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PONCELET PAIRS AND THE TWIST MAP ASSOCIATED TO THE PONCELET BILLIARD

Abstract

Consider a fixed differentiable curve K (which is the boundary of a convex domain) and a family indexed by $\lambda \in [0,1]$ of differentiable curves $L = L_\lambda$ such that they are the boundary of convex (connected) domains A_λ. Suppose that for $\lambda_1 < \lambda_2$ we have $A_{\lambda_1} \subset A_{\lambda_2}$. Then, the number of n-Poncelet pairs is given by $\frac{e(n)}{2}$, where $e(n)$ is the number of natural numbers m smaller than n and which satisfies $\text{mcd}(m,n) = 1$. The curve K does not have to be part of the family.

In order to show this result we consider an associated billiard transformation and a twist map which preserves area. We use Aubry-Mather theory and the rotation number of invariant curves to obtain our main result. In the last section we estimate the derivative of the rotation number of a general twist map using some properties of the continued fraction expansion.

1 Introduction – The Poncelet Billiard.

All results here are for a fixed differentiable C^∞ curve K (which is the boundary of a convex domain) and a family indexed by $\lambda \in [0,1]$ of differentiable C^∞

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curves $L = L_\lambda$ such that they are the boundary of strictly convex domains A_λ (a family of connected sets). Suppose that for $\lambda_1 < \lambda_2$ we have $A_{\lambda_1} \subset A_{\lambda_2}$.

We will consider here the real (not complex) billiard. We point out that the hypothesis of differentiability C^∞ (plus strict convexity) of the curve is enough to define the associated twist map in class C^∞ (a key ingredient in the proofs) [16] (sections 9.2 and 9.3), [7] (section 3.4), [3].

Some explicit results (not for the Poncelet counting problem) for billiards, in the case the curves are ellipses, were obtained in [5]. Our reasoning applies for a more general family of curves. We refer the reader to [6] and [4] for some applications of the Poncelet billiard problem.

In order to simplify the exposition we will refer to circles instead of curves. Some computations are explicit for circles but our reasoning applies to the setting we just described above. The estimates of Proposition 3 below are not contained in previous results which analyze the Poncelet problem.

The main obstructions for a general theory (not just the circle or ellipse) for the Poncelet's problem is to be well defined by the Poncelet billiard map. There are several examples of where it is defined. Once it is defined, all main results described here are good (i.e. there exists an invariant natural probability, or at least the map decreases the natural probability). In our general results this should be implicit.

Let K be a circle as in figure 1, and also let L be another circle interior to K with A and A' as variable points in K. Consider the Poncelet transformation associated with such a pair of circles K and L and the corresponding image of the point A being A', and then the image of A' is A'' and so on (see figure 1). A nice description of the problem appears in [17] and [18].

T, C, and o are shown in figure 2. Here R is the radius of the circle K and C is the center of K. We will use the variables θ and φ to describe the point A and its future hit as in figure 1 and 2. The point A' has coordinates θ' and φ'. The variable points B and B' on the x-axis are also described in figure 2. Denote $\theta' = G_1(\theta, \varphi)$ and $\varphi' = G_2(\theta, \varphi)$. The transformation $G = (G_1, G_2)$ can be consider also as a transformation of $(A, r) \to (A', r')$, where r and r' are tangent lines to the circles as in figure 1. Note that once A is fixed, if we take a variable line r, this means we are considering different circles L. The circle K is considered fixed in our setting. The reader can find several references about the Poncelet Billiard in [1], [5], [6], [4], [9], [11], [13], [14], [15], [18].

We say K, L is an n-Poncelet pair if there exists an A such that the successive iteration of the procedure described above, after n steps, returns to A. The existence of an n-periodic point (A, r) for G (for a certain pair of circles K, L) is equivalent to the existence of a n-Poncelet pair. We would like to count the number of possible Poncelet pairs of order n.

It is easy to obtain the analytic expression

$$G(\theta, \varphi) = (2\,\varphi - \theta + \pi,\, 3\,\varphi - 2\theta - B(\theta') + \pi),$$

where $B(\theta') = 2\ \arctan(\frac{C\ \sin(\theta')}{R+C\cos(\theta')})$.

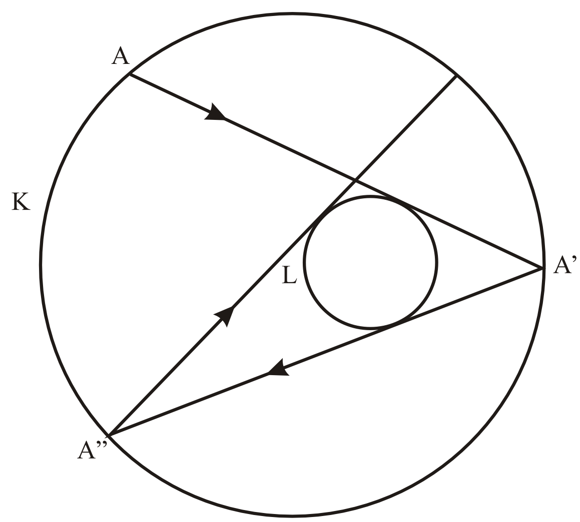

Figure 1: Poncelet's Billiard

We know that r is a periodic function of φ of period π. We change coordinates to get $x = \frac{\theta}{2\,\pi}$ and $y = \frac{\varphi}{\pi}$. Therefore, denoting by f the transformation G in the new coordinates

$$(x', y') = f(x, y) = (\,y\ - x\ +\ 1/2,\, 3\,y - 4x + Z(y - x + 1/2\,)\ +\ 1\,),$$

where $Z(s) = -B(2\pi s)$. The function Z is periodic of period 1. Therefore $f = (f_1, f_2)$ is a twist map on the torus (see [2],[5],[9] for references), because

$$\frac{\partial f_1(x, y)}{\partial y} = \frac{\partial x'}{\partial y} = 1 > 0,$$

and, also preserves area

$$\mathrm{Jac}\, D\, f \;=\; \mathrm{Det}\begin{pmatrix} -1 & 1 \\ -4 - Z'(y-x+1/2) & 3 + Z'(y-x+1/2) \end{pmatrix} = 1.$$

One can write $y = x' + x - 1/2$, and therefore the generating potential $h(x, x')$ for such f is

$$h(x, x') \;=\; -xx' - \frac{x^2 - x}{2} + \frac{3\,(x')^2 - x'}{2} + H(x'),$$

where H is such that $H'(x') = Z(x')$. The twist map is not exact. After this brief introduction we will present in the next section a sequence of results that will be used in the last section to show our main theorem:

Theorem 1. *For a fixed circle K and for a family of concentric circles L, the number of n-Poncelet pairs is $\frac{e(n)}{2}$, where $e(n)$ is the number of natural numbers m smaller than n and which satisfies* $\mathrm{mcd}(m, n) = 1$.

The general problem for conics is the following: given a conic K and a pencil of conics $\mathcal{L}$, obtain the number of conics $L \in \mathcal{L}$ such that K, L is an n-Poncelet pair. This problem was solved in [14] (see also [2] for an enriched version) for the complex plane for a generic pencil but under the condition that $K \in \mathcal{L}$. In our analysis we have that K is not in $\mathcal{L}$.

Consider a family $g_t : \mathbb{R} \to \mathbb{R}$, where $a \le t \le b$, of monotonous increasing homeomorphisms, such that

$$g_t(x+1) = g_t(x) \,+\, 1,\; x \in \mathbb{R},\; a \le t \le b.$$

From the identification $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$, from the family g_t, we obtain another one $f_t : S^1 \to S^1$, where $a \le t \le b$, of homeomorphisms of the circle which preserve orientation.

We know that the following function is well-defined:

$$r(t) \;=\; \lim_{k\to\infty} \frac{g_t^k(x) - x}{k},\; a \le t \le b,$$

and the value $r(t)$ indeed is independent of x. We are interested in properties of the function $r(t)$ and of the rotation number function $\rho(t) = r(t) \pmod 1 \in \frac{\mathbb{R}}{\mathbb{Z}}$.

In the last section we show the following two propositions:

Proposition 2. *The function $r : [a, b] \to \mathbb{R}$ is continuous and monotonically increasing. Moreover, if $a \le t_1 < t_2 \le b$ and $r(t_1) \notin \mathbb{Q}$ or $r(t_2) \notin \mathbb{Q}$, then $r(t_1) < r(t_2)$.*

Proposition 3. *There exists a set of full measure $X \subset \mathbb{R}$ (that is, $\mathbb{R} - X$ has Lebesgue measure zero) such that if $\tau \in (a, b)$ and $r(\tau) \in X$, then*

$$\limsup_{t_2 \to \tau^-,\, t_1 \to \tau^+} \frac{r(t_2) - r(t_1)}{(t_2 - t_1)^2} \geq \frac{m^2}{e^{2F}(1 + e^{2F})^2}.$$

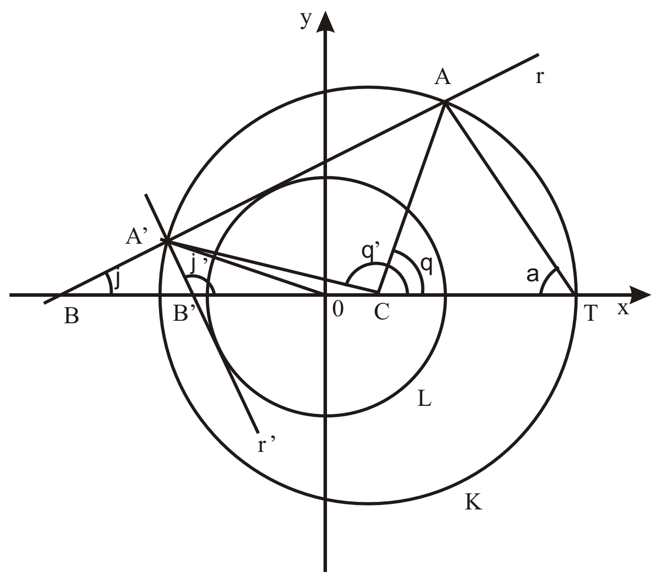

Figure 2: Coordinates

2 Results for Twist Maps.

Consider $f : S^1 \to S^1$, where $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$, a circle homeomorphism which preserves orientation. Denote by $g : \mathbb{R} \to \mathbb{R}$ a lifting of f. Denote by $r(g)$ the limit

$$\lim_{k \to \infty} \frac{g^k(x) - x}{k} = r(g),$$

which is independent of $x \in \mathbb{R}$ (see [12] chapter 11). The rotation number of f, denoted by $\rho(f)$, is the number $r(g)$ (mod 1) $\in S^1$. General references for the rotation number and Twist Maps are [16], [8], [12].

Lemma 4. *For f and its lifting g:*

a) $\rho(f)$ is an invariant of conjugation (by a homeomorphism which preserves orientation) for f,

b) if $\rho(f) = \frac{p}{q}$ (mod 1), where $q > 0$ and p, q are relatively prime, then there exists an x_0 such that $g^q(x_0) = x_0 + p$ (see [9]), and the orbit of x_0 (mod 1) is periodic of period exactly q.

The following Lemma is also well known:

Lemma 5. *Consider $g, h : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing such that $g(x+1) = g(x) + 1$ and $h(x+1) = h(x) + 1$, $\forall x \in \mathbb{R}$.*

Then,

a) If $h(x) \leq g(x)$, $\forall x \in \mathbb{R}$, then $r(h) \leq r(g)$.

b) $g \to r(g)$ is continuous in the C^0 topology.

c) If $h(x) < g(x)$, $\forall x \in \mathbb{R}$, and if $r(h)$ or $r(g)$ is irrational, then $r(h) < r(g)$.

Now, consider the cylinder $X = S^1 \times \mathbb{R} = \frac{\mathbb{R}}{\mathbb{Z}} \times \mathbb{R}$ and the natural projection $\pi : \mathbb{R}^2 \to X$.

Definition 6. The vector field u over the cylinder X is called the ascendent vector field if u is such that $u(a, b)$ equals the tangent vector to the curve $t \to (a, t + b)$, in the point $t = 0$.

We denote by $\pi_1 : X \to S^1$ and $\pi_2 : X \to \mathbb{R}$ the natural projections. We will consider here diffeomorphisms of class C^1, $f : X \to X$, such that

a) f preserves orientation and the ends of X,

b) f preserves area of a Riemannian metric, and

c) for each point $x \in X$ the vectors u and $df_x(u)$ are independent and coherent with the natural orientation.

Consider $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (F_1(x, y), F_2(x, y))$, a lifting of such f. Then F satisfies the conditions

1. $F_1(x + 1, y) = F_1(x, y) + 1$,

2. $F_2(x+1, y) = F_2(x, y)$,
3. for each x, we have $\lim_{y\to\infty} F_2(x, y) = \infty$,
4. F preserves area of a Riemannian metric which is invariant by translation $(x, y) \to (x + 1, y)$,
5. $\frac{\partial F_1}{\partial y}(x, y) > 0$ (the twist condition).

We point out that by condition 5 we have that given (x_0, y_0) and (x_0, y_1) in $\mathbb{R}^2$, such that, $y_1 > y_0$, then $F_1(x_0, y_1) > F_1(x_0, y_0)$.

Definition 7. Consider $\phi : \mathbb{R} \to \mathbb{R}$, a continuous function of period 1, and the curve $\Gamma \subset X$ with parametrization $x \to \pi(x, \phi(x))$, $0 \le x \le 1$. We say Γ is an invariant rotational circle for f if $f(\Gamma) = \Gamma$.

We point out that Γ is an oriented circle and $\pi^{-1}(\Gamma)$ is the graph of ϕ. Moreover $f : \Gamma \to \Gamma$ preserves the orientation of Γ. We denote by $\rho(\Gamma)$ the rotation number of $\rho(f|\Gamma)$. The transformation F also preserves the graph of ϕ and its orientation (we refer the reader to section 9.3 in [16]).

The usual definition of invariant rotational circle is more general ([7] chapter 3, definition 11) but a theorem due to Birkhoff ([7], 3.1) shows that, under quite general conditions, this definition coincides with our definition 7. The rotational invariant curve we are going to consider here (a subset of the torus) is a set of positions q on the circle K and respective tangent line tangent to the small circle L (passing through q, p) as shown in figure 3.

Consider a g associated to Γ by $g(x) = F_1(x, \phi(x)), \forall x \in \mathbb{R}$. The function g is continuous, strictly increasing and satisfies

$$g(x + 1) = F_1(x + 1, \phi(x + 1)) = F_1(x + 1, \phi(x)) = g(x) + 1.$$

We claim that

$$\rho(\Gamma) = r(g)\,(\mathrm{mod} 1).$$

Indeed, by definition $\pi_1|\Gamma : \Gamma \to S^1$ is a homeomorphism which preserves orientation. Consider $\varphi = (\pi_1|\Gamma) \circ (f|\Gamma) \circ (\pi_1|\Gamma)^{-1}$. Then φ preserves orientation and $\rho(\Gamma) = \rho(\varphi)$.

By the other side

$$\pi_1 \circ \pi(x, 0) = x\,(\mathrm{mod} 1), \forall x \in \mathbb{R}.$$

As

$$\varphi(\pi_1(\pi(x, 0))) = \varphi(\pi_1(\pi(x, \phi(x)))) =$$
$$\pi_1(f(\pi(x, \phi(x)))) = \pi_1(\pi(F(x, \phi(x)))) = \pi_1 \circ \pi(g(x), 0),$$

then g is a lifting of φ. Then, $\rho(\varphi) = r(g)\,(\mathrm{mod}\ 1)$ and the claim is true.

Theorem 8. *Consider Γ_1 and Γ_2, two rotational invariant circles associated respectively to ϕ_1 and ϕ_2. Suppose $\phi_1(x) < \phi_2(x), \forall x \in \mathbb{R}$. Denote*

$$g_1(x) = F_1(x, \phi_1(x)), \ \ g_2(x) = F_1(x, \phi_2(x)), \ \forall x \in \mathbb{R}.$$

Then,

$$r(g_1) < r(g_2).$$

Proof. Following [7] 3.3, from $\phi_1 < \phi_2$ we have that $g_1(x) < g_2(x) \forall x \in \mathbb{R}$. Therefore, $r(g_1) \le r(g_2)$. If $r(g_1)$ or $r(g_2)$ is irrational, then the claim is true (Lemma 5).

Suppose that $r(g_1) = r(g_2) = \frac{p}{q}$, where $q > 0$ and p and q are relatively prime. Consider the translation $T_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_n(x, y) = (x + n, y)$ and the diffeomorphism $G = T_{-p} \circ F^q : \mathbb{R}^2 \to \mathbb{R}^2$. We will show that $G(A)$, for a certain region A, is strictly inside the set A and we will get a contradiction with the fact that G preserves area.

We denote $m' = G(m)$, $m \in \mathbb{R}^2$. Note that G preserves the oriented graphs of ϕ_1 and ϕ_2. By Lemma 4 b) there exists $x_1, x_2 \in \mathbb{R}$ such that

$$g_1^q(x_1) = x_1 + p, \ g_2^q(x_2) = x_2 + p.$$

Without loss of generality suppose $x_2 > x_1$.

We denote

$$M_1 = (x_1, \phi_1(x_1)), M_2 = (x_2, \phi_2(x_2)), M_3 = (x_1, \phi_2(x_1)), M_4 = (x_2, \phi_1(x_1)).$$

As

$$F^q(x, \phi_1(x)) = (g^q(x), \phi_1(g^q(x)) = (x + p, \phi_1(x + p)) = (x + p, \phi_1(x)),$$

then $M_1' = G(M_1) = G(x_1, \phi_1(x)) = (x_1, \phi_1(x)) = M_1$. In the same way $M_2' = M_2$.

By the twist condition, M_3' follows M_3 in the orientation of the graph of ϕ_2 (see figure 4). As M_2 is fixed, then M_3' precedes in order to M_2. That is, M_3' is interior to the arc (which is part of the graph of ϕ_2) connecting M_3 and M_2 (in this order as shown in figure 4). In the same way, M_4' is interior to the arc (which is part of the graph of ϕ_1) connecting M_1 and M_4 (in this order).

Furthermore, by the twist condition, the image of the linear interval $L_1 = [M_1, M_3]$ by G is situated to the right of the line defined by M_1 and M_3. By the same reason, the image of the linear interval $L_2 = [M_4, M_2]$ by G is situated to the left of the line defined by M_4 and M_2.

Finally, as the graphs of ϕ_1 and ϕ_2 are invariant by G, then it follows from above that the region A delimited by them (and the lines M_1, M_3 and M_2, M_4) is mapped strictly inside A by the transformation G.

This is a contradiction to the fact that G preserves area.

□

Corollary 9. *Let $\phi_t : \mathbb{R} \to \mathbb{R}$, $t \in [a, b]$, be a family of continuous periodic functions of period 1. Suppose that $\phi_t(x)$ is continuous as a function of (t, x) and that, for each t, the transformation ϕ_t defines a rotational invariant circle Γ_t, such that they are disjoint two by two. Then there exists a continuous strictly monotone function $r : [a, b] \to \mathbb{R}$ such that*

$$\rho(\Gamma_t) = r(t)\,(mod\ 1),\ \forall t \in [a, b].$$

Proof. By hypothesis, for each $x \in \mathbb{R}$, the function $t \to \phi_t(x)$ is injective. Therefore, it is either strictly increasing or strictly decreasing. If it is increasing for some point x, it cannot be decreasing for other points.

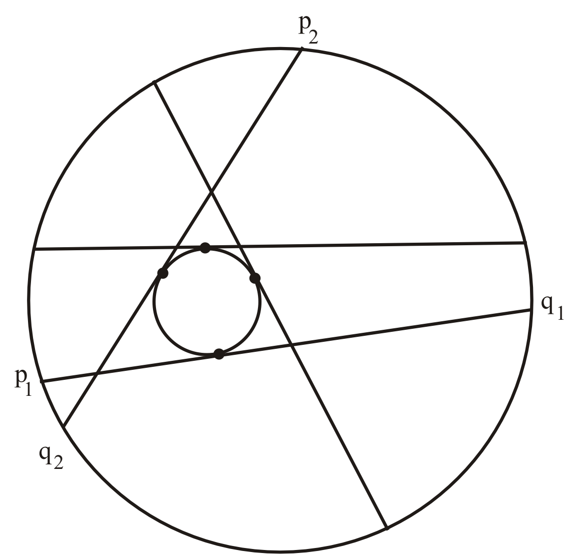

Figure 3: Points in the Billiard

Without loss of generality, suppose that $t \to \phi_t(x)$ is strictly increasing for all $x \in \mathbb{R}$. Denote $g_t(x) = F_1(x, \phi_t(x))$. Then, by the claim after Definition 7, for all $t \in [a,b]$,

$$\rho(\Gamma_t) = r(g_t) \, (\text{mod } 1).$$

Therefore, take $r(t) = r(g_t)$. Then, r is continuous (see [7], [16]) and strictly increasing (by Theorem 8).

□

Consider now the torus $Y = S^1 \times S^1$ with the natural orientation and let $\pi_1, \pi_2 : Y \to S^1$ be the canonical projections. Denote by v the unitary vector field which is tangent to the fibers of π_1 (compatible with the orientation of S^1). Consider a C^1 diffeomorphism $f : Y \to Y$ such that

a) f preserves orientation of Y,

b) f preserves area of a Riemannian metric on Y,

c) $v, df(v)$ are independent in each point and they are compatible (in this order) with the orientation of Y (twist condition).

Definition 10. Consider $\phi : S^1 \to S^1$, a continuous function and the oriented curve $\Gamma \subset X$, with parametrization $x \to \pi(x, \phi(x))$, $x \in S^1$. We say Γ is an invariant rotational circle for f if $f(\Gamma) = \Gamma$ and $f|\Gamma$ preserves orientation.

We point out that $\rho(f|\Gamma)$ is well defined and we denote $\rho(\Gamma) = \rho(f|\Gamma)$.

Corollary 11. *Let $\phi_t : S^1 \to S^1$, $t \in [a,b]$, be a family of continuous functions. Suppose that $\phi_t(x)$ is continuous as a function of (t,x) and that for each t the transformation ϕ_t defines a rotational invariant circle Γ_t, such that they are disjoint two by two.*

Then there exists a continuous strictly monotone function $r : [a,b] \to \mathbb{R}$ such that

$$\rho(\Gamma_t) = r(t) \, (mod\ 1), \ \forall t \in [a,b].$$

PROOF. Denote $Y_0 = Y - \Gamma_b$; then there exists a diffeomorphism $\theta : X \to Y_0$ which preserves orientation, compatible with the projections $\pi_1 : X \to S^1$, $\pi_1|Y_0 : Y_0 \to S^1$, and such that it preserves the orientation of the fibers of these projections.

The transformation $f_0 = \theta^{-1} \circ f \circ \theta : X \to X$ is a diffeomorphism of the cylinder X and which satisfies the properties of $f : X \to X$ described just after Definition 6 (f_0 preserves the ends of X because it preserves the orientation of Y and the orientation of Γ_b).

For each $t \in [a,b]$, we have that $\theta^{-1}(\Gamma_t)$ is a rotational invariant circle of f_0. In the same way as in Corollary 7, there exists $r[a,b) \to \mathbb{R}$, which is continuous, strictly monotone and such that

$$\rho(\Gamma_t) = r(t) \, (\text{mod}\, 1), \ \ t \in [a,b).$$

Note that $\pi_1|\Gamma_t : \Gamma_t \to S^1$ is a diffeomorphism which preserves orientation. By conjugacy,

$$\rho(\Gamma_t) = \rho((\pi_1|\Gamma_t) \circ (f|\Gamma_t) \circ (\pi_1|\Gamma_t)^{-1}).$$

Then, by Lemma 5 b), we have that

$$\lim_{t\to b^-} \rho(\Gamma_t) = \rho(\Gamma_b).$$

From this follows that

$$\lim_{t\to b^-} r(t) = l$$

is finite. We define $r(b) = l$ and $\rho(\Gamma_b) = r(b) \,(\text{mod } 1)$.

□

3 The Main Result.

Using the previous notation for the Poncelet billiard, denote by Y the torus

$$Y = \{(A, r) : A \in K,\ r \text{ a line by } A\} = K \times S^1.$$

The transformation $f : Y \to Y$ associated to the Poncelet Billiard was defined before in Section 1. f satisfies the hypothesis described in Section 2.

Consider a fixed circle L of center 0 and radius t, where $t \in [0, R-c]$. To each point $A \in K = S^1$, one can associate a tangent r to L passing by A, and which has the property that L is on the left of r when oriented from A to A'. In figure 3 we show the set of tangents to L defining the corresponding rotational invariant circle Γ_t. For a fixed circle L, we define in such way a transformation $\phi_t : S^1 \to S^1$, which depends continuously on t. These ϕ_t define a family of rotational invariant circles Γ_t, which are, two by two, disjoint.

By Corollary 11, there exists $r : [0, R-c] \to \mathbb{R}$, continuous, strictly monotonous and such that

$$\rho(\Gamma_t) = r(t) \,(\text{mod } 1).$$

As $f|\Gamma_{R-c}$ has a fixed point, then $\rho(\Gamma_{R-c}) = 0 \,(\text{mod } 1)$. As a continuous limit of the figure 1 when $t \to 0$, one can see that $f|\Gamma_0$ has period 2, and therefore $\rho(\Gamma_0) = 1/2 \,(\text{mod } 1)$. Therefore, $r(0) = \frac{1}{2} + n$, and $r(R-c) = m$, where $n, m \in \mathbb{Z}$.

We can suppose that $r(t)$ is such that $r(0) = 1/2$, and $r(R-c) = k \in \mathbb{Z}$. We point out that $f|\Gamma_t$ has no fixed points if $t \neq (R-c)$. Then, $r(t)$ is not in $\mathbb{Z}$ if $t \neq (R-c)$ (Lemma 4 b). The conclusion is that $k = 0$ or $k = 1$.

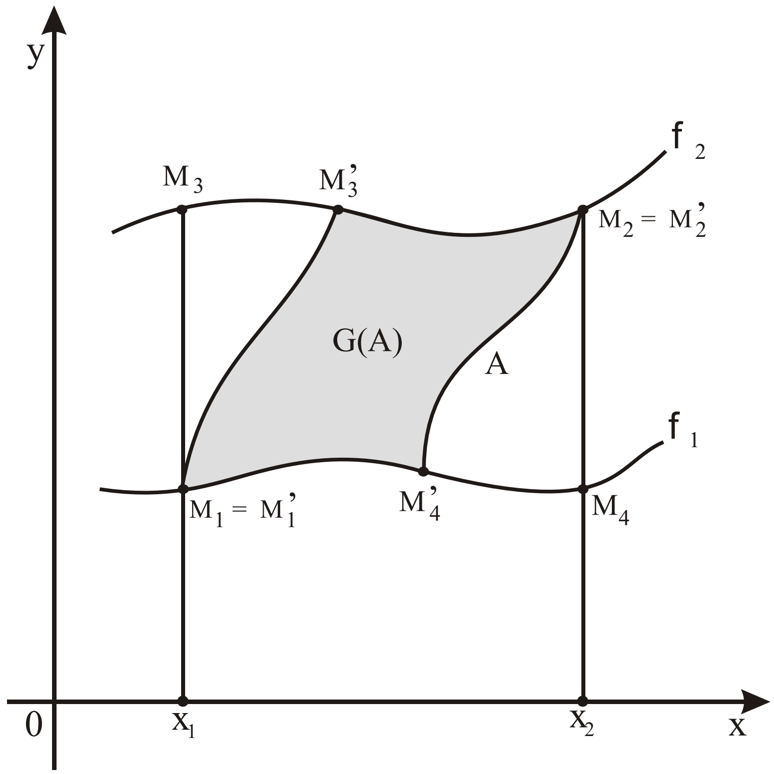

Figure 4: Phase Space

Therefore, r is continuous, strictly increasing with values in $[1/2, 1]$ or strictly decreasing with values in $[0, 1/2]$.

For each $q = 3, 4, 5, \ldots$, denote $e(q)$ the number of natural numbers $n < q$, which are relatively prime with q. Therefore, in any case, for a fixed q, there are exactly $\frac{e(q)}{2}$ numbers n in such a way that $\frac{n}{q}$ is attained in the image of r.

From this and Lemma 4 b) comes the following:

Lemma 12. *For each natural number* $q = 1, 2, 3, 4, \ldots$, *there are* $\frac{e(q)}{2}$ *rotational invariant circles* Γ_t *such that* $f|\Gamma_t$ *has a periodic orbit of period exactly* q.

By Poncelet Theorem [4] and [18], $f|\Gamma_t$ has a periodic orbit of period q, if and only if, $f|\Gamma_t$ is a q-periodic transformation. In this case the corresponding pair K, L is called a Poncelet pair. From this follows Theorem 1.

4 A Second Order Estimate of the Derivative of Twist Maps.

The results of this sections are for a more general class of twist maps.

Consider a family $g_t : \mathbb{R} \to \mathbb{R}$, where $a \leq t \leq b$, of monotonous increasing homeomorphisms, such that

$$g_t(x+1) = g_t(x) + 1, \; x \in \mathbb{R}, \; a \leq t \leq b.$$

From the identification $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$, from the family g_t, we obtain another one $f_t : S^1 \to S^1$, where $a \leq t \leq b$, of homeomorphisms of the circle which preserve orientation.

We know that the following function is well-defined:

$$r(t) \; = \lim_{k \to \infty} \frac{g_t^k(x) - x}{k}, \; a \leq t \leq b,$$

and the value $r(t)$ indeed is independent of x. We are interested in properties of the function $r(t)$ and of the rotation number function $\rho(t) = r(t) \pmod 1 \in \frac{\mathbb{R}}{\mathbb{Z}}$. We assume just the **twist condition**: there exists $\frac{\partial g_t(x)}{\partial t}$ and it is continuous and positive for all $x \in \mathbb{R}$, $t \in [a, b]$.

As $\frac{\partial g_t(x)}{\partial t}$ is periodic in x, we get

$$m = \inf_{a \leq t \leq b, \, x \in \mathbb{R}} \frac{\partial g_t(x)}{\partial t} \; > \; 0.$$

For a given irrational value x consider its development in continuous fraction and we call approximations of x the successive truncations of this infinite expansion. We denote by defect approximation one which is smaller than x and by excess approximation one which is larger than x. We refer the reader to [8] for general properties of continuous fraction expansion.

We denote by F the sum of the inverses of the Fibonacci sequence:

$$F = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + ...$$

Consider two monotonous increasing homeomorphisms $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that

$$g_1(x+1) = g_1(x) + 1 \, , \; g_2(x+1) = g_2(x) + 1 \, , \; x \in \mathbb{R}.$$

Denote

$$r_1 = \lim_{k\to\infty} \frac{g_1^k(x) - x}{k},\; r_2 = \lim_{k\to\infty} \frac{g_2^k(x) - x}{k}.$$

Suppose $g_1(x) < g_2(x)$, $\forall x \in \mathbb{R}$. As $g_1(x) - g_2(x)$ is periodic and continuous, we have that

$$\alpha = \inf_{x\in\mathbb{R}} (g_2(x) - g_1(x)) \,>\, 0.$$

Lemma 13. *a) $r_1 \le r_2$,*

b) Suppose that r_1 is irrational and denote by $\frac{p}{q}$ an excess continued fraction approximation of r_1 (that is, $\frac{p}{q} \ge r_1$) such that $q > \frac{1}{\alpha}$. Then,

$$r_1 \,<\, \frac{p}{q} \,\le r_2.$$

c) Suppose r_2 is irrational and denote by $\frac{p'}{q'}$ an excess continued fraction approximation of r_1 (that is, $\frac{p'}{q'} \ge r_1$) such that $q' > \frac{1}{\alpha}$. Then,

$$r_1 \,\le\, \frac{p'}{q'} \,< r_2.$$

For a proof of the lemma see section 11.1 in [16]

We need some results about approximation by truncation of continued fractions.

Lemma 14. *For each irrational x consider $\frac{p_n(x)}{q_n(x)}$, $n = 1, 2, 3, \ldots$, the n-truncation of the continuous fractional expansion of x. Given $0 < \epsilon < 1$, then for Lebesgue almost every irrational $x \in \mathbb{R}$, there exists a sequence $n_1 < n_2 < n_3, \ldots$ of natural numbers (which depend on x and ϵ) such that*

$$\frac{2e^{-2F}}{1+\epsilon} < \frac{q_{n_k+1}(x)}{q_{n_k}(x)} < \frac{2e^{2F}}{1-\epsilon},\; k = 1, 2, 3, \ldots$$

Proof. We can suppose $x \in (0, 1)$. Denote by $T : [0, 1) \to [0, 1)$ the Gauss transformation $T(x) = \frac{1}{x} - [\frac{1}{x}]$, if $x \neq 0$, and $T(0) = 0$.

For each irrational number x, from section 3.5 in [8]

$$-\log q_n(x) \,=\, \log(x) + \log(T(x)) + \ldots + \log(T^{n-1}(x)) + R(n, x),$$

where $|R(n, x)| \le F$, for all x and $n = 1, 2, 3, \ldots$ Therefore,

$$\log\left(\frac{q_{n+1}(x)}{q_n(x)}\right) = -\log T^n(x) + R(n, x) - R(n + 1, x),$$

and finally

$$-2\,F - \log T^n(x) \,\leq\, \log\left(\frac{q_{n+1}(x)}{q_n(x)}\right) \,\leq\, 2F - \log T^n(x).$$

As T is ergodic [8], for Lebesgue almost every irrational $x \in (0,1)$, we have that $\frac{1}{2}$, is a limit of a certain subsequence of $T^n(x)$. Therefore, there exists natural numbers $n_1 < n_2 < n_3 < \ldots$ such that

$$\frac{1}{2} - \frac{\epsilon}{2} < T^{n_k}(x) < \frac{1}{2} + \frac{\epsilon}{2},\; k = 1, 2, 3, \ldots$$

From this we get

$$-2F - \log\left(\frac{1}{2} + \frac{\epsilon}{2}\right) \,<\, \log\left(\frac{q_{n_k+1}(x)}{q_{n_k}(x)}\right) \,<\, 2F - \log\left(\frac{1}{2} - \frac{\epsilon}{2}\right).$$

□

Corollary 15. *Given* $0 < \epsilon < 1$, *for almost irrational* x *there exists an infinite number of excess approximations* $\frac{p}{q}$, *and infinite number of excessive approximations* $\frac{p'}{q'}$, *such that*

$$\frac{2\,e^{-2\,F}}{1+\epsilon} < \frac{q'}{q} < \frac{2\,e^{2\,F}}{1-\epsilon},\;\; or\;\; \frac{2\,e^{-2\,F}}{1+\epsilon} < \frac{q}{q'} < \frac{2\,e^{2\,F}}{1-\epsilon}.$$

PROOF. In the Lemma 14 above take

$$p = p_{n_k}(x),\; q = q_{n_k}(x),\; p' = p_{n_k+1}(x),\; q' = q_{n_k+1}(x)$$

if n_k is odd and the opposite if n_k is even.

□

Corollary 16. *Given* $0 < \epsilon < 1$, *denote*

$$K_\epsilon = \frac{1-\epsilon}{e^{2F}\,(1 + (1+\epsilon)e^{2F})^2}.$$

Then, for almost every irrational $x \in \mathbb{R}$ *there exists infinite excess approximations* $\frac{p}{q}$ *and infinite defect approximations* $\frac{p'}{q'}$ *of* x *such that:*

$$\frac{p}{q} - \frac{p'}{q'} \,\geq\, K_\epsilon\left(\frac{1}{q} + \frac{1}{q'}\right)^2.$$

PROOF. Denote

$$L = \frac{e^{-2F}}{1+\epsilon}, \; M = \frac{e^{2F}}{1-\epsilon}.$$

From Corollary 15 we can suppose that:

a) $L < \frac{q'}{q} < M$, or,

b) $L < \frac{q}{q'} < M$.

In the first case:

$$\left(\frac{1}{q} + \frac{1}{q'}\right)^2 < \left(\frac{1}{q} + \frac{1}{L\,q}\right)^2 = \frac{1}{q^2}\left(1 + \frac{1}{L}\right)^2,$$

and,

$$\frac{p}{q} - \frac{p'}{q'} = \frac{p\,q' - q\,p'}{q\,q'} \geq \frac{1}{q\,q'} \geq \frac{1}{M\,q^2}.$$

Therefore,

$$\frac{p}{q} - \frac{p'}{q'} > \frac{1}{M}\frac{1}{(1+\frac{1}{L})^2}\left(\frac{1}{q} + \frac{1}{q'}\right)^2.$$

The claim follows from the fact that

$$K_\epsilon = \frac{1}{M}\frac{1}{(1+\frac{1}{L})^2}.$$

The second case follows in a similar way.

□

Now we will apply the results above for the function $r(t)$.

Proposition 17. *The function $r : [a,b] \to \mathbb{R}$ is continuous and monotonically increasing. Moreover, if $a \leq t_1 < t_2 \leq b$ and $r(t_1) \notin \mathbb{Q}$ or $r(t_2) \notin \mathbb{Q}$, then $r(t_1) < r(t_2)$.*

PROOF. For continuity and other properties see [16]. The rest follows from Lemma 13. □

Lemma 18. *If $a \leq t_1 < t_2 \leq b$, then $g_{t_2}(x) - g_{t_1}(x) \geq m\,(t_2 - t_1)$.*

PROOF. It follows easily from the twist condition and the mean value theorem.

□

Proposition 19. *There exist a set of full measure $X \subset \mathbb{R}$ (that is, $\mathbb{R} - X$ has Lebesgue measure zero) and such that if $\tau \in (a, b)$ and $r(\tau) \in X$, then*

$$\limsup_{t_2 \to \tau^-,\, t_1 \to \tau^+} \frac{r(t_2) - r(t_1)}{(t_2 - t_1)^2} \geq \frac{m^2}{e^{2F}(1 + e^{2F})^2}.$$

Proof. Given $0 < \epsilon < 1$, by Corollary 16 there exists a set of full measure $X_\epsilon \subset \mathbb{R}$ of irrational numbers such that if $r(\tau) \in X_\epsilon$, then the claim of this corollary is true for K_ϵ. Let δ be such that $(\tau - \delta, \tau + \delta) \subset (a, b)$.

The function $g_\tau(x) - g_{\tau-\delta}(x)$ is continuous, periodic and positive by Lemma 3. Therefore,

$$\inf_{x \in \mathbb{R}} (g_\tau(x) - g_{\tau-\delta}(x)) > 0.$$

In the same way,

$$\inf_{x \in \mathbb{R}} (g_{\tau+\delta}(x) - g_\tau(x)) > 0.$$

By corollary 16, there exists approximations by excess $\frac{p}{q}$ and by defect $\frac{p'}{q'}$ of $r(\tau)$ such that

$$\frac{1}{q'} < \inf_{x \in \mathbb{R}} (g_\tau(x) - g_{\tau-\delta}(x)),$$
$$\frac{1}{q} < \inf_{x \in \mathbb{R}} (g_{\tau+\delta}(x) - g_\tau(x))$$

and

$$\left(\frac{p}{q} - \frac{p'}{q'}\right) > K_\epsilon \left(\frac{1}{q} + \frac{1}{q'}\right)^2.$$

As,

$$\inf_{x \in \mathbb{R}} (g_t(x) - g_\tau(x)) = \min_{0 \leq x \leq 1} (g_t(x) - g_\tau(x)),$$

there exist τ_1, τ_2 such that $\tau - \delta < \tau_1 < \tau < \tau_2 < \tau + \delta$ and

$$\inf_{x \in \mathbb{R}} (g_\tau(x) - g_{\tau_1}(x)) = \frac{1}{q'}, \text{ and } \inf_{x \in \mathbb{R}} (g_{\tau_2}(x) - g_\tau(x)) = \frac{1}{q}.$$

Now, from Lemma 18, if $t_1 < \tau_1 < \tau < \tau_2 < t_2$, then

$$\inf_{x \in \mathbb{R}} (g_\tau(x) - g_{t_1}(x)) > \frac{1}{q'}, \text{ and } \inf_{x \in \mathbb{R}} (g_{t_2}(x) - g_\tau(x)) > \frac{1}{q}.$$

By Lemma 13,

$$r(t_1) \leq \frac{p'}{q'} < r(\tau) < \frac{p}{q} \leq r(t_2).$$

Therefore,

$$r(t_2) - r(t_1) \geq \frac{p}{q} - \frac{p'}{q'} > K_\epsilon \left(\frac{1}{q} + \frac{1}{q'} \right)^2 .$$

As r is continuous, we get

$$r(\tau_2) - r(\tau_1) \geq K_\epsilon \left(\frac{1}{q} + \frac{1}{q'} \right)^2 .$$

Now, by Lemma 18

$$g_{\tau_2}(x) - g_\tau(x) \geq m(\tau_2 - \tau), \ \forall x \in \mathbb{R},$$

and

$$g_\tau(x) - g_{\tau_1}(x) \geq m\,(\tau - \tau_1), \forall x \in \mathbb{R}.$$

It follows that

$$\frac{1}{q} \geq m(\tau_2 - \tau), \text{ and } \frac{1}{q'} \geq m(\tau - \tau_1).$$

Therefore,

$$r(\tau_2) - r(\tau_1) \geq K_\epsilon\, m^2\, (\tau_2 - \tau_1)^2,$$

and then

$$\limsup_{t_1 \to \tau^-,\, t_1 \to \tau^+} \frac{r(t_2) - r(t_1)}{(t_2 - t_1)^2} \geq K_\epsilon\, m^2.$$

The claim of the proposition follows from taking

$$X = \cap_{n=2}^{\infty} X_{\frac{1}{n}}.$$

□

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