RESEARCH

Real Analysis Exchange
Vol. 34(2), 2008/2009, pp. 471–482

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ALMOST CONTINUOUS MULTI-MAPS AND M-RETRACTS

In memory of Professor O. G. Harrold

Abstract

We give results about almost continuous multi-valued functions and a characterization of compact almost continuous M-retracts of the Hilbert cube Q, where almost continuity is in the sense of Stallings instead of Husain. For instance, each connectivity or almost continuous point to closed-set valued multi-function $f : I \to I$, where $I = [0, 1]$, has a fixed point; i.e., a point $x \in I$ such that $x \in f(x)$. When Y is a compact subset of Q, a sufficient condition is given for a continuous multifunction $r : Y \to Y$, with $x \in r(x)\ \forall x \in Y$, to have an almost continuous multi-valued extension $r : Q \to Y$.

Given a metric space (X, d), let $S(X)$, $CB(X)$ and 2^X denote, respectively, the collection of all nonempty closed subsets of X, the collection of all nonempty closed and bounded subsets of X and the collection of all nonempty compact subsets of X, each with the Hausdorff metric H on it. By definition,

$$N(A, \epsilon) = \{x \in X \ : \ d(x, a) < \epsilon \ \text{ for some } a \in A\},$$

and for $A, B \in CB(X)$,

$$H(A, B) = \inf\{\epsilon > 0 \ : \ A \subset N(B, \epsilon) \text{ and } B \subset N(A, \epsilon)\}.$$

A single-valued function $f : X \to Y$ has a *fixed point* if X is a subset of Y and there exists x such that $x = f(x)$. Given arbitrary metric spaces X and Y, a *multi-valued function* $T : X \to Y$ maps each point x of X to a

Mathematical Reviews subject classification: Primary: 54C05, 54H25, 26E25, 54C60
Key words: M-retracts, fixed points, continuous, connectivity, almost continuous multi-valued functions
Received by the editors November 16, 2008
Communicated by: Udayan B. Darji

unique nonempty subset $T(x)$ of Y, and if each $T(x)$ is closed in Y, T can be treated as a single-valued function $T : X \to S(Y)$. A multi-valued function $T : X \to X$ (or its corresponding single-valued function $T : X \to S(X)$ when each $T(X)$ is closed in X) is said to have a *fixed point* x_0 if $x_0 \in T(x_0)$. Schauder's theorem [8] states that every compact convex nonempty subset X of a normed space has the fixed point property for single-valued continuous maps $T : X \to X$ (abbr. f.p.p.), and in [3, Cor. 2], Girolo shows such a space X has the fixed point property for single-valued connectivity functions $T : X \to X$. Strother [10, Thm. 1] shows that I has the fixed point property for point to closed set continuous multi-functions (abbr. F.p.p.) but gives an example showing I^2 does not have the F.p.p. His example can be modified to hold also for I^n, $n \geq 3$, by replacing the $90°$ rotation of the unit circle S there with the antipodal map of S^{n-1}. Plunkett actually shows that a Peano continuum has the F.p.p. if and only if it is a dendrite [6]. Also non-Peano arc-like continua have the F.p.p. [12]. Smithson shows that a biconnected point-closed multi-valued function F on a tree into itself has a fixed point. (A multi-valued function $F : X \to Y$ is called *biconnected* if

$$F(C) = \bigcup \Big\{ F(x) \ : \ x \in C \Big\} \quad \text{and} \quad F^{-1}(D) = \Big\{ x \in X \ : \ F(X) \cap D \neq \emptyset \Big\}$$

are connected sets whenever C and D are connected subsets of X and Y respectively.) We show that each connectivity or almost continuous $f : I \to 2^I$ has a fixed point.

For $M \subset X$, M is a *retract* of X if there exists a single-valued continuous function $f : X \to M$ such that $f(x) = x \ \forall x \in M$. Wojdyslawski [13] proves that M is a retract of a compact space X implies $S(M)$ is a retract of $S(X)$. The converse is false. For, in [11], Strother defines $M \subset X$ to be an *M-retract* of X if there exists a continuous multi-valued function $F : X \to M$ such that $F(x) = \{x\} \ \forall x \in M$ and then uses his construction in [10] to show that the unit circle S^1 is an M-retract of the unit disc B^2 and 2^{S^1} is a retract of 2^{B^2} even though S^1 is, of course, not a retract of B^2. He also shows in [11, Thm. 8] that for a metric continuum, these are equivalent:

1) X is a Peano space;

2) X is an $MCAR^*$ (i.e., $\forall$ Hausdorff space Y, closed set $Y_0 \subset Y$, and continuous multi-valued function $F : Y_0 \to X$, $\exists$ continuous extension $F_1 : Y \to X$);

3) X is homeomorphic to an M-retract of a Tychonoff cube.

We give results about fixed points of connectivity or almost continuous multifunctions and a characterization of compact almost continuous M-retracts of the Hilbert cube Q, where the M-retraction $F : Q \to M$ is required to be almost continuous in place of continuous. We deal with multifunctions obeying Stallings' definition of almost continuity given below instead of obeying Husain's nonequivalent definition.

If $A \subset X$, a multifunction $F : X \to A$ is called an *ϵ-multi-retraction* if $\forall x \in A$, $d\big(x\,,\,F(x)\big) < \epsilon$ and *diam* $F(x) < \epsilon$, and A is called an *ϵ-multi-retract* of X. It is well known that if X has the f.p.p. and Y is a retract of X, then Y has the f.p.p., too. For completeness, we verify the known generalization of this to ϵ-multi-retracts.

Lemma 1. *If A is a compact subset of a metric space (X, d) and $T : A \to 2^A$ is continuous and if for every $\epsilon > 0$ there is $x(\epsilon) \in A$ such that*

$$d\Big(x(\epsilon)\,,\,T\big(x(\epsilon)\big)\Big) < \epsilon\,,$$

then T has a fixed point x.

PROOF. Since $T(A)$ is compact, there exists a sequence $\epsilon_n \to 0$ such that $T\big(x(\epsilon_n)\big) \to Y \in 2^A$. Therefore

$$H\Big(Y\,,\,T\big(x(\epsilon_n)\big)\Big) \to 0 \quad \text{as} \quad n \to \infty\,, \qquad \text{and} \qquad d\Big(x(\epsilon_n)\,,\,T\big(x(\epsilon_n)\big)\Big) < \epsilon_n\,.$$

Let $y_1 \in Y$ and $y_2 \in T\big(x(\epsilon_n)\big)$ such that

$$d\Big(x(\epsilon_n)\,,\,T\big(x(\epsilon_n)\big)\Big) = d\big(x(\epsilon_n)\,,\,y_2\big) \quad \text{and} \quad d(y_2\,,\,Y) = d(y_2\,,\,y_1)\,.$$

Then

$$\begin{aligned} d\big(x(\epsilon_n)\,,\,Y\big) &\le d\big(x(\epsilon_n)\,,\,y_1\big) && (1)\\ &\le d\big(x(\epsilon_n)\,,\,y_2\big) + d(y_2\,,\,y_1) && (2)\\ &= d\Big(x(\epsilon_n)\,,\,T\big(x(\epsilon_n)\big)\Big) + d(y_2\,,\,Y) && (3)\\ &\le d\Big(x(\epsilon_n)\,,\,T\big(x(\epsilon_n)\big)\Big) + H\Big(T\big(x(\epsilon_n)\big)\,,\,Y\Big)\,. && (4) \end{aligned}$$

Therefore $d\big(x(\epsilon_n)\,,\,Y\big) \to 0$ as $n \to \infty$. Since A is compact, some subsequence $x(\epsilon_{n_k})$ converges to some $x \in A$. Since Y is closed, $x \in Y$, and since T is continuous, $T\big(x(\epsilon_{n_k})\big) \to T(x) = Y$. This shows $x \in T(x)$. □

Theorem 1. *If A is a compact subset of a metric space (X, d), if X has the F.p.p., and if $\forall \epsilon > 0$, $\exists$ a continuous ϵ-multi-retraction $r : X \to A$, then A has the F.p.p.*

PROOF. Suppose $T : A \to A$ is a continuous multi-function and $t : 2^A \to 2^A$ is its united extension defined whenever $B \subset A$ by $t(B) = \cup_{b \in B} T(b)$. Since $tr : X \to X$ is a continuous multi-valued function and X has the F.p.p., there exists $w \in X$ such that

$$w \in tr(w) = \bigcup_{b \in r(w)} T(b)$$

and so $w \in T(b)$ for some $b \in r(w)$. There exists $b' \in r(w)$ such that $d(w\,,\,b') = d\big(w\,,\,r(w)\big)$. Therefore

$$d\big(b\,,\,T(b)\big) \le d(b\,,\,w) \le d(b\,,\,b') + d(b'\,,\,w) < 2\epsilon$$

because $w \in T(b)$, $b \in r(w)$, $d\big(w\,,\,r(w)\big) < \epsilon$ and $diam\ r(w) < \epsilon$. By Lemma 1, T has a fixed point. $\square$

For topological spaces X and Y, we define the following "Darboux-like" classes of functions $f : X \to Y$ (where Y could possibly equal $S(X)$, $CB(X)$, or 2^X):

f is *Darboux* (abbr. $f \in D$) if $f(C)$ is connected for each connected $C \subset X$.

f is *almost continuous* ($f \in AC$) if each open subset of $X \times Y$ containing the graph of f also contains the graph of a continuous function $g : X \to Y$.

f is a *connectivity function* ($f \in Conn$) if the graph of the restriction $f_{|C}$ is a connected subset of $X \times Y$ for each connected subset C of X.

f is *extendable* ($f \in Ext$) if there is a connectivity function $F : X \times I \to Y$ such that $F(x\,,\,0) = f(x)$ for every $x \in X$.

f is *peripherally continuous* ($f \in PC$) if for every $x \in X$ and for all open sets U containing x and V containing $f(x)$, there exists an open set W containing x such that $W \subset U$ and $f\big(bd(W)\big) \subset V$.

According to [4], if $X = I^n$, then $Y = 2^{I^n}$ is a Peano space and is uniformly locally p-connected for all $p > 0$, which means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for each $y \in Y$ and for each integer $k = 0, 1, 2, \ldots, p$, every continuous $\varphi : S^k \to N(y\,,\,\delta)$ can be extended to a continuous $\Phi : B^{k+1} \to N(y\,,\,\epsilon)$, where S^k is the boundary of the closed unit ball B^{k+1} in Euclidean $(k+1)$-space $\mathbb{R}^{k+1}$. This helps to see that for any $n \geq 1$, the relationships given in [2, pp. 496 and 513] between the above classes of Darboux-like single-valued functions $I^n \to I^n$ are exactly the same for Darboux-like closed-set valued multi-functions $I^n \to I^n$. In particular, for any $n \geq 2$, in the class of all functions $f : I^n \to 2^{I^n}$, we have $PC \subset AC$. This follows from Stallings' Theorem 5 in [9] which states that if X is a locally peripherally connected polyhedron of dimension n, Y is a uniformly locally $(n-1)$-connected metric space, and $f : X \to Y$ is a peripherally continuous function, then f is almost continuous. What is left to verify next is that in the class of all functions $f : I \to 2^I$, $AC \subset Conn$.

We list these four propositions from [9]:

Stallings' Proposition 1. *If $f : X \to Y$ is almost continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is almost continuous.*

In fact, he shows that for each open set N containing the graph of $g \circ f$, there exists a continuous function $F : X \to Y$ such that $g \circ F \subset N$.

Stallings' Proposition 2. *If $f : X \to Y$ is almost continuous and C is closed in X, then $f_{|C} : C \to Y$ is almost continuous.*

Stallings' Proposition 3. *If $X \times Y$ is a completely normal T_2 space, X is connected, and $f : X \to Y$ is almost continuous, then the graph of f is connected.*

Stallings' Proposition 4. *If X is a compact T_2 space, Y a T_2 space, and Z a topological space and if $f : X \to Y$ is continuous and $g : Y \to Z$ is almost continuous, then $g \circ f : X \to Z$ is almost continuous.*

Theorem 2. *Each almost continuous function $f : I \to 2^I$ is a connectivity function.*

Proof. For each closed subinterval K of I, $f_{|K}$ is almost continuous and therefore connected by Stallings' Propositions 2 and 3. Every subinterval J of I is the union of a sequence $J_1 \subset J_2 \subset J_3 \subset \ldots$ of closed subintervals of I. Since each $f_{|J_i}$ is connected and $f_{|J_1} \subset f_{|J_i}$ for $i \geq 1$, then

$$f_{|J} = f_{|\cup_{i=1}^{\infty} J_i} = \bigcup_{i=1}^{\infty} f_{|J_i}$$

is connected. This shows f is a connectivity function. $\square$

The next result generalizes Strother's Theorem 1 in [10] from continuous functions to connectivity functions, and a referee for an earlier version of my paper gives this simpler proof.

Theorem 3. *Each connectivity function $f : I \to 2^I$ has a fixed point.*

PROOF. This follows from the fact that if $F, g : C \to X$ are continuous functions where F is onto and C is connected, then there exists $x \in C$ such that $F(x) = g(x)$. Pick F to be the projection from the connected graph C of the given connectivity function f onto $X = I$ and define $g : f \to I$ by $g\big(x\,,\, f(x)\big) = \min f(x)$. This shows that there exists a point $x \in C$ such that $x = \min f(x)$ and so $x \in f(x)$. □

Example 1. Let $g : I \to I$ be the almost continuous function

$$g(x) = \begin{cases} \dfrac{1}{2}\Big(1 + \sin\dfrac{1}{x}\Big) & \text{if} \quad 0 < x \leq 1 \\ \quad 0 & \text{if} \quad x = 0\,. \end{cases}$$

Define the almost continuous discontinuous function $f : I \to 2^I$ by $f(x) = \big[0\,,\, g(x)\big]$ for each $x \in I$. (We let $[0\,,\,0] = \{0\}$.) Since g has infinitely many fixed points, so does f. We could have applied either Theorem 3 or Theorem 4 below to conclude that this almost continuous function f has at least one fixed point.

Next, interior and boundary of a cell are its combinatorial ones.

Lemma 2. [7, Thm. 3] *Suppose $D_1, D_2, D_3, \ldots$ are topological n-cells in I^n with pairwise disjoint interiors such that each BdD_i is the union of $(n-1)$-cells E_i and B_i with*

$$B_i = Bd(D_i) - Int(E_i) \quad \textit{and} \quad E_i \subset BdI^n\,.$$

Let

$$M = I^n - \bigcup_{i=1}^{\infty}(D_i - B_i)\,.$$

Then there exists an almost continuous retraction $r : I^n \to I^n$ of I^n onto M.

Example 2. Let g be the function in Example 1, $X = I^2$, and

$$M = cl(g) \bigcup \left(\left[\frac{1}{2\pi}, \frac{1}{\pi}\right] \times \left\{\frac{1}{2}\right\} \right).$$

M contains a simple closed curve

$$J = g\Big|_{[\frac{1}{2\pi}, \frac{1}{\pi}] \cup \left([\frac{1}{2\pi}, \frac{1}{\pi}] \times \{\frac{1}{2}\}\right)},$$

which is the boundary of a disk D in X. M is not an M-retract of X because M is not locally connected [11, Thm. 8], and M is not an almost continuous single-valued retract of X because M separates $\mathbb{R}^2$ [7, Thm.1]. However, M is an almost continuous M-retract of X due to the multifunction $F : X \to M$ defined by

$$F(x) = \begin{cases} F_1(x) & \text{if} \quad x \in D \\ \\ \big\{F_2(x)\big\} & \text{if} \quad x \in X \setminus D, \end{cases}$$

where F_1 is a J-retraction of D given by [10] and F_2 is an almost continuous single-valued retraction of X onto $M \cup D$ given by Lemma 2.

Example 3. We construct an almost continuous function $f : I \to 2^I$ with graph dense in $I \times 2^I$. Let $\big\{F_\alpha : \alpha < c\big\}$ be a well ordering of all blocking sets of $I \times 2^I$ such that each F_α has less than c-many predecessors. A *blocking* set K of $I \times 2^I$ is a closed subset of $I \times 2^I$ that misses the graph of some function $I \to 2^I$ but meets the graph of every continuous function $I \to 2^I$, and as in the proof of [5, Thm. 5.2] and using 2^I is an AR because of [14], one can show that the projection $p(F_\alpha)$ of each F_α into I contains a nondegenerate interval. A function $f : I \to 2^I$ is almost continuous if and only if there exists no blocking set of $I \times 2^I$ missing f. For each α, pick a point $x_\alpha \in p(F_\alpha) \setminus \{x_\xi : \xi < \alpha\}$ and pick $f(x_\alpha) \in 2^I$ such that $\big(x_\alpha, f(x_\alpha)\big) \in F_\alpha$. Define f arbitrarily on $I \setminus \{x_\alpha : \alpha < c\}$. Assume f were not almost continuous. Then there is an open neighborhood U of f in $I \times 2^I$ such that each continuous function $g : I \to 2^I$ meets $(I \times 2^I) \setminus U$. Therefore $(I \times 2^I) \setminus U$ misses f and is one of these blocking sets F_α for some $\alpha < c$, a contradiction. Therefore f must be almost continuous and, by construction, is dense in $I \times 2^I$.

Theorem 4. *If the metric space X has the F.p.p. and $f : X \to CB(X)$ is almost continuous, then f has a fixed point.*

PROOF. Assume f has no fixed point. To see that the "diagonal"

$$\Delta = \big\{(x, A) \in X \times CB(X) \, : \, x \in A\big\}$$

is closed in $X \times CB(X)$, suppose $(x_n, A_n) \in \Delta$ and $(x_n, A_n) \to (x_0, A_0)$ in $X \times CB(X)$. Then $x_n \in A_n$, $x_n \to x_0$ in X, and $A_n \to A_0$ in $CB(X)$. For every $\epsilon > 0$, there exists N such that for all $n > N$, $d(x_n, x_0) < \frac{\epsilon}{2}$ and $H(A_n, A_0) < \frac{\epsilon}{2}$. Pick $n > N$. There exists $y \in A_0$ such that $d(x_n, y) < \frac{\epsilon}{2}$. Therefore

$$d(y, x_0) \leq d(y, x_n) + d(x_n, x_0) < \epsilon\,.$$

Since A_0 is a closed subset of X, $x_0 \in A_0$ and so $(x_0, A_0) \in \Delta$.
The open set $\big(X \times CB(X)\big) \setminus \Delta$ contains f and therefore contains a continuous $g : X \to CB(X)$. So g has no fixed point, a contradiction to X having the F.p.p. $\square$

Cornette shows that each single-valued connectivity retract of a unicoherent Peano continuum is again a unicoherent Peano continuum [1, Thm. 3]. According to [5] or Lemma 2, there is a single-valued almost continuous retraction $r : I^2 \to I^2$ of I^2 onto Knaster's indecomposable continuum with one endpoint, but it is an unsolved problem whether there is a single-valued almost continuous retraction of I^2 onto a pseudoarc. Does there exist an almost continuous M-retraction $r : I^2 \to I^2$ of I^2 onto a pseudoarc M?

A compactum Y is an ϵAR means that whenever Y is homeomorphic to a closed subset Y' of a space X, then Y' is an ϵ-retract of X; i.e., $\forall\ \epsilon > 0\,,\ \exists$ continuous single-valued function $r : X \to Y'$ such that

$$d\big(x\,,\, r(x)\big) < \epsilon \ \ \forall x \in Y'\,.$$

According to Kellum [5], for single-valued functions, a compactum Y is an $\epsilon AR \iff$ whenever $f' : X' \to Y$ is continuous where X' is a closed subset of a space X, then $\exists$ continuous function $f : X \to Y$ such that

$$d\big(f(x)\,,\, f'(x)\big) < \epsilon \ \ \forall x \in X'\,.$$

Our final result is based on his arguments given there.

Theorem 5. *Suppose a compact subset Y of Q obeys this general Tietze multi-valued approximate extension property:*

(1) *If $f' : X' \to Y$ is a continuous multi-valued function, where X' is a closed subset of a space X, then for each $\epsilon > 0$ there exists a continuous multi-valued function $f : X \to Y$ such that $H\big(f(x)\,,\, f'(x)\big) < \epsilon\ \forall x \in X'$, where H is the Hausdorff metric on 2^Y.*

Then

(2) *each continuous multi-valued function* $r : Y \to Y$*, such that* $x \in r(x)$ $\forall x \in Y$*, has an almost continuous multi-valued extension* $r : Q \to Y$*.*

Proof. Let Θ be the collection of all closed subsets S of $Q \times 2^Y$ such that the projection $p(S)$ of S into Q contains c-many points not in Y. So we can by transfinite induction define $r : Q \to 2^Y$ such that if $x \in Y$ then $x \in r(x)$ as already defined, and if $S \in \Theta$ then $r \cap S \neq \emptyset$. Assume r is not almost continuous. Then there exists a minimal blocking set K of $Q \times 2^Y$ that misses r, and $p(K)$ is nondegenerate because K meets every constant function from Q into 2^Y. Assume $p(K)$ is not connected. Then $p(K) = A \cup B$ for some separated sets A and B.

$$K_1 = K \setminus \Big(K \cap p^{-1}(B)\Big) \quad \text{and} \quad K_2 = K \setminus \Big(K \cap p^{-1}(A)\Big)$$

are closed proper subsets of K and so cannot be blocking sets. Therefore there are continuous functions $g_1, g_2 : Q \to 2^Y$ such that $g_1 \cap K_2 = \emptyset$ and $g_2 \cap K_1 = \emptyset$, $p(g_1 \cap K) \subset A$, and $p(g_2 \cap K) \subset B$. Let $X = Q$ and $X' = p(K)$. The function

$$f' = (g_{1|B}) \cup (g_{2|A}) \ : p(K) \to 2^Y$$

is continuous, and f' and K are disjoint closed subsets of the compact space $X \times 2^Y$. There exists $\epsilon > 0$ such that if $g' : X' \to 2^Y$ is continuous and

$$H\big(g'(x)\,,\, f'(x)\big) < \epsilon \ \ \forall x \in X'\,,$$

then $g' \cap K = \emptyset$, too. By hypothesis, for this ϵ, there exists a continuous function $f : Q \to 2^Y$ such that

$$H\big(f(x)\,,\, f'(x)\big) < \epsilon \ \ \forall x \in X'\,.$$

Therefore $f \cap K = \emptyset$, which is a contradiction. Since $p(K)$ is connected and $K \notin \Theta$, $p(K) \subset Y$. Since $r \cap K = \emptyset$, there exists $\epsilon > 0$ such that if $g : p(K) \to 2^Y$ is continuous and

$$H\big(g(x)\,,\, r(x)\big) < \epsilon \ \ \forall x \in p(K)\,,$$

then $g \cap K = \emptyset$. By hypothesis for this ϵ, there exists a continuous function $f : Q \to 2^Y$ such that

$$H\big(f(x)\,,\, r(x)\big) < \epsilon \ \ \forall x \in p(K)\,.$$

Therefore $f \cap K = \emptyset$, a contradiction. So r is almost continuous after all. $\square$

By letting $r(x) = x \; \forall x \in Y$ in Theorem 5, it follows that for a compact subset Y of Q, (1) $\Rightarrow$ (3) Y is an almost continuous M-retract of Q.
A straightforward proof that (3) $\Rightarrow$ (1) for a compact subset Y of Q can be given based on Kellum's proof of sufficiency for Theorem 3.1 in [5] and using Stallings' Propositions 1 and 4.

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