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LIMITS OF TRANSFINITE SEQUENCES OF BAIRE-2 FUNCTIONS

Abstract

It is consistent that CH fails and every function which is the pointwise limit of an ω_2-sequence of Baire-2 functions is Baire-2. It is also consistent that CH fails and there is a function which is not such a limit.

1 Introduction

W. Sierpiński initiated the investigation of pointwise convergent transfinite sequences of Baire-1 functions [4]. It is easy to observe that the convergence of transfinite sequences of reals is somewhat trivial; $x = \lim\{x_\alpha : \alpha < \kappa\}$ holds for some κ of uncountable cofinality iff $x_\alpha = x$ is true for $\alpha < \kappa$ large enough. Sierpiński himself proved that the ω_1-limit of continuous functions is continuous and the ω_1-limit of Baire-1 functions is Baire-1 again. M. Laczkovich pointed out that this no longer holds for Baire-2 functions. Namely, if $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ has cardinality ω_1, then f can be written as $f = \lim\{f_\alpha : \alpha < \omega_1\}$ for some Baire-2 functions by the following argument.

Enumerate A as $A = \{a_\alpha : \alpha < \omega_1\}$ and let $f_\alpha(a_\beta) = f(a_\beta)$ for $\beta < \alpha$, otherwise let f_α be identically 0. Clearly the functions $\{f_\alpha : \alpha < \omega_1\}$ are Baire-2 and their limit is f. As the characteristic function of a non-Borel set can be obtained in this way, (There is always a non-Borel set of cardinal ω_1.) we get that the limits can be functions which are not Baire. Also, if CH (the Continuum Hypothesis) holds, then every function is the ω_1-limit of Baire-2 functions. In Theorem 1 we show that if the cofinality of 2^ω (continuum

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cardinality) is not ω_1, then there is a real function which is not the limit of Baire-2 functions. If $\mathrm{cf}(2^\omega) = \omega_1$, then both possibilities may occur.

Laczkovich also asked what happens if we are interested in ω_2 limits of Baire-2 functions. He remarked that in this case there is no problem if CH is assumed as, then every convergent sequence of functions eventually stabilizes. We show that if the continuum is ω_2, then both cases may occur; that is, it is consistent that every real function is the ω_2 limit of Baire-2 functions, it is also consistent that only Baire-2 functions can be so obtained.

2 Notation

We use the standard axiomatic set theory notation. Specifically, cardinals are identified with initial ordinals. 2^ω denotes the least ordinal of cardinality continuum, therefore, if we well order a set of cardinal continuum into ordinal 2^ω, then in that ordering every element is preceded by less than continuum many elements.

When we force with a partial order $(P, \leq)$, $G \subseteq P$ is generic, and τ is some P-name, then we let τ^G be the realization of τ.

For a set A of ordinals we let $F(A)$ be the notion of forcing adding Cohen reals for the elements of A. That is, $p \in F(A)$ iff p is a function with a domain that is a finite subset of $A \times \omega$ and range that is $\subseteq \{0, 1\}$. $p \leq q$ iff p extends q as a function. If $G \subseteq F(A)$ is a generic subset, then we define the Cohen reals as follows; for $\alpha \in A$ let $c_\alpha : \omega \to \{0, 1\}$ be the function satisfying $c_\alpha(n) = p\big((\alpha, n)\big)$ for some $p \in G$. (Standard forcing facts give that c_α is a totally defined function.) We notice that if $A \subseteq B$, then the inclusion $F(A) \subseteq F(B)$ is an order preserving inclusion.

If $A, A' \subseteq B$ are disjoint sets of ordinals, $\pi : A \to A'$ is a bijection, then π can be lifted to an isomorphism $\overline{\pi} : F(B) \to F(B)$ as follows. $\overline{\pi}(p(\alpha, n)) = p(\alpha, n)$ if $\alpha \notin A \cup A'$, $\overline{\pi}(p(\pi(\alpha), n)) = p(\alpha, n)$ if $\alpha \in A$, $\overline{\pi}\big(p(\pi^{-1}(\alpha), n)\big) = p(\alpha, n)$ if $\alpha \in A'$.

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3 The Results

Theorem 1. *If* $\mathrm{cf}(2^\omega) > \omega_1$*, then there is a real function which is not the pointwise limit of an* ω_1*-sequence of Baire-2 functions.*

PROOF. Enumerate $\mathbb{R}$ as $\mathbb{R} = \{r_\alpha : \alpha < 2^\omega\}$. Enumerate also the Baire-2 (or even Borel) functions as $\{f_\alpha : \alpha < 2^\omega\}$. Construct $f : \mathbb{R} \to \mathbb{R}$ in such a way that $f(r_\alpha)$ is different from the less than continuum many values $\{f_\beta(r_\alpha) : \beta < \alpha\}$. We claim that f is as required. Assume not, and so f is the pointwise limit of some functions $\{f_{\gamma_\tau} : \tau < \omega_1\}$. As $\mathrm{cf}(2^\omega) > \omega_1$ there is an $\alpha < 2^\omega$ with $\sup\{\gamma_\tau : \tau < \omega_1\} < \alpha$ and by the way f was constructed $f(r_\alpha)$ is different from all the values $\{f_{\gamma_\tau}(r_\alpha) : \tau < \omega_1\}$; so f is not the limit of those functions. □

Theorem 2. *It is consistent that* $2^\omega = \omega_{\omega_1}$ *and there is a real function which is not the pointwise limit of an* ω_1*-sequence of Baire-2 functions.*

PROOF. Let V be a model of CH and let the poset $(P, \leq)$ add ω_{ω_1} Cohen reals, $\{c_\alpha : \alpha < \omega_{\omega_1}\}$. If $G \subseteq P$ is generic, let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(c_{2\alpha}) = c_{2\alpha+1}$ for $\alpha < \omega_{\omega_1}$. We claim that f is not the ω_1-limit of Baire-2 functions. Assume it is, $f = \lim\{f_\gamma : \gamma < \omega_1\}$. As f_γ is a Baire-2 function there is a real number in $V[G]$ from which it can be defined. There is a countable set $A_\gamma \subseteq \omega_{\omega_1}$ such that this real is an element of the model $V[G \cap F(A_\gamma)]$. Set $A = \bigcup\{A_\gamma : \gamma < \omega_1\}$. Select $\alpha < \omega_{\omega_1}$ such that $2\alpha + 1 \notin A$. Then $f(c_{2\alpha})$ is an element of $V\big[G \cap F\big(A \cup \{2\alpha\}\big)\big]$ which contradicts the standard forcing theory fact $c_{2\alpha+1} \notin V\big[G \cap F\big(A \cup \{2\alpha\}\big)\big]$. □

Theorem 3. *It is consistent that* $2^\omega = \omega_{\omega_1}$ *and every real function is the pointwise limit of an* ω_1*-sequence of Baire-2 functions.*

PROOF. Let V be a model of GCH (the Generalized Continuum Hypothesis). We are going to construct a finite support iterated forcing of length ω_1, $\{P_\alpha : \alpha \leq \omega_1\}$. Assume that we have constructed P_α, $2^\omega = \omega_{\alpha+1}$ in V^{P_α}, and GCH holds above $\kappa = \omega_{\alpha+1}$. Let $\mathbb{R}_\alpha$ be the set of reals in V^{P_α}. Enumerate, in V^{P_α}, all subsets of $\mathbb{R}_\alpha$ as $\{X_\xi : \xi < \kappa^+\}$. Let R_ξ be a ccc forcing of cardinality κ making X_ξ a relative F_σ subset of $\mathbb{R}_\alpha$ (see [1,3]). (Notice that after the first step $\mathbb{R}_\alpha$ will cease being the set of all reals.) Let Q_α be the finite support iteration of these posets. Notice that if $X = H \cap \mathbb{R}_\alpha$ (with H an F_σ set) is once achieved, then it will survive later extensions even though we have to redefine H (but not $\mathbb{R}_\alpha$). As Q_α is the iteration of ccc posets, it is ccc as well. $|Q_\alpha| = \kappa^+$; so the number of reals in V^{P_α} is $\kappa(\kappa^+)^\omega = \kappa^+ = \omega_{\alpha+1}$ and we can continue the definition.

Our final model is V^P with $P = P_{\omega_1}$. It suffices to show that if $f : \mathbb{R} \to [0,1]$ is a function in V^P, then it is the limit of an ω_1-sequence of Baire-2 functions. We first show this for two-valued functions; that is, for $f : \mathbb{R} \to \{0,1\}$. As in the intermediate model $V[G \cap P_\alpha]$ the set $\mathbb{R}_\alpha$ of all reals has cardinality $\omega_{\alpha+1}$ we can find an enumeration of the set of reals in the final model as $\mathbb{R} = \{r_\xi : \xi < \omega_{\omega_1}\}$ such that $\mathbb{R}_\alpha = \{r_\xi : \xi < \omega_{\alpha+1}\}$, this part of enumeration is in $V[G \cap P_\alpha]$, and $\mathbb{R}_\alpha \setminus \bigcup\{\mathbb{R}_\beta : \beta < \alpha\}$ is mapped onto the ordinal interval $[\omega_\alpha, \omega_{\alpha+1})$.

Fix a name τ for f. For every $\xi < \omega_{\omega_1}$ choose a maximal antichain $\{p_i^\xi : i < \omega\} \subseteq P$ of conditions determining the value of $f(r_\xi)$. (This antichain is countable as $(P, \leq)$ is a ccc forcing.) Pick an ordinal $\alpha(\xi) < \omega_1$ such that $\xi < \omega_{\alpha(\xi)+1}$ and also $\{p_i^\xi : i < \omega\} \subseteq P_{\alpha(\xi)}$. Then r_ξ and $f(r_\xi)$ are determined in $V[G \cap P_{\alpha(\xi)}]$.

We now define the functions $\{f_\alpha : \alpha < \omega_1\}$ as follows. The domain of f_α is the set $\{r_\xi : \alpha(\xi) \leq \alpha\}$ and $f_\alpha(r_\xi) = 0$ (or 1) if the unique $p_i^\xi \in G$ forces that value. The function f_α is in $V[G \cap P_\alpha]$ and the forcing Q_α will make the set $f_\alpha^{-1}(0)$ a relative F_σ subset of $\mathbb{R}_\alpha$. Then f_α is the restriction of a Baire-2 function to $\mathbb{R}_\alpha$ and so f is the limit of Baire-2 functions.

Having proved the result for two-valued functions let $f : \mathbb{R} \to [0,1]$ be an arbitrary function in $V[G]$. So f can be written as $f(x) = g_1(x) + g_2(x) + \cdots$ with $g_n(x) \in \{0, 2^{-n}\}$. As g_n is a two-valued function it can be written as $g_n = \lim\{g_\alpha^n : \alpha < \omega_1\}$ where the functions $\{g_\alpha^n : \alpha < \omega_1\}$ are Baire-2 functions. Now $f_\alpha = \sum\{g_\alpha^n : 1 \leq n < \omega\}$ is a Baire-2 function as it is the uniform limit of Baire-2 functions. And finally, $f = \lim\{f_\alpha : \alpha < \omega_1\}$. $\square$

Theorem 4. *It is consistent with $2^\omega = \omega_2$ that every real function is the pointwise limit of an ω_2-sequence of Baire-2 functions.*

PROOF. We deduce the statement from the axiom MA_{ω_1} and $2^\omega = \omega_2$. Assume that $f : \mathbb{R} \to [0,1]$ and enumerate $\mathbb{R}$ as $\{r_\alpha : \alpha < \omega_2\}$. A well known corollary of MA_{ω_1} is that in every set of reals of cardinality at most ω_1 every subset is a relative F_σ set (i.e., every set of cardinality at most ω_1 is a Q-set, see [1]). With the argument as in the proof of Theorem 3 we can find a Baire-2 function f_α which agrees with f on $\{r_\beta : \beta < \alpha\}$ for every $\alpha < \omega_2$; so f is the limit of the f_α's. $\square$

Theorem 5. *It is consistent that the pointwise limit of an ω_2-sequence of Baire-2 functions is Baire-2 again.*

PROOF. We add ω_2 Cohen reals to a model of CH. Let $P = F(\omega_2)$ be the applied notion of forcing, $V[G]$ the enlarged model and $\{c_\alpha : \alpha < \omega_2\}$ the

Cohen reals. Assume that $\mathbf{1}_P$ forces that $\{f_\alpha : \alpha < \omega_2\}$ is a set of Baire-2 functions converging to $f : \mathbb{R} \to \mathbb{R}$.

For every $\alpha < \omega_2$ there is a countable set $A_\alpha \subseteq \omega_2$ such that the behavior of f_α is completely determined by the restriction of G to A_α. Every function f_α can be written as $f_\alpha = \lim_m \lim_n g^\alpha_{m,n}$, with $g^\alpha_{m,n}$ continuous, and let, for q, q' rational numbers, $\{p(\alpha, m, n, q, q', i) : i < \omega\}$ be a maximal antichain of conditions determining the truth value of the statement $g^\alpha_{m,n}(q) < q'$.

By shrinking the index set and using the Δ-system lemma (p. 49 in [2]) we can assume that our sets form a Δ-system; that is, $A_\alpha = A \cup B_\alpha$ with the sets $\{A, B_\alpha : \alpha < \omega_2\}$ disjoint. We can also assume that $A = \emptyset$ (by passing to the model $V[G \cap F(A)]$). Using CH again and again shrinking the index set we can also assume that the above structures on the sets B_α are isomorphic. This means that if $\alpha < \beta < \omega_2$ are given, then the isomorphism of the ordered sets $\pi : (B_\alpha, <) \to (B_\beta, <)$ naturally extends to an isomorphism π' between the parts of P with supports in B_α and B_β, respectively such that $\pi'\big(p(\alpha, m, n, q, q', i)\big) = p(\beta, m, n, q, q', i)$ holds for all values of m, n, q, q', and i.

If $x \in V[G]$ is a real, then there is a countable set $T(x) \subseteq \omega_2$ such that x is determined in $V[G \cap F(T(x))]$. By the disjointness assumption the set $d(x) = \{\alpha < \omega_2 : T(x) \cap B_\alpha \neq \emptyset\}$ is countable. We claim that if α, $\beta \notin d(x)$, then $f_\alpha(x) = f_\beta(x)$. In any case, the value of $f_\alpha(x)$ is determined in the model $V[G \cap F(T(x) \cup B_\alpha)]$ while the value of $f_\beta(x)$ is likewise determined in the model $V[G \cap F(T(x) \cup B_\beta)]$. This implies that the status of $f_\alpha(x) = f_\beta(x)$ is determined in $V[G \cap F(T(x) \cup B_\alpha \cup B_\beta)]$. Assume that our claim fails and so $p \Vdash f_\alpha(x) \neq f_\beta(x)$ for some condition $p \in F\big(T(X) \cup B_\alpha \cup B_\beta\big)$. If we now select α', β' in such a way that $B_{\alpha'}$, $B_{\beta'}$ are disjoint from $T(x)$, and $\{\alpha', \beta'\} \cap \{\alpha, \beta\} = \emptyset$, then there is an automorphism $\pi : P \to P$ which is the identity on $P|T(x)$ and carries B_α to $B_{\alpha'}$, B_β to $B_{\beta'}$, $\pi(p)$ is compatible with p. As the structures are isomorphic $\pi(p) \Vdash f_{\alpha'}(x) \neq f_{\beta'}(x)$. This way, working in $V[G \cap F(T(x)]$, we can find ω_2 such pairs $\big\{\{\alpha'_\xi, \beta'_\xi\} : \xi < \omega_2\big\}$ with the corresponding isomorphisms

$$\pi_\xi : F\big(T(x) \cup B_\alpha \cup B_\beta\big) \to F\big(T(x) \cup B_{\alpha_\xi} \cup B_{\beta_\xi}\big).$$

Then

$$p_\xi = \pi_\xi(p) \Vdash f_{\alpha_\xi}(x) \neq f_{\beta_\xi}(x).$$

If we show that ω_2 of conditions p_ξ are in G, then we get that $f_\alpha(x)$ does not stabilize in $V[G]$, and so we reach a contradiction. So assume that some $q \leq p$ forces that $\big\{\xi < \omega_2 : p_\xi \in G\big\}$ is of cardinal $\leq \omega_1$. We can as well assume that q forces that $\sup\big\{\xi < \omega_2 : p_\xi \in G\big\} = \gamma$ for some $\gamma < \omega_2$. Then there is some

$\xi > \gamma$ such that p_ξ is compatible with q and so a common extension forces a contradiction.

We now make a *further* extension of $V[G]$ by adding countably many (to be more exact, $|B_\alpha|$ many for any $\alpha < \omega_2$) Cohen reals. This makes it possible to construct a further Baire-2 function, f_{ω_2} in the following way. Let the index set of the extra Cohen reals be $B_{\omega_2} = [\omega_2, \omega_2 + \nu)$ where $\nu = |B_\alpha|$ (any α) is either ω or some natural number. We define f_{ω_2} as the B_{ω_2} counterpart of any f_α. That is, choose some $\alpha < \omega_2$, set $\pi : B_\alpha \to B_{\omega_2}$ a bijection. Let π' be the corresponding isomorphism between the parts of P with supports in B_α and B_{ω_2}. Define $f_{\omega_2} = \lim_m \lim_n g^{\omega_2}_{m,n}$ where the continuous functions $g^{\omega_2}_{m,n}$ are determined by the conditions $p(\omega_2, m, n, q, q', i) = \pi'\bigl(p(\alpha, m, n, q, q', i)\bigr)$ for the suitable values of m, n, q, q', i.

Using our previous claim, if $x \in V[G]$, $\alpha \notin d(x)$, then $f_\alpha(x) = f_{\omega_2}(x)$. That is, our function $f \in V[G]$ is extended to a Baire-2 function in the further extension. We show that then f is already Baire-2 in $V[G]$ (and this concludes the proof). It is well known that a function is Baire-2 if and only if all the level sets are of the form $\bigcap_i \bigcup_j F_{i,j}$ for some closed sets $F_{i,j}$; so it suffices to show the following claim.

Assume that V is a model of set theory, $X \subseteq \mathbb{R}$, P is a countable notion of forcing, and in V^P there is a set $H = \bigcap_i \bigcup_j F_{i,j}$ with $F_{i,j}$ closed, such that $X = H \cap \mathbb{R}^V$. Then there is such a set already in V.

Assume that $\mathbf{1}_P$ forces that H, F_{ij} satisfy the requirements. We argue that $X = \bigl\{x : \forall p \forall i \exists p' \le p \exists j, p' \Vdash x \in F_{i,j}\bigr\}$. Indeed, if $x \in X$, then $1 \Vdash x \in H$; so for every $p \in P$ and $i < \omega$ there are some $p' \le p$ and $j < \omega$ that $p' \Vdash x \in F_{i,j}$. On the other hand, if $x \notin X$, then there are $p \in P$ and $i < \omega$ that $p \Vdash x \notin F_{i,j}$. But then no $p' \le p$ can force with some $j < \omega$ that $x \in F_{ij}$.

Having proved the above formula for X as the indicated unions and intersections are countable, we only need to show that the sets $\{x : p' \Vdash x \in F_{i,j}\}$ are closed (for fixed p', i, j). Indeed, if $x_n \to x$ and $p' \Vdash x_n \in F_{i,j}$ for every n then if $G \subseteq P$ is some generic set with $p' \in G$, then in $V[G]$ the convergence $x_n \to x$ still holds, and F_{ij} is a closed set containing every x_n, containing therefore x as well. That is, p' forces $x \in F_{i,j}$. $\square$

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