat{M}_x\right] = \left[(\{x\} \times Y) \cap \hat{M}\right] = \Sigma_{y \in Y} [(\{x\} \times Y) \cap (X \times \{y\})]$, and (ii) follows. □

Another characterization of $\hat{M}$ is

(i) For every horizontal cross-section H of $X \times Y$, $H \setminus \hat{M}$ has property (s_0), and

(ii) For any $N \subset X \times Y$, if for every horizontal cross-section H of $X \times Y$, $H \setminus N$ has property (s_0), then $\hat{M} \setminus N$ has (s_0).

Let $\sigma(s(X)\times s(Y))$ denote the smallest σ-algebra containing all sets $A\times B$ where $A\subset X$ and $B\subset Y$ have property (s). Our main goal here is to prove that the (s)-set $\hat{M}$ does not belong to $\sigma(s(X)\times s(Y))$. To this end, we introduce two new properties, (s^2) and (s_0^2).

Define a *perfect rectangle* to be a set of the form $P\times Q$, where $P\subset X$ and $Q\subset Y$ are perfect sets. We say that $M\subset X\times Y$ has property (s^2) if every perfect rectangle has a subset which is a perfect rectangle and is either a subset of M or is disjoint from M. We say that M has property (s_0^2) if every perfect rectangle has a subset which is a perfect rectangle and is disjoint from M.

Proposition 2.

(i) If $A\subset X$ and $B\subset Y$ have property (s), then $A\times B$ has property (s^2).

(ii) The class of (s_0^2)-sets is a σ-ideal on $X\times Y$.

(iii) The class of (s^2)-sets is a σ-algebra on $X\times Y$.

Proof. For (i), let $P\times Q$ be a perfect rectangle. If $P\cap A$ and $Q\cap B$ contain perfect sets P',Q', respectively, then $(P\times Q)\cap(A\times B)\supset P'\times Q'$. On the other hand, if $P\setminus A$ contains the perfect set P' (or $Q\setminus B$ contains the perfect set Q'), then $(P\times Q)\setminus(A\times B)\supset(P'\times Q)$ (or $(P\times Q)\setminus(A\times B)\supset(P\times Q')$, respectively).

For (ii), we use a standard "dyadic schema" argument. Let $A_1,A_2,\ldots$ have property (s_0^2), and let $P\times Q$ be a perfect rectangle. We build recursively two dyadic schemata $\{P_{i_1,\ldots,i_n}\}$ and $\{Q_{i_1,\ldots,i_n}\}$ of perfect sets, such that, for all binary sequences $(i_1,\ldots,i_n)$,

$$\begin{gathered}P_{i_1,\ldots,i_n}\subset P_{i_1,\ldots,i_{n-1}}\subset X,\quad Q_{i_1,\ldots,i_n}\subset Q_{i_1,\ldots,i_{n-1}}\subset Y,\\ P_{i_1,\ldots,i_n}\text{ and }Q_{i_1,\ldots,i_n}\text{ have diameter }<1/n,\\ P_{i_1,\ldots,i_n,0}\cap P_{i_1,\ldots,i_n,1}=Q_{i_1,\ldots,i_n,0}\cap Q_{i_1,\ldots,i_n,1}=\emptyset,\text{ and}\\ (P_{i_1,\ldots,i_n}\times Q_{i_1,\ldots,i_n})\cap A_n=\emptyset.\end{gathered}\tag{1}$$

Indeed, let $P_\emptyset=P$, $Q_\emptyset=Q$. Now suppose that the perfect sets $P_{i_1,\ldots,i_{n-1}}$ and $Q_{i_1,\ldots,i_{n-1}}$ have been constructed so as to satisfy (1). By hypothesis, there exist $P^*\subset P_{i_1,\ldots,i_{n-1}}$ and $Q^*\subset Q_{i_1,\ldots,i_{n-1}}$ such that $(P^*\times Q^*)\cap A_n=\emptyset$. Finally, choose $P_{i_1,\ldots,i_{n-1},0}$ and $P_{i_1,\ldots,i_{n-1},1}$ to be disjoint perfect subsets of P^* each of diameter $<1/n$, and $Q_{i_1,\ldots,i_{n-1},0}$ and $Q_{i_1,\ldots,i_{n-1},1}$ to be disjoint perfect subsets of Q^* each of diameter $<1/n$. Then $P_{i_1,\ldots,i_{n-1},i_n}$ and $Q_{i_1,\ldots,i_{n-1},i_n}$ satisfy (1), as well.

Let $P'=\bigcap_n\bigcup_{i_1,\ldots,i_n}P_{i_1,\ldots,i_n}$ and $Q'=\bigcap_n\bigcup_{i_1,\ldots,i_n}Q_{i_1,\ldots,i_n}$. It is now a routine matter to verify that $(P' \times Q') \subset (P \times Q)$ is a perfect rectangle that does not intersect $\bigcup_n A_n$.

For (iii), it is clear from the symmetry of the definition that the complement of an (s^2)-set is another (s^2)-set. To show that (s^2) is closed under countable unions, let $A_1, A_2, \ldots$ have property (s^2), and let $P \times Q$ be a perfect rectangle. If for some n $A_n \cap (P \times Q)$ has a subset which is a perfect rectangle, then so does $(\bigcup_n A_n) \cap (P \times Q)$. If not, it is not hard to show that for all n, $A_n \cap (P \times Q)$ has property (s_0^2), and so by part (ii) $(\bigcup_n A_n) \cap (P \times Q)$ has property (s_0^2). Thus there exists a perfect rectangle $P' \times Q' \subset (P \times Q) \setminus (\bigcup_n A_n)$, and we are done. □

Corollary 3. *The set $\hat{M}$ has property (s), but is not an element of the σ-algebra $\sigma(s(X) \times s(Y))$.*

Proof. Indeed, by 1(i) no perfect rectangle misses $\hat{M}$, and by 1(ii), no perfect rectangle is a subset of $\hat{M}$. Thus $\hat{M}$ does not have property (s^2). However, it follows from Proposition 2 that the class of (s^2)-sets contains the σ-algebra in question. We are done. □

Remark. The construction of the set $\hat{M}$ depends on the result (cBa), whose proof belongs to John Morgan's theory of Category Bases. (An introduction to the theory may be found in [2], or in [4]. The proof of (cBa) is found only in [4].) A *category base* is a pair $(X, \mathcal{C})$ in which $\mathcal{C}$ is a class of subsets of X, satisfying axioms that guarantee that the classes of $\mathcal{C}$-Baire property sets and $\mathcal{C}$-meager sets (which are defined in the natural way) are well-behaved. For example, every topological space $(X, \mathcal{C})$ is a category base, and the classes of $\mathcal{C}$-Baire property sets and $\mathcal{C}$-meager sets are exactly as usual.

The fact (cBa) and therefore the construction of $\hat{M}$ depend on the following facts (also proved in [2] and [4]): If Z is a complete separable metric space and $\mathcal{P}$ is the class of perfect sets in Z, then $(Z, \mathcal{P})$ is a category base, the $\mathcal{P}$-Baire property coincides with property (s), and the $\mathcal{P}$-meager sets are exactly the sets with property (s_0).

We implicitly considered here another category base here. If X and Y are complete separable metric spaces and $\mathcal{P}^2$ the class of perfect rectangles in $X \times Y$, then $(X \times Y, \mathcal{P}^2)$ is a category base. (This fact is proved in [3].) The present Proposition 2 shows, in the language of category bases, that property (s^2) coincides with the $\mathcal{P}^2$-Baire property, and the sets with property (s_0^2) are exactly the $\mathcal{P}^2$-meager sets.

References

[1] H. Elalaoui-Talibi, *A remark on graphs of Marczewski measurable functions*, Real Anal. Exchange **23**(2) (1997/'98), 813–816.

[2] J. Morgan, *Baire category from an abstract viewpoint*, Fund. Math. **94** (1977), 13–23.

[3] J. Morgan, *On product bases*, Pac. J. Math. **99(1)** (1982), 105–126.

[4] J. Morgan, *Point set theory*, M. Dekker, New York, 1990.

[5] R. Sikorski, *Boolean algebras*, Springer-Verlag, N.Y., 1969.

[6] K. Schilling, *Some category bases which are equivalent to topologies*, Real Anal. Exchange **14**(1) (1988), 210–214.

[7] E. Szprilrajn, *Sur une classe de fonction de M. Sierpinski et la classe correspondante d'ensembles*, Fund. Math. **24** (1935), 17–34.