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THE SET OF CONTINUOUS FUNCTIONS WITH ZERO TOPOLOGICAL ENTROPY

Abstract

Let $I = [0,1]$. We show that those functions in $C(I,I)$ possessing zero topological entropy form a nowhere dense perfect subset of the continuous self maps of the interval. We also show that every function with zero topological entropy that possesses an infinite ω-limit set is the uniform limit of functions having only finite ω-limit sets.

1 Introduction

In the study of chaotic and dynamical systems, those functions possessing zero topological entropy have received considerable attention. Topological entropy, as introduced in [1], was initially used to provide a numerical measure for the complexity of an endomorphism of a compact topological space. The notion has since been extended to provide a way of describing the chaotic behavior of a self map of a compact interval. As the following results of Misiurewicz and Sarkovskii indicate, those functions with zero topological entropy must have relatively benign iterative structures [8], [9], [7].

Theorem 1. *Let $f : [0,1] \to [0,1]$ be continuous. The following conditions are equivalent.*

- *f has positive topological entropy.*
- *For some $x \in I = [0,1]$, the ω-limit set $\omega(x,f)$ is infinite and contains a periodic point.*
- *f has a cycle of order not a power of two.*
- *There are closed intervals J and K in I having at most one point in common, and positive integers m, n such that $J \cup K \subset f^m(J) \cap f^n(K)$.*

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Four years later V. Fedorenko, A. Sarkovskii and J. Smítal published a more extensive list of conditions equivalent to a function having zero topological entropy; the interested reader is referred to Theorem A of [6]. From Theorem 1, one sees that each periodic orbit of f must have order 2^n for some nonnegative integer n whenever f has zero topological entropy. Our next theorem, due to Smítal, sheds considerable light onto the structure of the infinite ω-limit sets of a function with zero topological entropy [10].

Theorem 2. *If ω is an infinite ω-limit set of $f \in C(I,I)$ possessing zero topological entropy, then there exists a sequence of closed intervals $\{J_k\}_{k=1}^{\infty}$ in $[0,1]$ such that*

- *for each $k, \{f^i(J_k)\}_{i=1}^{2^k}$ are pairwise disjoint, and $J_k = f^{2^k}(J_k)$.*
- *for each $k, J_{k+1} \cup f^{2^k}(J_{k+1}) \subset J_k$.*
- *for each $k, \omega \subset \cup_{i=1}^{2^k} f^i(J_k)$.*
- *for each k and $i, \omega \cap f^i(J_k) \neq \emptyset$.*

Given the very specific behavior that functions of zero topological entropy must demonstrate on their infinite ω-limit sets, it may not be too surprising that Bruckner and Smital have been able to characterize these sets [4].

Theorem 3. *An infinite compact set $W \subset (0,1)$ is an ω-limit set of a map $f \in C(I,I)$ with zero topological entropy if and only if $W = Q \cup P$ where Q is a Cantor set and P is empty or countably infinite, disjoint with Q, and satisfies the following conditions:*

- *every interval contiguous to Q contains at most two points of P;*
- *each of the intervals $[0, \min Q), (\max Q, 1]$ contains at most one point of P, and*
- *$\overline{P} = Q \cup P$.*

More recently in [2], Block and Coppel have turned their attention to the structure of the collection $\mathcal{E} = \{f \in C(I,I) : f \text{ has zero topological entropy}\}$ as a subset of $C(I,I)$. We extend their results by first recalling that $\mathcal{E}$ is a nowhere dense perfect subset of $C(I,I)$. We then go on to prove that every function in $\mathcal{E}$ possessing an infinite ω-limit set is the uniform limit of functions possessing only finite ω-limit sets. That $\mathcal{E}$ is a nowhere dense perfect subset of $C(I,I)$ follows from earlier work of Block and Misiurewicz; we present an alternate proof of this fact since it complements the second part of Theorem 4, which answers a specific query of [2].

2 Preliminaries

We write $\mathbf{h}(f) = 0$ in order to indicate that f in $C(I, I)$ possesses zero topological entropy, so that $\mathcal{E} = \{f \in C(I, I) : \mathbf{h}(f) = 0\}$. While we can take I to be any compact interval of the real line, for convenience we set $I = [0, 1]$. We let $\|f - g\| = \sup\{|f(x) - g(x)| : x \in I\}$, and work in the complete metric space $(C(I, I), \| \circ \|)$. Also, let $\mathbb{P}_n$ denote the set of all continuous functions $f : I \to I$ which possess a point of period n, and let $\mathbb{P}_{2^\infty}$ represent those functions with zero topological entropy that possess a point of period 2^n for all natural numbers n. From the Sarkovskii ordering on periodic orbits of continuous self-maps of a compact interval, one sees that $\mathbb{P}_{2^\infty} \subset \cdots \subset \mathbb{P}_{2^{n+1}} \subset \mathbb{P}_{2^n} \subset \cdots \subset \mathbb{P}_4 \subset \mathbb{P}_2 \subset \mathbb{P}_1$.

We make the following definitions with Smítal's Theorem 2 in mind. Let ω be an infinite compact subset of I, and let f map ω into itself. We call f a simple map on ω if ω has a decomposition $S \cup T$ into compact portions that f exchanges, and f^2 is simple on each of these portions. From Smítal's Theorem one sees that every map f with zero topological entropy is simple on each of its infinite ω-limit sets. Let $\{J_k\}_{k=1}^{\infty}$ be a nested sequence of compact periodic intervals of ω and f as described in Smítal's Theorem. Every set of the form $\omega \cap f^i(J_k)$ is periodic of period 2^k, and we call each such set a periodic portion of rank k. This system of periodic portions of ω, or of the corresponding periodic intervals, is called the simple system of ω with respect to f. We now recall a device from [5] that allows us to code the sets $f^i(J_k)$ with finite tuples of zeros and ones. Let $\mathbb{N}$ denote the natural numbers, and take $\mathcal{N}$ to be the set of sequences composed of zeros and ones. If $\mathbf{n} \in \mathcal{N}$ and $\mathbf{n} = \{n_i\}_{i=1}^{\infty}$, we let $\mathbf{n}|k = (n_1, n_2, \ldots, n_k)$. Set $\mathbf{0} = \{0, 0, \ldots\}$ and $\mathbf{1} = \{1, 1, \ldots\}$. Now, define a function $\mathcal{A} : \mathcal{N} \longrightarrow \mathcal{N}$ given by $\mathcal{A}(\mathbf{n}) = \mathbf{n} + 1\mathbf{0}$, where addition is modulus two from left to right. For each $k \in \mathbb{N}$ and $i \in \mathbb{N}$ put $F_{\mathbf{1}|k} = J_k$ and $F_{\mathcal{A}^i(\mathbf{1}|k)} = f^i(J_k)$. Thus, for each $\mathbf{m}$ and $\mathbf{n}$ in $\mathcal{N}$ and $k \in \mathbb{N}$ there is a $j \in \mathbb{N}$ such that $\mathcal{A}^j(\mathbf{m}|k) = \mathbf{n}|k$; the above relations define $F_{\mathbf{n}|k}$ for all $\mathbf{n} \in \mathcal{N}$ and $k \in \mathbb{N}$. Now, set $F_{\mathbf{n}} = \cap_{k=1}^{\infty} F_{\mathbf{n}|k}$, and let $K = \cup_{\mathbf{n} \in \mathcal{N}} \cap_{k=1}^{\infty} F_{\mathbf{n}|k}$. Then K and each $F_{\mathbf{n}}$ are compact, and the components of K consist of the $F_{\mathbf{n}}$ sets. Let G be the component of $[0, 1] - K$ which contains the interval between F_1 and F_0. In general, let $G_{\mathbf{n}|k}$ be that component of $[0, 1] - K$ which contains the interval between $F_{\mathbf{n}|k0}$ and $F_{\mathbf{n}|k1}$. We set $\mathcal{G} = \{G\} \cup \{G_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in \mathbb{N}\}$, $G^0 = G \cup [0, \inf K) \cup (\sup K, 1]$ and $G^k = \cup\{G_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}$. Thus, $[\inf K, \sup K] = K \cup \mathcal{G}$ and $[0, 1] = K \cup (\cup_{j=0}^{\infty} G^j)$.

3 Results

We are now in a position to state precisely as well as prove our main result; this is the content of the following theorem.

Theorem 4. *Let* $\mathcal{E} = \{f \in C(I,I) : \mathbf{h}(f) = 0\}$*. Then*

- $\mathcal{E}$ *is a nowhere dense perfect subset of* $C(I,I)$*.*
- $\overline{\cup_{k\geq 0}(\mathbb{P}_{2^k} \cap \mathcal{E})} = \mathcal{E}$*.*

Proof. We first prove that $\mathcal{E}$ is a nowhere dense closed subset of $C(I,I)$. From [2] we know that the function $\mathbf{h} : C(I,I) \to \mathbb{R}^+ \cup \{+\infty\}$, given by $f \to \mathbf{h}(f)$, is lower semicontinuous. Thus, if $\mathbf{h}(f) > \alpha > 0$, then $\mathbf{h}(g) > \alpha$ for all $g \in C(I,I)$ sufficiently close to f. In particular, the set $\mathcal{S} = \{f \in C(I,I) : \mathbf{h}(f) > 0\}$ is open. Now, suppose that $f \in \mathcal{E}$. Since f must have a fixed point in I, for any $\varepsilon > 0$ we can find g in $C(I,I)$ so that $\|f - g\| < \varepsilon$, yet $\mathbf{h}(g) > 0$. In fact, we can take g to equal f outside of a neighborhood of the fixed point, and define g on a subinterval of that neighborhood so that it is an appropriately scaled copy of the hat map $h : I \to I$ given by $x \longmapsto 2x$ for $x \in [0, \frac{1}{2}]$, and $x \longmapsto 2(1-x)$ for $x \in [\frac{1}{2}, 1]$. To show that $\mathcal{E}$ is perfect in addition to being nowhere dense and closed in $C(I,I)$, we let $\{\varphi_n\}$ be a sequence of homeomorphisms from I to I that converge uniformly to the identity map on I. Let f be an element of $\mathcal{E}$, and set $f_n = \varphi_n f \varphi_n^{-1}$. Since f_n is topologically conjugate to $f \in \mathcal{E}$, one sees that $\mathbf{h}(f_n) = 0$ for each n, too. Moreover, $f_n \to f$ since the uniform limit of $\{\varphi_n\}$ is the identity map.

It remains to show that $\overline{\cup_{k\geq 0}(\mathbb{P}_{2^k} \cap \mathcal{E})} = \mathcal{E}$. To this end, it suffices to show that for any f in $\mathbb{P}_{2^\infty}$, there exists $\{f_n\}$ contained in $\cup_{k\geq 0}(\mathbb{P}_{2^k} \cap \mathcal{E})$ for which $f_n \to f$. Let $f \in \mathbb{P}_{2^\infty}$. We proceed through two cases. First, let us suppose that f is not chaotic in the sense of Li and Yorke, so that $|F_{\mathbf{n}}| = 0$ for all the components $F_{\mathbf{n}}$ in a simple system of f, should the function possess an infinite ω-limit set [3]. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ with the property that $|F_{\mathbf{n}|k}| < \varepsilon$ for all $\mathbf{n}$ in $\mathcal{N}$, whenever $k \geq N$. We let g equal f on $I - (\cup_{\mathbf{n}\in\mathcal{N}} F_{\mathbf{n}|N})$, and extend g linearly on each of the 2^N intervals of the form $F_{\mathbf{n}|N}$. Since $g^{2^N} : F_{\mathbf{n}|N} \to F_{\mathbf{n}|N}$ is linear for each $\mathbf{n}$ in $\mathcal{N}$, it follows that g has periodic points of order no more that 2^{N+1}. In a similar fashion one constructs g a 2^{N+1} function so that $\|f - g\| < \varepsilon$ whenever f is a 2^∞ function that does not possess an infinite ω-limit set.

For our second case, let us suppose that f is chaotic in the sense of Li and Yorke, so that $\operatorname{int} F_{\mathbf{n}} \neq \emptyset$ for some component of the simple system of f. Let $x \in \operatorname{int} F_{\mathbf{n}}$, and take $\varepsilon > 0$. Since f is uniformly continuous, there exists N a natural number such that $|G_{\mathbf{m}|n}| < \varepsilon$ and $|f(G_{\mathbf{m}|n})| < \varepsilon$ whenever $n > N$

for all $\mathbf{m}$ in $\mathcal{N}$. Also, there exists $\mathbf{m}$ in $\mathcal{N}$ and M in $\mathbb{N}$ so that $|F_{\mathbf{m}|n}| < \varepsilon$ and $|F_{\mathcal{A}(\mathbf{m})|n}| < \varepsilon$ for any $n > M$. Let $n > \max\{M, N\}$, and without loss of generality we may presume that $F_{\mathbf{m}|n+2}$ lies between $G_{\mathbf{m}|n}$ and $G_{\mathbf{m}|n+1}$. There exists a $k \in \mathbb{N}$ for which $f^k(x) \in F_{\mathbf{m}|n+2}$, and since $f^{2^{n+2}-k}(\,F_{\mathbf{m}|n+2}) = F_{\mathbf{n}|n+2}$, there is x^* in $F_{\mathbf{m}|n+2}$ such that $f^{2^{n+2}-k}(\,x^*) = x$. We modify f on $F_{\mathbf{m}|n+2}, G^n$ and G^{n+1} to get a function g so that $g \in \mathbb{P}_{2^{n+2}}$, and $\|f-g\| < \varepsilon$. If y is in $I - (F_{\mathbf{m}|n+2} \cup G^n \cup G^{n+1})$, we let $g(y) = f(y)$. If $y \in F_{\mathbf{m}|n+2}$, let $g(y) = f(x^*)$. We extend g in a linear fashion on G^n and G^{n+1} to get $g \in C(I,I)$ for which $\|f-g\| < \varepsilon$. Since $g^k(x) = f^k(x) \in F_{\mathbf{m}|n+2}$, $g(g^k(x)) = f(x^*)$ and $g^{2^{n+2}-(k+1)}(g(g^k(x))) = f^{2^{n+2}-(k+1)}(f(x^*)) = x$, it follows that x has period 2^{n+2}. Since g possesses a unique periodic point of period 2^n in G^n, and a unique periodic point of period 2^{n+1} in G^{n+1}, it follows that g is a 2^{n+2} function. $\square$

References

[1] R. Adler, A. Konheim and M. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.

[2] L. Block and W. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, vol. **1513**, Springer-Verlag, 1991.

[3] A. M. Bruckner and J. G. Ceder, *Chaos in terms of the map $x \longmapsto \omega(x, f)$*, Pac. J. Math. **156** (1992), 63–96.

[4] A. M. Bruckner and J. Smítal, *The structure of ω-limit sets of maps of the interval with zero topological entropy*, Ergod. Th. and Dynam. Sys. **13** (1993), 7–19.

[5] R. L. Devaney, *Chaotic dynamical systems*, Benjamin/Cummings Publ. Co., 1986.

[6] V. Fedorenko, A. Sarkovskii and J. Smítal, *Characterizations of weakly chaotic maps of the interval*, Proc. Amer. Math. Soc. **110** (1990), 141–148.

[7] M. Misiurewicz, *Horseshoe mappings of the interval*, Bull. Acad. Polon. Sci. Ser. Math. **27** (1979), 167–169.

[8] A. N. Sarkovskii, *The behavior of a map in a neighborhood of an attracting set*, Ukrain. Mat Z. **18** (1966), 60–83. (in Russian)

[9] A. N. Sarkovskii, *On cycles and the structure of continuous mappings*, Ukrain. Mat Z. **17** (1965), 104–111. (in Russian)

[10] J. Smítal, *Chaotic functions with zero topological entropy*, Trans. Amer. Math. Soc. **297** (1986), 269–282.