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WEAK TYPE INEQUALITY FOR LOGARITHMIC MEANS OF WALSH-KACZMARZ-FOURIER SERIES

Abstract

The main aim of this paper is to prove that the Nörlund logarithmic means $t_n^{\kappa} f$ of one-dimensional Walsh-Kaczmarz-Fourier series is weak type (1,1), and this fact implies that $t_n^{\kappa} f$ converges in measure on I for every function $f \in L(I)$ and $t_{n,m}^{\kappa} f$ converges in measure on I^2 for every function $f \in L \ln^+ L(I^2)$.

Moreover, the maximal Orlich space such that Nörlund logarithmic means of two-dimensional Walsh-Kaczmarz-Fourier series for the functions from this space converge in two-dimensional measure is found.

1 Introduction.

In 1948 Šneider [18] showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^{\kappa}(x)}{\log n} \geq C > 0$$

holds a.e. for the Walsh-Kaczmarz Dirichlet kernel. This inequality shows that the behavior of the Walsh-Kaczmarz system is worse than the behavior of the Walsh system in the Paley enumeration. This "spreadness" property of

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the kernel makes it easier to construct examples of divergent Fourier series [1]. On the other hand, Schipp [13] and Young [20] in 1974 proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [17] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous function f. For any integrable function Gát [2] proved that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. Recently, Gát's result was generalized by Simon [15, 16].

The partial sums $S_n^w(f)$ of the Walsh-Fourier series of a function $f \in L(I),\ I = [0,1)$ converge in measure on I [5]. The condition $f \in L\ln^+ L(I^2)$ provides convergence in measure on I^2 of the rectangular partial sums $S_{n,m}^w(f)$ of double Walsh-Fourier series [21]. The first example of a function from classes wider than $L\ln^+ L(I^2)$ with $S_{n,n}^w(f)$ divergent in measure on I^2 was obtained in [4, 10]. Moreover, in [19] Tkebuchava proved that in each Orlicz space wider than $L\ln^+ L(I^2)$ the set of functions which quadratic Walsh-Fourier sums converge in measure on I^2 is of first Baire category (see Goginava [8] for Walsh-Kaczmarz series).

The main aim of this paper is to prove that the Nörlund logarithmic means $t_n^\kappa f$ of one-dimensional Walsh-Kaczmarz-Fourier series is weak type (1,1), and this fact implies that $t_n^\kappa f$ converges in measure on I for every function $f \in L(I)$ and $t_{n,m}^\kappa f$ converges in measure on I^2 for every function $f \in L\ln^+ L(I^2)$. On the other hand, the logarithmic means $t_{n,m}^\kappa f$ of the double Fourier series with respect to Walsh-Kaczmarz system does not improve the convergence in measure. In particular, we prove that for any Orlicz space, which is not a subspace of $L\ln^+ L(I^2)$, the set of the functions that quadratic logarithmic means of the double Fourier series with respect to the Walsh-Kaczmarz system converge in measure is of first Baire category.

At last, we note that the Walsh-Nörlund logarithmic means are closer to the partial sums than to the classical logarithmic means or the Fejér means. Namely, it was proved that there exists a function in a certain class of functions and a set with positive measure, such that the Walsh-Nörlund logarithmic means of the function diverge on the set [3].

2 Definitions and Notations.

We denote the set of non-negative integers by $\mathbf{N}$.

By a dyadic interval in $I := [0,1)$ we mean one of the form $\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right)$ for some $p \in \mathbf{N}$, $0 \le p < 2^n$. Given $n \in \mathbf{N}$ and $x \in [0,1)$, let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains the point x.

Every point $x \in I$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} := (x_0, x_1, ..., x_n, ...), \quad x_k \in \{0, 1\}.$$

In the case when there are two different forms we choose the one for which $\lim\limits_{k\to\infty} x_k = 0$.

Denote

$$e_j := \frac{1}{2^{j+1}} = (0, ..., 0, x_j = 1, 0, ...).$$

It is well-know that [5]

$$I_n(x_0, ..., x_{n-1}) := I_n(x) = \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right),$$

where

$$p = \sum_{j=0}^{n-1} x_j 2^{n-1-j}.$$

We denote by $L^0 = L^0(I^2)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $I^2 = [0, 1) \times [0, 1)$. $\mu(A)$ is the Lebesgue measure of the set $A \subset I^2$. The constants appearing in this article are denoted by c.

Let $L_\Phi = L_\Phi(I^2)$ be the Orlicz space [11] generated by Young function Φ; i.e. Φ is a convex, continuous, even function such that $\Phi(0) = 0$ and

$$\lim_{u\to+\infty} \frac{\Phi(u)}{u} = +\infty, \quad \lim_{u\to 0} \frac{\Phi(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_\Phi(I^2)} = \inf\left\{k > 0 : \int_{I^2} \Phi(|f(x,y)|/k)\, dx\, dy \le 1\right\}.$$

In particular, if $\Phi(u) = u\ln(1+u),\ u > 0$, then the corresponding space will be denoted by $L\ln L(I^2)$.

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2), \\ -1, & \text{if } x \in [1/2, 1), \end{cases} \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \ge 0 \text{ and } x \in [0, 1).$$

Let $w_0, w_1, \ldots$ represent the Walsh functions; i.e. $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s \geq 0$ then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{n_1}(x) \prod_{k=0}^{n_1-1} (r_{n_1-1-k}(x))^{n_k}.$$

For $A \in \mathbf{N}$ and $x \in I$ define the transformation $\tau_A : I \to I$ by

$$\tau_A(x) := \sum_{k=0}^{A-1} x_{A-k-1} 2^{-(k+1)} + \sum_{j=A}^{\infty} x_j 2^{-(j+1)}.$$

By the definition of τ_A we have (see [17])

$$\kappa_n(x) = r_{n_1}(x) w_{n-2^{n_1}}(\tau_{n_1}(x)) \quad (n \in \mathbf{N}, x \in I).$$

The Dirichlet kernels are defined by

$$D_n^\alpha(x) := \sum_{k=0}^{n-1} \alpha_k(x),$$

where $\alpha_k = w_k$ or κ_k.

It is well-known that [5, 17]

$$D_n^\kappa(x) = D_{2^{n_1}}(x) + w_{2^{n_1}}(x) D_{n-2^{n_1}}^w(\tau_{n_1}(x)), \tag{1}$$

and

$$D_n^w(x) = D_{2^{n_1}}(x) + w_{2^{n_1}}(x) D_{n-2^{n_1}}^w(x). \tag{2}$$

Recall that

$$D_{2^n}(x) := D_{2^n}^w(x) = D_{2^n}^\kappa(x) = \begin{cases} 2^n, & \text{if } x \in [0, 1/2^n), \\ 0, & \text{if } x \in [1/2^n, 1). \end{cases} \tag{3}$$

The Fejér means of the Walsh-(Kaczmarz-)Fourier series of function f is given by the equality

$$\sigma_n^\alpha(f, x) := \frac{1}{n} \sum_{j=0}^{n} S_j^\alpha(f, x),$$

where

$$S_j^{\alpha}(f,x)=\sum_{k=0}^{n-1}\hat{f}^{\alpha}(k)\alpha_k(x).$$

$\hat{f}^{\alpha}(n) := \int\limits_I f\alpha_n$ $(n\in\mathbf{N})$ is said to be the nth Walsh-(Kaczmarz-)Fourier coefficient of f.

The Nörlund logarithmic (simply we say logarithmic) means and kernels of one dimensional Walsh-(Kaczmarz-)Fourier series are defined as follows

$$t_n^{\alpha}(f,x)=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k^{\alpha}(f,x)}{n-k},\quad F_n^{\alpha}(t)=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k^{\alpha}(t)}{n-k},$$

where

$$l_n=\sum_{k=1}^{n-1}\frac{1}{k}.$$

The Kronecker product $(\alpha_{m,n}: n,m\in\ \mathbf{N})$ of two Walsh(-Kaczmarz) systems is said to be the two-dimensional Walsh(-Kaczmarz) system. Thus,

$$\alpha_{m,n}(x,y)=\alpha_m(x)\,\alpha_n(y).$$

If $f\in L\left(I^2\right)$, then the number $\hat{f}^{\alpha}(m,n) := \int\limits_{I^2} f\alpha_{m,n}$ $(n,m\in\mathbf{N})$ is said to be the (m,n)th Walsh-(Kaczmarz-)Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh(-Kaczmarz) system are defined by

$$S_{m,n}^{\alpha}(f,x,y)=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\hat{f}^{\alpha}(i,j)\,\alpha_i(x)\alpha_j(y).$$

The logarithmic means of double Walsh-(Kaczmarz-)Fourier series is defined as follows

$$t_{n,m}^{\alpha}(f,x,y)=\frac{1}{l_nl_m}\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\frac{S_{i,j}^{\alpha}(f,x,y)}{(n-i)(m-j)}.$$

It is evident that

$$t_{n,m}^{\alpha}(f,x,y)-f(x,y)=\int_0^1[f(x\oplus t,y\oplus s)-f(x,y)]F_n^{\alpha}(t)\,F_m^{\alpha}(s)\ dt\ ds,$$

where $\oplus$ denotes the dyadic addition [14].

3 Main Results.

The main results of this paper are presented in the following propositions.

Theorem 1. *Let $\lambda > 0$ and $f \in L(I)$. Then*

$$\lambda\mu\left\{x \in I : \left|t_n^{\kappa}(f,x)\right| > \lambda\right\} \le c\left\|f\right\|_1,$$

and c is an absolute constant independent of n and f.

Corollary 1. *Let $0 < p < 1$. Then for $f \in L\ln^+ L(I^2)$*

a)

$$\left(\int_{I^2}\left|t_{n,m}^{\kappa}(f,x,y)\right|^p dxdy\right)^{1/p} \le c_p\int_{I^2}\left|f(x,y)\right|\ln^+\left|f(x,y)\right|dxdy + c_p$$

b)

$$\int_{I^2}\left|t_{n,m}^{\kappa}(f,x,y) - f(x,y)\right|^p dxdy \to 0 \text{ as } n,m\to\infty.$$

Corollary 2. *Let $f \in L\ln^+ L(I^2)$. Then*

a)

$$\mu\left\{(x,y)\in I^2 : \left|t_{n,m}^{\kappa}(f,x,y)\right| > \lambda\right\} \le \frac{c}{\lambda}\int_{I^2}\left|f(x,y)\right|\ln^+\left|f(x,y)\right|dxdy + c$$

b)

$$\left|t_{n,m}^{\kappa}(f,x,y) - f(x,y)\right| \to 0 \text{ in measure on } I^2, \text{ as } n,m\to\infty.$$

Classical regular summation methods often improve the convergence of Walsh-Fourier series. For instance, the Fejér means $\sigma_{n,m}^{w}f$ of the two-dimensional Walsh-Fourier series of the function $f \in L(I)$ converge in $L(I)$ norm to the function f, as $n,m\to\infty$. In [7] the method of Nörlund logarithmic means $t_{n,m}^{w}f$ was investigated, which is weaker than the Cesàro method of any positive order and it was proved that the class $L\ln^+ L(I^2)$ provides convergence in measure of logarithmic means of two-dimensional Walsh-Fourier series. It was also proved ([6]) that in each Orlicz space wider than $L\ln^{+}L(I^2)$ the set of functions which quadratic Walsh-Fourier sums converge in measure on I^2 is of first Baire category.

Now, we show that the logarithmic means $t^{\kappa}_{n,m}f$ of the double Fourier series with respect to the Walsh-Kaczmarz system does not improve the convergence in measure. In particular, we prove the following theorem

Theorem 2. *Let $L_{\Phi}(I^2)$ be an Orlicz space, such that*

$$L_{\Phi}(I^2) \nsubseteq L\ln L^{+}(I^2).$$

Then the set of the functions from the Orlicz space $L_{\Phi}(I^2)$ with quadratic logarithmic means of the Fourier series with respect to the Walsh-Kaczmarz system converge in measure on I^2 is of first Baire category in $L_{\Phi}(I^2)$.

Corollary 3. *Let $\varphi : [0,\infty[\to [0,\infty[$ be a nondecreasing function satisfying the condition*

$$\varphi(x) = o(x\log x)$$

for $x \to +\infty$. Then there exists a function $f \in L(I^2)$ such that

a)

$$\int_{I^2} \varphi(|f(x,y)|)\ dx\ dy < \infty;$$

b) the quadratic logarithmic means of the Walsh-Kaczmarz-Fourier series of f diverges in measure on I^2.

4 Auxiliary Results.

It is well-known [5, 14] for the Dirichlet kernel function that

$$|D^{w}_{n}(x)| < \frac{1}{x}$$

for any $0 < x < 1$. Then for these x's we also get

$$|F^{w}_{n}(x)| < \frac{1}{x},$$

where $n \in \mathbf{N}$ is a nonnegative integer. The following lower bound is also well-known for the Walsh-Paley-Dirichlet kernel functions. Let $p_A = 2^{2A} + \cdots + 2^2 + 2^0$ ($A \in \mathbf{N}$). Then for any $2^{-2A-1} \le x < 1$ and $A \in \mathbf{N}$ we have

$$|D^w_{p_A}(x)| \ge \frac{1}{4x}.$$

This inequality plays a prominent role in the proofs of some divergence results concerning the partial sums of the Fourier series. Then it seems that it would be useful to get a similar inequality also for the logarithmic kernels. In [6] the first author, Gát and Tkebuchava proved the inequality

$$|F^w_{p_A}(x)| \ge c\frac{\log(1/x)}{x\log p_A}$$

for all $1 \le A \in \mathbf{N}$, but not for every x in the interval $(0,1)$. We have an exceptional set, such that it is "rare around zero". For $t = t_0, t_0+1, \ldots, 2A, t_0 = \inf\{t : \left\lfloor \frac{l_{p_{[t/2]-1}}}{16} - 2^{15} \right\rfloor > 1\}$ set $\tilde{t} := \left\lfloor \frac{l_{p_{[t/2]-1}}}{16} - 2^{15} \right\rfloor$ (where $\lfloor u \rfloor$ denotes the lower integral part of u), and we take a "small part" of the interval $I_t \setminus I_{t+1} = [2^{-t-1}, 2^{-t})$. This way we define the intervals

$$J_t := \left[\frac{1}{2^{t+1}}, \frac{1}{2^{t+1}} + \frac{1}{2^{t+\tilde{t}}}\right).$$

We define the exceptional set as:

$$J := \bigcup_{t=t_0}^{\infty} J_t.$$

The following are proved:

Lemma 1 (Gát, Goginava, Tkebuchava [6]). *For $x \in \left(2^{-2A-1}, 1\right) \setminus J$ we have*

$$|F^w_{p_A}(x)| \ge c\frac{\log(1/x)}{x\log p_A}.$$

Corollary 4 (Gát, Goginava, Tkebuchava [6]). *For $x \in \left(2^{-2A-1}, 2^{-A}\right) \setminus J$ we have the estimation $|F^w_{p_A}(x)| \ge \frac{c}{x}$.*

Lemma 2 (Gát, Goginava, Tkebuchava [6]). *Let L_Φ be an Orlicz space and let $\varphi : [0,\infty) \to [0,\infty)$ be a measurable function with condition $\varphi(x) = o(\Phi(x))$ as $x \to \infty$. Then there exists an Orlicz space L_ω, such that $\omega(x) = o(\Phi(x))$ as $x \to \infty$, and $\omega(x) \ge \varphi(x)$ for $x \ge c \ge 0$.*

Now, for the Walsh-Kaczmarz logarithmic kernels we will prove the following:

Lemma 3. *Let* $x \in I_{2A}(1, x_1, \ldots, x_{t-1}, 1, 1, 0, \ldots, 0) =: I_{2A}^t, t = 2, 3, \ldots, A$. *Then*

$$\left|F_{p_A}^{\kappa}(x)\right| \geq cA2^{2A-t}.$$

Proof. Set $x \in I_{2A}^t$. Let

$$G_{p_A}^{\alpha}(x) := l_{p_A} F_{p_A}^{\alpha}(x)$$

for $\alpha = w$ or κ. Thus, we have

$$G_{p_A}^{\kappa}(x) = \sum_{j=1}^{2^{2A}} \frac{D_j^{\kappa}(x)}{p_A - j} + \sum_{j=2^{2A}+1}^{p_A - 1} \frac{D_j^{\kappa}(x)}{p_A - j} := I + II. \tag{4}$$

First, by the help of (1) we discuss II.

$$II = \sum_{j=1}^{p_{A-1}-1} \frac{D_{j+2^{2A}}^{\kappa}(x)}{p_{A-1} - j} = l_{p_{A-1}} D_{2^{2A}}(x) + r_{2A}(x)\, G_{p_{A-1}}^{w}(\tau_{2A}(x)).$$

If $x \in I_{2A}^t$, then (see (3))

$$D_{2^{2A}}(x) = 0$$

and

$$\tau_{2A}(x) = (0, \ldots, 0, 1, 1, x_{t-1}, \ldots, x_1, x_0 = 1, x_{2A}, \ldots).$$

Moreover, by Lemma 1 we have

$$|II| = \left|G_{p_{A-1}}^{w}(\tau_{2A}(x))\right| \geq c\,(2A - t)\, 2^{2A-t}. \tag{5}$$

Now, we discuss I. We use the equation (1)

$$\begin{aligned} I &= \sum_{l=0}^{2A-1} \sum_{j=2^l}^{2^{l+1}-1} \frac{D_j^{\kappa}(x)}{p_A - j} + \frac{D_{2^{2A}}(x)}{p_{A-1}} \\ &= \sum_{l=0}^{2A-1} \sum_{j=0}^{2^l - 1} \frac{D_{j+2^l}^{\kappa}(x)}{p_A - j - 2^l} + \frac{D_{2^{2A}}(x)}{p_{A-1}} \\ &= \sum_{l=0}^{2A-1} \sum_{j=0}^{2^l - 1} \frac{D_{2^l}(x) + r_l(x)\, D_j^{w}(\tau_l(x))}{p_A - j - 2^l}. \end{aligned}$$

Since, $x_0 = 1$, $D_{2^l}(x) = 0$ for all $l \geq 1$. Thus,

$$\begin{aligned} I =& \frac{1}{p_A - 1} + \sum_{l=1}^{2A-1} \sum_{j=0}^{2^l-1} \frac{r_l(x) D_j^w(\tau_l(x))}{p_A - j - 2^l} \\ =:& \frac{1}{p_A - 1} + \sum_{l=1}^{2A-1} I_l. \end{aligned} \tag{6}$$

We use Abel's transformation for $I_l \, (l \geq 1)$

$$\begin{aligned} I_l = r_l(x) \sum_{j=1}^{2^l-2} \left(\frac{1}{p_A - j - 2^l} - \frac{1}{p_A - j - 2^l - 1} \right) j K_j^w(\tau_l(x)) \\ + \frac{r_l(x) \left(2^l - 1\right) K_{2^l-1}^w(\tau_l(x))}{p_A - 2^{l+1} + 1} =: I_l^1 + I_l^2, \end{aligned}$$

$$\left| I_l^1 \right| \leq \frac{c}{2^{4A}} \sum_{j=1}^{2^l-1} j \left| K_j^w(\tau_l(x)) \right|.$$

Since, ([14])

$$n \left| K_n^w(x) \right| \leq 2 \sum_{j=0}^{m-1} 2^j \sum_{i=j}^{m-1} D_{2^i}(x + e_j) \text{ for } 2^{m-1} \leq n < 2^m, \tag{7}$$

and for I_l^1 we can write

$$\begin{aligned} \left| I_l^1 \right| &\leq \frac{c}{2^{4A}} \sum_{m=1}^{l} \sum_{j=2^{m-1}}^{2^m-1} j \left| K_j^w(\tau_l(x)) \right| \\ &\leq \frac{c}{2^{4A}} \sum_{m=1}^{l} 2^m \left(\sum_{s=0}^{m-1} 2^s \sum_{q=s}^{m-1} D_{2^q}(\tau_l(x) + e_s) \right) \\ &\leq \frac{c 2^l}{2^{4A}} \sum_{s=0}^{l-1} 2^s \sum_{q=s}^{l-1} D_{2^q}(\tau_l(x) + e_s). \end{aligned}$$

Since, $D_{2^n} \leq 2^n$ and $t \leq A$ we obtain that

$$\sum_{l=0}^{t+2} \left| I_l^1 \right| \leq \frac{c}{2^{4A}} \sum_{l=0}^{t+2} 2^{3l} \leq \frac{c 2^{3t}}{2^{4A}} < c. \tag{8}$$

By the inequality (7) we obtain again

$$\left|I_l^2\right| \leq \frac{c}{2^{2A}} \sum_{s=0}^{l-1} 2^s \sum_{q=s}^{l-1} D_{2^q}\left(\tau_l(x)+e_s\right)$$

and

$$\sum_{l=0}^{t+2} \left|I_l^2\right| \leq \frac{c}{2^{2A}} \sum_{l=0}^{t+2} 2^{2l} \leq \frac{c2^{2t}}{2^{2A}} \leq c. \tag{9}$$

Let $t+2<l<2A$. Then we have

$$\tau_l(x) = \left(0,\ldots,0,1,1,x_{t-1},\ldots,x_1,1,0,\ldots,0,x_{2A},\ldots\right).$$

Hence,

$$D_{2^q}\left(\tau_l(x)+e_s\right) = \begin{cases} 0, & \text{if } s \geq l-t \text{ or } 0 \leq s \leq l-t-1, q>s, \\ 2^s, & \text{if } 0 \leq s \leq l-t-1, q=s, \end{cases}$$

so we can write

$$\begin{aligned} \sum_{l=t+3}^{2A-1} \left|I_l^1\right| &\leq \frac{c}{2^{4A}} \sum_{l=t+3}^{2A-1} 2^l \sum_{s=0}^{l-t-1} 2^{2s} \\ &\leq \frac{c}{2^{4A}} \sum_{l=t+3}^{2A-1} 2^{3l-2t} \leq \frac{c}{2^{4A}} 2^{6A-2t} < c2^{2A-t}, \end{aligned} \tag{10}$$

$$\sum_{l=t+3}^{2A-1} \left|I_l^2\right| \leq \frac{c}{2^{2A}} \sum_{l=t+3}^{2A-1} \sum_{s=0}^{l-t-1} 2^{2s} \leq c2^{2A-2t} < c2^{2A-t}. \tag{11}$$

Combining (4)-(11) we complete the proof of Lemma 3. □

During the proof of Theorem 1 we will use the following Lemma:

Lemma 4 (Gát, Goginava, Tkebuchava [7])**.** *Let $\lambda>0$ and $f \in L^1(I)$. Then*

$$\lambda\mu\{x \in I : |t_n^w(f,x)| > \lambda\} \leq c\|f\|_1,$$

where c is an absolute constant independent of n and f.

5 Proofs of the Theorems.

Proof of Theorem 1. Define the maximal function f^* by

$$f^* := \sup_{n\in\mathbf{P}} |S_{2^n} f|.$$

It is well-known that f^* is of weak type (1,1). During the proof of Theorem 1 we will use the equation (1) and

$$D^{\kappa}_{2^A-j}(x) = D_{2^A}(x) - \omega_{2^A-1}(x) D^{\omega}_j(\tau_{A-1}(x)), \quad j = 0,1,...,2^{A-1} \tag{12}$$

(see [12]).

For $n \in \mathbf{P}$ set $n_1 := A \in \mathbf{N}$ (that is, $2^A \leq n < 2^{A+1}$). To prove Theorem 1 we decompose the kernel F^{κ}_n in the following way:

$$\begin{aligned} l_n F^{\kappa}_n &= \sum_{k=1}^{n-1} \frac{D^{\kappa}_k}{n-k} = \sum_{k=1}^{2^{A-1}-1} \frac{D^{\kappa}_k}{n-k} + \sum_{k=2^{A-1}}^{2^A-1} \frac{D^{\kappa}_k}{n-k} + \sum_{k=2^A}^{n-1} \frac{D^{\kappa}_k}{n-k} \\ &=: \ l_n(F^{\kappa,1}_n + F^{\kappa,2}_n + F^{\kappa,3}_n). \end{aligned}$$

First, we discuss $f * F^{\kappa,1}_n$. The equation (1) and Abel's transformation immediately give

$$\begin{aligned} l_n F^{\kappa,1}_n &= \sum_{k=0}^{A-2} \sum_{l=2^k}^{2^{k+1}-1} \frac{D^{\kappa}_l}{n-l} = \sum_{k=0}^{A-2} \sum_{l=0}^{2^k-1} \frac{D^{\kappa}_{2^k+l}}{n-2^k-l} \\ &= \sum_{k=0}^{A-2} D_{2^k} \sum_{l=0}^{2^k-1} \frac{1}{n-2^k-l} + \sum_{k=0}^{A-2} \sum_{l=0}^{2^k-1} \frac{r_k D^w_l \circ \tau_k}{n-2^k-l} \\ &= \sum_{k=0}^{A-2} D_{2^k} (l_{n-2^k+1} - l_{n-2^{k+1}+1}) \\ &\quad + \sum_{k=0}^{A-2} \sum_{l=0}^{2^k-2} \left(\frac{1}{n-2^k-l} - \frac{1}{n-2^k-l-1} \right) r_k l K^w_l \circ \tau_k \\ &\quad + \sum_{k=0}^{A-2} \frac{2^k-1}{n-2^{k+1}+1} r_k K^w_{2^k-1} \circ \tau_k \\ &=: \ l_n(F^{\kappa,1,1}_n + F^{\kappa,1,2}_n + F^{\kappa,1,3}_n). \end{aligned}$$

This means that

$$|f * F^{\kappa,1,1}_n| \leq c f^*. \tag{13}$$

The equation (see [5])

$$\|f * (r_k K_l^w \circ \tau_k)\|_1 \leq \|f\|_1 \|r_k K_l^w \circ \tau_k\|_1 \leq \|f\|_1 \|K_l^w\|_1 \leq c\|f\|_1$$

immediately gives

$$\|f * F_n^{\kappa,1,2}\|_1 \leq \frac{c\|f\|_1}{l_n} \left(\sum_{k=0}^{A-2} \sum_{l=0}^{2^k-2} \frac{1}{n-2^k-l} \right) \leq c\|f\|_1 \tag{14}$$

and

$$\|f * F_n^{\kappa,1,3}\|_1 \leq \frac{c\|f\|_1}{l_n} \sum_{k=0}^{A-2} \frac{2^k-1}{n-2^{k+1}+1} \leq c\|f\|_1. \tag{15}$$

Second, to discuss $f * F_n^{\kappa,2}$ we use equation (12).

$$\begin{aligned} l_n F_n^{\kappa,2} &= \sum_{l=1}^{2^{A-1}} \frac{D^{\kappa}_{2^A-l}}{n-2^A+l} \\ &= \sum_{l=1}^{2^{A-1}} \frac{D_{2^A}}{n-2^A+l} - \sum_{l=1}^{2^{A-1}} \frac{w_{2^A-1} D_l^w \circ \tau_{A-1}}{n-2^A+l} \\ &=: l_n(F_n^{\kappa,2,1} - F_n^{\kappa,2,2}). \end{aligned}$$

This means that

$$|f * F_n^{\kappa,2,1}| \leq cf^*. \tag{16}$$

Abel's transformation yields

$$\begin{aligned} l_n F_n^{\kappa,2,2} = w_{2^A-1} &\sum_{l=1}^{2^{A-1}-1} \left(\frac{1}{n-2^A+l} - \frac{1}{n-2^A+l+1} \right) l K_l^w \circ \tau_{A-1} \\ &+ \frac{w_{2^A-1} 2^{A-1}}{n-2^{A-1}} K^w_{2^{A-1}} \circ \tau_{A-1}. \end{aligned}$$

The equation (see [5])

$$\|f * (w_{2^A-1} K_l^w \circ \tau_{A-1})\|_1 \leq \|f\|_1 \|w_{2^A-1} K_l^w \circ \tau_{A-1}\|_1 \leq \|f\|_1 \|K_l^w\|_1 \leq c\|f\|_1$$

gives again

$$\|f * F_n^{\kappa,2,2}\|_1 \leq \frac{c\|f\|_1}{l_n} \left(\sum_{l=1}^{2^{A-1}} \frac{1}{n-2^A+l} + 1 \right) \leq c\|f\|_1. \tag{17}$$

At last, we discuss $f * F_n^{\kappa,3}$. The equation (1) implies

$$l_n F_n^{\kappa,3} = \sum_{k=0}^{n-2^A-1} \frac{D^{\kappa}_{2^A+k}}{n-2^A-k} = l_{n-2^A} D_{2^A} + r_A l_{n-2^A} F^w_{n-2^A} \circ \tau_A.$$

$$|f * \frac{l_{n-2^A}}{l_n} D_{2^A}| \leq cf^* \tag{18}$$

means that we have to discuss $t'_{n-2^A}(f,x) := (f * (r_A F^w_{n-2^A} \circ \tau_A))(x)$. The transformation $\tau_A : I \to I$ is measure-preserving and such that $\tau_A(\tau_A(x)) = x$ (that is, $\tau_A^{-1} = \tau_A$) for all $x \in I$ [17]. Thus, Theorem 39.C in [9] allows us to write

$$\begin{aligned} t'_{n-2^A}(f,x) &= \int_I f(x \oplus y) r_A(y) F^w_{n-2^A}(\tau_A(y)) dy \\ &= \int_I f(x \oplus \tau_A(y)) r_A(\tau_A(y)) F^w_{n-2^A}(y) d\tau_A(y) \\ &= \int_I f(x \oplus \tau_A(y)) r_A(\tau_A(y)) F^w_{n-2^A}(y) \frac{d\tau_A(y)}{dy} dy. \end{aligned}$$

Theorem 32.B in [9] and the fact that the transformation $\tau_A : I \to I$ is measure-preserving give for the Radon-Nikodym derivative $\frac{d\tau_A(y)}{dy}$ that $\frac{d\tau_A(y)}{dy} = 1$ almost everywhere. Thus,

$$t'_{n-2^A}(f,x) = \int_I f(x \oplus \tau_A(y)) r_A(y) F^w_{n-2^A}(y) dy$$

and

$$\begin{aligned} t'_{n-2^A}(f,\tau_A(x)) &= r_A(x) \int_I f(\tau_A(x \oplus y)) r_A(x \oplus y) F^w_{n-2^A}(y) dy \\ &= r_A(x)((r_A f \circ \tau_A) * F^w_{n-2^A})(x) = r_A(x) t_{n-2^A}(r_A f \circ \tau_A, x). \end{aligned}$$

Now, by the help of Lemma 4 we show that the operator t'_{n-2^A} is of weak type (1,1).

$$\begin{aligned} \lambda\mu\{x \in I : |t'_{n-2^A}(f,x)| > \lambda\} &= \lambda\mu\{x \in I : |t'_{n-2^A}(f,\tau_A(x))| > \lambda\} \\ &= \lambda\mu\{x \in I : |r_A(x) t_{n-2^A}(r_A(f \circ \tau_A), x)| > \lambda\} \\ &\leq c\|r_A(f \circ \tau_A)\|_1 \leq c\|f\|_1. \end{aligned} \tag{19}$$

Summarising our results on (13)-(19) we could complete the proof of Theorem 1. □

The proof of Corollary 1 and 2 follow from Theorem 1 in the same way as it was done in [7].

Now, we will prove Theorem 2.

Proof of Theorem 2. The proof of Theorem 2 will be complete if we show that there exists $c>0$ such that (for more details see the proof of Theorem 1 from [6])

$$\mu\{(x,y)\in I^2:\left|t^{\kappa}_{p_A,p_A}\left(D_{2^{2A+1}}\otimes D_{2^{2A+1}},x,y\right)\right|>2^{3A}\}>c\frac{A}{2^{3A}}. \tag{20}$$

Denote

$$\Omega_A:=\bigcup_{l=A+2}^{2A-2}\bigcup_{s=A+2}^{2A-2}I^{2A-l}_{2A}\times I^{2A-s}_{2A}.$$

Since,

$$t^{\kappa}_{p_A}\left(D_{2^{2A+1}},x\right)=S_{2^{2A+1}}\left(F^{\kappa}_{p_A},x\right)=F^{\kappa}_{p_A}(x)$$

for $(x,y)\in I^{2A-l}_{2A}\times I^{2A-s}_{2A}$ we have the following estimation from Lemma 3 for quadratic logarithmic means of the function $D_{2^{2A+1}}(x)\,D_{2^{2A+1}}(y)$

$$\left|F^{\kappa}_{p_A}(x)\,F^{\kappa}_{p_A}(y)\right|=\left|t^{\kappa}_{p_A,p_A}\left(D_{2^{2A+1}}\otimes D_{2^{2A+1}},x,y\right)\right|\geq c2^{l+s}.$$

Consequently,

$$\begin{aligned}\mu&\left\{(x,y)\in I^2:\left|t^{\kappa}_{p_A,p_A}\left(D_{2^{2A+1}}\otimes D_{2^{2A+1}},x,y\right)\right|\geq c2^{3A}\right\}\\&\geq c\sum_{l=A+2}^{2A-2}\sum_{s=3A-l}^{2A-2}\frac{2^{2A-l}2^{2A-s}}{2^{4A}}\geq\frac{cA}{2^{3A}}.\end{aligned}$$

Hence, (20) is proved and the proof of Theorem 2 is complete. □

The validity of Corollary 3 follows immediately from Theorem 2 and Lemma 2.

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