

A. Boccuto, Dipartimento di Matematica e Informatica, via Vanvitelli 1,
I-06123 Perugia, Italy. email: boccuto@dipmat.unipg.it,
boccuto@yahoo.it

D. Candeloro, Dipartimento di Matematica e Informatica, via Vanvitelli 1,
I-06123 Perugia, Italy. email: candelor@dipmat.unipg.it

B. Riečan, Katedra Matematiky, Fakulta Prírodných Vied, Univerzita
Mateja Bela, Tajovského, 40 SK-97401 Banská Bystrica, Slovakia. email:
riecan@fpv.umb.sk

ABSTRACT GENERALIZED KURZWEIL-HENSTOCK-TYPE INTEGRALS FOR RIESZ SPACE-VALUED FUNCTIONS

Abstract

Some convergence theorems have been obtained for the GH_k integral for functions defined in abstract topological spaces and with values in Riesz spaces.

1 Introduction.

In a previous paper ([9]) a kind of integral (GH_k integral) has been introduced for Riesz space-valued functions, defined on (possibly) unbounded subintervals of the real line, together with some versions of convergence theorems. This integral is a generalization of the Kurzweil-Henstock and the Henstock-Stieltjes integrals (concerning the literature, see [12, 13, 18, 19, 20, 22] and for related topics we refer to the bibliography of [9]). In [12, 13, 23] one can find concrete examples and illustrations of functions and integrals which can be considered as particular cases of this theory.

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In this paper we investigate the case of Riesz space-valued functions defined in abstract topological spaces and prove some versions of convergence theorems, which in the case of the Kurzweil-Henstock integral were stated in [4, 5, 8, 22]. The spaces on which our involved functions are defined can be for example compact topological spaces, or locally compact topological spaces associated with their Alexandroff one-point compactification: so, the (bounded or not) intervals of the real line and even the whole (extended) real line can be viewed as particular cases of our spaces.

We follow the approach and the techniques of [4, 5, 8, 21, 22].

2 Preliminaries.

Definition 2.1. A Riesz space R is said to be *Dedekind complete* if every nonempty subset of R , bounded from above, has supremum in R .

Definition 2.2. A bounded double sequence $(a_{i,j})_{i,j}$ in R is called *regulator* or *(D)-sequence* if, for each $i \in \mathbb{N}$, $a_{i,j} \downarrow 0$, that is $a_{i,j} \geq a_{i,j+1} \forall j \in \mathbb{N}$ and $\bigwedge_{j \in \mathbb{N}} a_{i,j} = 0$.

Given a sequence $(r_n)_n$ in R , we say that $(r_n)_n$ *(D)-converges* to an element $r \in R$ if there is a regulator $(a_{i,j})_{i,j}$, such that:

for all maps $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists an integer n_0 such that

$$|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for all $n \geq n_0$. In this case, we write $(D) \lim_n r_n = r$.

Analogously, given $l \in R$, a function $f : A \rightarrow R$, where $\emptyset \neq A \subset \tilde{\mathbb{R}}$, and a limit point x_0 for A , we will say that $(D) \lim_{x \rightarrow x_0} f(x) = l$ if there exists a (D) -sequence $(a_{i,j})_{i,j}$ in R such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a neighborhood \mathcal{U} of x_0 such that for any $x \in \mathcal{U} \cap A \setminus \{x_0\}$ we get

$$|f(x) - l| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Definition 2.3. We say that R is *weakly σ -distributive* if, for every (D) -sequence $(a_{i,j})_{i,j}$,

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0. \quad (1)$$

Throughout the paper, we shall always assume that R is a Dedekind complete weakly σ -distributive Riesz space, and all limits in R will be considered as (D) limits.

The following lemma, due to D. H. Fremlin, will be useful in the sequel (see [15, 22]).

Lemma 2.4. *Let $\{(a_{i,j}^{(p)})_{i,j} : p \in \mathbb{N}\}$ be any countable family of regulators. Then for each fixed element $u \in R$, $u \geq 0$, there exists a regulator $(a_{i,j})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$,*

$$u \wedge \sum_{p=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+p)}^{(p)} \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

3 The Abstract Integral.

We begin with introducing the space where our involved Riesz space-valued functions are defined.

Let T be a Hausdorff compact topological space. If $A \subset T$, then its interior and its boundary are denoted by $\text{int } A$ and ∂A respectively.

We shall deal with a family \mathcal{F} of compact subsets of T such that $T \in \mathcal{F}$ and closed under arbitrary intersections and finite unions, and a monotone and additive mapping $\lambda : \mathcal{F} \rightarrow [0, +\infty]$, where in this context additivity means that

$$\lambda(A \cup B) + \lambda(A \cap B) = \lambda(A) + \lambda(B) \tag{2}$$

whenever $A, B \in \mathcal{F}$.

By *partition* (or *k-partition*) of a set $W \in \mathcal{F}$ we mean a finite collection

$$\Pi = \{(t_1; F_{1,1}, \dots, F_{1,k}), \dots, (t_q; F_{q,1}, \dots, F_{q,k})\} = \{(t_1; E_1), \dots, (t_q; E_q)\} \tag{3}$$

such that

(i) $F_{i,j} \in \mathcal{F}$, $F_{i,j} \subset W$ for all $i = 1, \dots, q$ and $j = 1, \dots, k$;

(ii) $\bigcup_{j=1}^k F_{i,j} = E_i$ for all $i = 1, \dots, q$;

(iii) $\bigcup_{i=1}^q E_i = W$;

(iv) $t_i \in E_i$ ($i = 1, \dots, q$);

(v) $\lambda(F_{i,j} \cap F_{l,s}) = 0$ for $i, l = 1, \dots, q$ and $j, s = 1, \dots, k$, whenever $i \neq l$ or $j \neq s$.

A finite collection Π as in (3), satisfying conditions (i), (ii), (iv) and (v), but not necessarily (iii), is said to be a *decomposition* (or *k-decomposition*) of W .

Definition 3.1. We say that \mathcal{F} is *separating*, if there is a sequence $(\Pi_n)_n$ of partitions of T such that Π_{n+1} is a refinement of Π_n ($n \in \mathbb{N}$) and, for any $x, y \in T, x \neq y$, there exists an integer n such that, as soon as $x \in \overline{E}$ for some $E \in \Pi_n$, then $y \in T \setminus \overline{E}$ (see also [21]).

From now on, we always assume that \mathcal{F} is separating.

A *gauge* on a set $A \subset T$ is a mapping δ assigning to every point $x \in A$ a neighborhood $\delta(x)$ of x . If Π is a decomposition of A as in (3) and δ is a gauge on A , then we say that Π is δ -*fine* if $E_i \subset \delta(t_i)$ for any $i = 1, \dots, q$.

A classical example is obtained by setting $T = [a, b] \subset \mathbb{R}$ with the usual topology, \mathcal{F} = the family of all finite unions of closed subintervals of T , $\lambda([\alpha, \beta]) = \beta - \alpha$, $a \leq \alpha < \beta \leq b$; $\delta(x) = (x - \omega(x), x + \omega(x))$, where $\omega : [a, b] \rightarrow \mathbb{R}^+$ is any fixed mapping. Another example is the unbounded interval $[a, +\infty] = [a, +\infty) \cup \{+\infty\}$ with $a \in \mathbb{R}$, considered as the one-point compactification of the locally compact space $[a, +\infty)$. The base of open sets is the usual one in this space, and \mathcal{F} is the collection of the finite unions of closed (bounded or not) subintervals of $[a, +\infty)$. An example of gauge in $[a, +\infty]$ is: $\delta(x) = (x - \omega(x), x + \omega(x))$, if $x \in [a, +\infty) \cap \mathbb{R}$, $\delta(+\infty) = (b, +\infty]$, where ω denotes a positive real-valued function defined on $[a, +\infty)$, and b is a fixed real number (see also [9]).

Note that for every gauge δ and for every set $A \in \mathcal{F}$ there always exists a δ -fine partition (see [21] and also [12]).

Now, our aim is to give a definition of "GH_k-type" integral, inspired at the corresponding ones of [12] and [13], for suitable R -valued functions defined on $T \times \mathcal{F}^k$.

Given any decomposition Π of T as in (3) and a map $U : T \times \mathcal{F}^k \rightarrow R$, we call *Riemann sum* of U (and we write $\sum_{\Pi} U$) the quantity

$$\sum_{i=1}^q U(t_i; F_{i,1}, \dots, F_{i,k}). \quad (4)$$

Definition 3.2. We say that a function $U : T \times \mathcal{F}^k \rightarrow R$ is *GH_k integrable* on T if there exist an element $I \in R$ and a (D) -sequence $(a_{i,j})_{i,j}$ in R such

that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge δ such that

$$\left| \sum_{\Pi} U - I \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{5}$$

whenever Π is a δ -fine partition of T . In this case we say that I is the GH_k integral of U , denoting the element I by the symbol $(GH_k) \int_T U$, and we often write $U \in GH_k(T)$. Similarly it is possible to define the integral $(GH_k) \int_A U$ for every $A \in \mathcal{F}$.

Remark 3.3. We note that the GH_k integral is well-defined, that is there exists at most one element I , satisfying condition (5) (see also [7]).

We now turn to some elementary properties of our integral, whose proof is straightforward.

Proposition 3.4. *If $U_1, U_2 \in GH_k(T)$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 U_1 + c_2 U_2 \in GH_k(T)$, and*

$$(GH_k) \int_T (c_1 U_1 + c_2 U_2) = c_1 (GH_k) \int_T U_1 + c_2 (GH_k) \int_T U_2.$$

If $U, V \in GH_k(T)$ and $U \leq V$, then

$$(GH_k) \int_T U \leq (GH_k) \int_T V;$$

in particular, if $U, |U| \in GH_k(T)$, then

$$\left| (GH_k) \int_T U \right| \leq (GH_k) \int_T |U|.$$

We now state the Cauchy criterion.

Theorem 3.5. *A map $U : T \times \mathcal{F}^k \rightarrow R$ is GH_k integrable if and only if there exists a (D)-sequence $(a_{i,j})_{i,j}$ in R such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge $\delta = \delta(\varphi)$ with the property that*

$$\left| \sum_{\Pi} U - \sum_{\Pi'} U \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

holds, as soon as Π, Π' are δ -fine partitions of T .

PROOF. The technique is analogous to the corresponding one used in [9]. \square

We now investigate GH_k integrability on subsets.

Proposition 3.6. *If $l \in \mathbb{N}$, $E = \cup_{i=1}^l E_i$, $E_i \in \mathcal{F}$ for all $i = 1, \dots, l$, $\lambda(E_i \cap E_j) = 0$ whenever $i \neq j$ and $U \in GH_k(E)$, then $U \in GH_k(E_i)$ for every $i = 1, \dots, l$ and*

$$(GH_k) \int_E U = \sum_{i=1}^l (GH_k) \int_{E_i} U.$$

PROOF. The technique is similar to the one in [21]. \square

We now are looking forward to a "viceversa" of Proposition 3.6, under suitable conditions.

We begin by formulating the following property, which is a kind of "sub-additivity" of the involved set functions.

Fix $E_1, E_2 \in \mathcal{F}$ with $E_1 \cap \text{int } E_2 = E_2 \cap \text{int } E_1 = \emptyset$. We say that $U : T \times \mathcal{F}^k \rightarrow R$ satisfies condition **H1** w. r. to E_1 and E_2 if either $E_1 \cap E_2 = \emptyset$ or there exists a (D) -sequence $(c_{i,j})_{i,j}$ (depending in general on the chosen sets E_1 and E_2) with the property that to all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a gauge η , defined on $\partial E_1 \cap \partial E_2$, such that, for each finite nonempty subset $D \subset \partial E_1 \cap \partial E_2$,

$$\left| \sum_{x \in D} [U(x; F_{0,1}, \dots, F_{0,k}) - U(x; F_{1,1}, \dots, F_{1,k}) - U(x; F_{2,1}, \dots, F_{2,k})] \right| \leq \bigvee_{i=1}^{\infty} c_{i, \varphi(i)} \tag{6}$$

whenever $\lambda(F_{j,s} \cap F_{l,t}) = 0$ ($(j, s) \neq (l, t)$) and both

$$\bigcup_{s=1}^k F_{0,s} = \bigcup_{j=1}^2 \left(\bigcup_{s=1}^k F_{j,s} \right) \subset \eta(x), \text{ and } \bigcup_{s=1}^k F_{j,s} \subset E_j, j = 1, 2.$$

In many cases, when $R = \mathbb{R}$, $T = \tilde{\mathbb{R}}$, \mathcal{F} is the family of all finite (non-overlapping) unions of closed bounded intervals, λ is the Lebesgue measure, the function U is defined by means of suitable "differences" (for example, $U(t; [u, v]) = V(t; v) - V(t; u)$ when $k = 1$ or

$$U(t; [w_0, w_1], \dots, [w_{k-1}, w_k]) = V(t; w_1, \dots, w_k) - V(t; w_0, \dots, w_{k-1})$$

for $k \geq 2$). In these particular cases, since the intersection of the boundaries of any two non-overlapping intervals is obviously empty or a one-point set, the sums in (3) consist of a single summand, and thus **H1**) assumes a simpler form (given in [9]): if $k = 1$, property **H1**) is automatically satisfied (see also [23], Theorem 1.11, pp. 10-12); while for $k \geq 2$ it is implied by the condition of "existence of the iterated limit J " used by A. G. Das and S. Kundu (see [12], Definition 2.9., p. 69).

In general, we observe that **H1**) is fulfilled whenever U is an *additive set function*, that is

$$U(x; F_{0,1}, \dots, F_{0,k}) = U(x; F_{1,1}, \dots, F_{1,k}) + U(x; F_{2,1}, \dots, F_{2,k})$$

whenever $x \in T$, $\lambda(F_{j,s} \cap F_{l,t}) = 0$ ($(j, s) \neq (l, t)$) and $\bigcup_{s=1}^k F_{0,s} = \bigcup_{j=1}^2 \left(\bigcup_{s=1}^k F_{j,s} \right)$.

Indeed, in this case, the first member of (3) is obviously zero.

We now prove the following result, which utilizes **H1**).

Proposition 3.7. *Let $E_1, E_2 \in \mathcal{F}$, $\lambda(E_1 \cap E_2) = 0$, $E_1 \cap \text{int } E_2 = E_2 \cap \text{int } E_1 = \emptyset$, $U \in GH_k(E_1) \cap GH_k(E_2)$. Moreover suppose that condition **H1**) w. r. to E_1 and E_2 holds. Then $U \in GH_k(E_1 \cup E_2)$.*

PROOF. We note that, if $\partial E_1 \cap \partial E_2 = \emptyset$, then the assertion holds without assuming **H1**). Suppose now $\partial E_1 \cap \partial E_2 \neq \emptyset$. By hypothesis, there is a (D) -sequence $(e_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge δ^* for which

$$\left| \sum_{\Pi_j} U - (GH_k) \int_{E_j} U \right| \leq \bigvee_{i=1}^{\infty} e_{i, \varphi(i)}$$

holds, as soon as Π_j is a δ^* -fine k -partition of E_j , $j = 1, 2$. Let $(c_{i,j})_{i,j}$ and η be related with condition **H1**), and define a gauge δ on $E_1 \cup E_2$ by setting $\delta(x) = \delta^*(x) \cap \text{int } E_j$ if $x \in \text{int } E_j$, $j = 1, 2$, and $\delta(x) = \delta^*(x) \cap \eta(x)$ whenever $x \in \partial E_1 \cap \partial E_2$. Pick now any δ -fine k -partition

$$\Pi = \{(\xi_i; F_{i,1}, \dots, F_{i,k}) : i = 1, \dots, n\} = \{(\xi_i; G_i) : i = 1, \dots, n\}$$

of $E_1 \cup E_2$. By virtue of δ -finess of Π and the structure of the sets E_1 , E_2 and δ , if the set G_i has nonempty intersection both with E_1 and with E_2 , then the corresponding *tag* ξ_i belongs to $\partial E_1 \cap \partial E_2$. We have:

$$\begin{aligned} \sum_{\Pi} U &= \sum_{\xi_i \in \text{int } E_1} U(\xi_i; F_{i,1}, \dots, F_{i,k}) + \sum_{\xi_i \in \partial E_1 \cap \partial E_2} U(\xi_i; F_{i,1}, \dots, F_{i,k}) \\ &+ \sum_{\xi_i \in \text{int } E_2} U(\xi_i; F_{i,1}, \dots, F_{i,k}). \end{aligned}$$

Let $j = 1, 2$, pick the points $\xi_i \in \partial E_1 \cap \partial E_2$, and set $Z_{i,l}^{(j)} = F_{i,l} \cap E_j$, $l = 1, \dots, k$ (clearly, we take only the nonempty $Z_{i,l}^{(j)}$'s). We get:

$$\begin{aligned} \left| \sum_{\Pi} U - \sum_{\Pi_1} U - \sum_{\Pi_2} U \right| &= \left| \sum_{\xi_i \in \partial E_1 \cap \partial E_2} U(\xi_i; F_{i,1}, \dots, F_{i,k}) \right. \\ &\quad \left. - \sum_{j=1}^2 \left(\sum_{\xi_i \in \partial E_1 \cap \partial E_2} U(\xi_i; Z_{i,1}^{(j)}, \dots, Z_{i,k}^{(j)}) \right) \right| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}, \end{aligned}$$

thanks to **H1**). Thus we obtain:

$$\begin{aligned} \left| \sum_{\Pi} U - (GH_k) \int_{E_1} U - (GH_k) \int_{E_2} U \right| &\leq \sum_{j=1}^2 \left| \sum_{\Pi_j} U - (GH_k) \int_{E_j} U \right| \\ &\quad + \left| \sum_{\Pi} U - \sum_{\Pi_1} U - \sum_{\Pi_2} U \right| \\ &\leq 2 \bigvee_{i=1}^{\infty} e_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \end{aligned}$$

From this it follows that $U \in GH_k(E_1 \cup E_2)$ and

$$(GH_k) \int_{E_1 \cup E_2} U = (GH_k) \int_{E_1} U + (GH_k) \int_{E_2} U.$$

□

4 Convergence Theorems.

We begin with a version of the Saks-Henstock lemma in our abstract setting.

Lemma 4.1. *Let $U : T \times \mathcal{F}^k \rightarrow R$ be a map, GH_k integrable on T . Then there is a (D) -sequence $(a_{i,j})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge δ with*

$$\left| \sum_{s \in L} \left[U(\eta_s; Z_{s,1}, \dots, Z_{s,k}) - (GH_k) \int_{Y_s} U \right] \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \quad (7)$$

whenever $\Pi := \{(\eta_s; Z_{s,1}, \dots, Z_{s,k}) : s = 1, \dots, m\} = \{(\eta_s; Y_s) : s = 1, \dots, m\}$ is a δ -fine partition of T and $\emptyset \neq L \subset \{1, \dots, m\}$, with $Y_s = \cup_{t=1}^k Z_{s,t}$, $s = 1, \dots, m$.

PROOF. First of all we note that, by GH_k integrability of U on T , there is a (D) -sequence $(a_{i,j})_{i,j}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a gauge δ according to (5).

So, fix arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$, and let $\Pi := \{(\eta_s; Z_{s,1}, \dots, Z_{s,k}) : s = 1, \dots, m\} = \{(\eta_s; Y_s) : s = 1, \dots, m\}$ be a δ -fine partition of T , where $Y_s = \bigcup_{t=1}^k Z_{s,t}$ for all $s = 1, \dots, m$. By Proposition 3.6, $U \in GH_k(Y_s)$. There exists a (D) -sequence $(b_{i,j})_{i,j}$ such that for every $\psi \in \mathbb{N}^{\mathbb{N}}$, $\emptyset \neq L \subset \{1, \dots, m\}$ and $s \in L$, a gauge δ_s on Y_s can be found, such that $\delta_s(x) \subset \delta(x)$ for all $s = 1, \dots, m$ and $x \in Y_s$, and such that

$$\left| \sum_{s \notin L} \left[\sum_{\Pi_s} U - (GH_k) \int_{Y_s} U \right] \right| \leq \bigvee_{l=1}^{\infty} b_{l,\psi(l)},$$

whenever the involved partitions Π_s of Y_s are all δ_s -fine. Take now any δ_s -fine partition Π'_s of Y_s , for all $s \notin L$, and set $\Pi' = \{(\eta_s, Y_s) : s \in L\} \cup (\cup_{s \notin L} \Pi'_s)$. Then Π' is a δ -fine partition of T , hence

$$\left| (GH_k) \int_T U - \sum_{\Pi'} U \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

and

$$\left| \sum_{s \notin L} (GH_k) \int_{Y_s} U - \sum_{s \notin L} \sum_{\Pi'_s} U \right| \leq \bigvee_{l=1}^{\infty} b_{l,\psi(l)}.$$

Now

$$\begin{aligned} & \left| \sum_{s \in L} (GH_k) \int_{Y_s} U - \sum_{s \in L} U(\eta_s; Z_{s,1}, \dots, Z_{s,k}) \right| \\ &= \left| (GH_k) \int_T U - \sum_{s \notin L} (GH_k) \int_{Y_s} U - \sum_{\Pi'} U + \sum_{s \notin L} \sum_{\Pi'_s} U \right| \\ &\leq \left| (GH_k) \int_T U - \sum_{\Pi'} U \right| + \left| \sum_{s \notin L} \sum_{\Pi'_s} U - \sum_{s \notin L} (GH_k) \int_{Y_s} U \right| \\ &\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{l=1}^{\infty} b_{l,\psi(l)}. \end{aligned}$$

Since

$$\left| \sum_{s \in L} (GH_k) \int_{Y_s} U - \sum_{s \in L} U(\eta_s; Z_{s,1}, \dots, Z_{s,k}) \right| - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \leq \bigvee_{l=1}^{\infty} b_{l,\psi(l)}$$

for every $\psi \in \mathbb{N}^{\mathbb{N}}$, by weak σ -distributivity of R we obtain:

$$\left| \sum_{s \in L} (GH_k) \int_{Y_s} U - \sum_{s \in L} U(\eta_s; Z_{s,1}, \dots, Z_{s,k}) \right| - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \leq 0.$$

This concludes the proof. \square

We remark that the regulator $(a_{i,j})_{i,j}$ in (7) is the same which works for (GH_k) integrability in T .

We now formulate two properties, which will be useful in the sequel.

Let $x_0 \in T$, $T_0 := T \setminus \{x_0\}$, $U : T \times \mathcal{F}^k \rightarrow R$ be GH_k integrable for all $A \in \mathcal{F}$ with $A \subset T_0$ (here x_0 can be viewed as the *point at infinity* of the locally compact space T_0); fix a regulator $(a_{i,j})_{i,j}$ and $I \in R$, and let us introduce the following condition:

H2) for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds an open neighborhood \mathcal{V} of x_0 with

$$\left| (GH_k) \int_A U - I + U(x_0; \Xi_1, \dots, \Xi_k) \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \quad (8)$$

whenever $A \in \mathcal{F}$, $T \setminus A \subset \mathcal{V}$ and Ξ_1, \dots, Ξ_k are elements of \mathcal{F} with $\bigcup_{t=1}^k \Xi_t \subset \mathcal{V}$.

Observe that, in case **H2)** is satisfied, I is uniquely determined.

In the literature several situations are investigated, in which

$$U(x_0; \Xi_1, \dots, \Xi_k) = 0 \quad (9)$$

for every choice of $\Xi_j \in \mathcal{F}$, $j = 1, \dots, k$. In this case, **H2)** can be automatically replaced by a simpler condition (see for instance [4, 5]): in the classical situation when $T = [a, +\infty]$, with $a \in \mathbb{R}$, and $x_0 = +\infty$, this condition turns out to be equivalent to the existence in R of the limit

$$\lim_{c \rightarrow +\infty} (GH_k) \int_a^c U \quad (10)$$

(see also [4]). Finally, we note that, when $R = \mathbb{R}$: in the case $k = 1$, **H2** is equivalent to the existence in \mathbb{R} of the limit in [23], (1.11), p. 15; while for $k > 1$ **H2** is implied by the two conditions of existence in \mathbb{R} of the limit as in (10) and of "existence of the iterated limit J^- " formulated by A. G. Das and S. Kundu (see [12]).

We say that a function U , an element $u \in R$, $u \geq 0$ and a gauge γ_0 on T satisfy **H3**) if

$$\left| \sum_{\Pi} U - (GH_k) \int_{\cup_{i=1, \dots, q, t_i \neq x_0} E_i} U \right| \leq u$$

for every γ_0 -fine partition of T

$$\Pi := \{(t_1; F_{1,1}, \dots, F_{1,k}), \dots, (t_q; F_{q,1}, \dots, F_{q,k})\} = \{(t_1; E_1), \dots, (t_q; E_q)\}$$

with $\bigcup_{j=1}^k F_{i,j} = E_i, i = 1, \dots, q$.

Observe that, when $R = \mathbb{R}$ or $R = L^0(X, \mathcal{B}, \mu)$ with μ σ -additive and σ -finite, it is easy to find functions U satisfying **H3**) (see also [5]).

We now prove a version of the extension Cauchy theorem, which generalizes the corresponding result in [4].

Theorem 4.2. *Let: $T = T_0 \cup \{x_0\}$ be the one-point compactification of a locally compact space T_0 ; $U : T \times \mathcal{F}^k \rightarrow R$; $(A_n)_n$ be a sequence of sets, with $A_n \in \mathcal{F}$, $A_n \subset A_{n+1}$ ($n \in \mathbb{N}$), $\cup_{n=1}^{\infty} A_n = T_0$. Moreover, suppose that U is GH_k integrable on each subset $A \subset T_0$, with $A \in \mathcal{F}$, and that there are a regulator $(a_{i,j})_{i,j}$ and an element $I \in R$ satisfying **H2**).*

*Finally, assume that there exist $u \in R$, $u \geq 0$, and a gauge γ_0 , satisfying **H3**) together with U .*

Then U is GH_k integrable on T and $(GH_k) \int_T U = I$.

PROOF. Let $(a_{i,j})_{i,j}$ be as in the hypotheses of the theorem, γ_0 be according to **H3**), and choose arbitrarily an element $\varphi \in \mathbb{N}^{\mathbb{N}}$.

For every $n \in \mathbb{N}$ there exists a (D) -sequence $(b_{i,j}^{(n)})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gauge δ_n on A_n with $\delta_n(x) \subset \gamma_0(x)$ for each $n \in \mathbb{N}$ and $x \in A_n$, and

$$\left| (GH_k) \int_{A_n} U - \sum_{\Pi_n} U \right| \leq \bigvee_{i=1}^{\infty} b_{i, \varphi(i+n)}^{(n)} \tag{11}$$

for any δ_n -fine partition Π_n of A_n .

Put $C_1 = A_1$, $C_n = A_n \setminus A_{n-1}$, $n \geq 2$. For every $\xi \in T_0$ there exists exactly one $n = n(\xi) \in \mathbb{N}$ with $\xi \in C_n$. Choose now a gauge δ on T with the property that $\delta(x_0) \subset \mathcal{V}$ and $\delta(\xi) \subset \delta_{n(\xi)}(\xi) \cap \gamma_0(\xi)$, $\delta(\xi) \cap T_0 \subset \text{int } A_{n(\xi)}$ whenever $\xi \in T_0$.

Let

$$\Pi = \{(t_1; G_{1,1}, \dots, G_{1,k}), \dots, (t_q; G_{q,1}, \dots, G_{q,k})\} = \{(t_1; \mathcal{U}_1), \dots, (t_q; \mathcal{U}_q)\}$$

be any δ -fine partition of T , where $\bigcup_{j=1}^k G_{i,j} = \mathcal{U}_i$ for all $i = 1, \dots, q$. For

the sake of simplicity, as no confusion can arise, we sometimes write $U(t_i; \mathcal{U}_i)$ instead of $U(t_i; G_{i,1}, \dots, G_{i,k})$, $i = 1, \dots, q$. There exists $(t_{i_0}, \mathcal{U}_{i_0}) \in \Pi$, with $i_0 \in \{1, 2, \dots, q\}$, such that $x_0 \in \mathcal{U}_{i_0}$. Then $t_{i_0} = x_0$ (otherwise $x_0 \in \mathcal{U}_{i_0} \subset \delta(t_{i_0}) \subset \delta_n(t_{i_0})$ for some n ; but $\delta_n(t) \subset T_0$ for $t \neq x_0$, and thus we'd obtain $x_0 \in T_0$, a contradiction).

The Riemann sum $\sum_{\Pi} U$ has the form

$$\sum_{i \neq i_0} U(t_i; \mathcal{U}_i) + U(x_0; \mathcal{U}_{i_0}),$$

with $t_i \in T_0$ ($i = 1, \dots, q, i \neq i_0$). Let $A = \cup_{i \neq i_0} \mathcal{U}_i$: since Π is a δ -fine partition of T , we get $T \setminus A \subset \mathcal{V}$. By hypothesis, thanks to **H2**), we have

$$\left| (GH_k) \int_A U - I + U(x_0; \mathcal{U}_{i_0}) \right| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \quad (12)$$

for a suitable regulator $(a_{i,j})_{i,j}$ (independent on the choice of A) and $I \in R$. By virtue of Lemma 4.1,

$$\left| \sum_{t_i \in C_n} \left[U(t_i; \mathcal{U}_i) - (GH_k) \int_{\mathcal{U}_i} U \right] \right| \leq \bigvee_{i=1}^{\infty} b_{i, \varphi(i+n)}^{(n)}$$

for all $n \in \mathbb{N}$. By Proposition 3.6, we get

$$(GH_k) \int_A U = \sum_{i \neq i_0} (GH_k) \int_{\mathcal{U}_i} U.$$

Hence,

$$\begin{aligned} & \left| \sum_{i \neq i_0} U(t_i; \mathcal{U}_i) - (GH_k) \int_A U \right| = \left| \sum_{i \neq i_0} \left[U(t_i; \mathcal{U}_i) - (GH_k) \int_{\mathcal{U}_i} U \right] \right| \\ & \leq \sum_{n=1}^{\infty} \left| \sum_{t_i \in C_n} \left[U(t_i; \mathcal{U}_i) - (GH_k) \int_{\mathcal{U}_i} U \right] \right| \leq \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} b_{i, \varphi(i+n)}^{(n)} \right). \end{aligned}$$

Furthermore, thanks to **H2**) and **H3**), since the involved k -partition Π is γ_0 -fine, there is an element $0 \leq u^* \in R$ with

$$\left| \sum_{i \neq i_0} U(t_i; \mathcal{U}_i) - (GH_k) \int_A U \right| \leq u^*.$$

By the Fremlin Lemma 2.4, a (D) -sequence $(c_{i,j})_{i,j}$ can be found, satisfying

$$u^* \wedge \left(\sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} b_{i, \varphi(i+n)}^{(n)} \right) \right) \leq \bigvee_{i=1}^{\infty} c_{i, \varphi(i)} \quad \text{for all } \varphi \in \mathbb{N}^{\mathbb{N}}. \quad (13)$$

Thus we obtain:

$$\begin{aligned} \left| \sum_{\Pi} U - I \right| & \leq \left| \sum_{i \neq i_0} U(t_i; \mathcal{U}_i) + U(x_0; \mathcal{U}_{i_0}) - I \right| \\ & \leq \left| \sum_{i \neq i_0} U(t_i; \mathcal{U}_i) - (GH_k) \int_A U \right| + \left| (GH_k) \int_A U - I + U(x_0; \mathcal{U}_{i_0}) \right| \\ & \leq \left| \sum_{i \neq i_0} U(t_i; \mathcal{U}_i) - (GH_k) \int_A U \right| + \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}; \\ & \left| \sum_{i \neq i_0} U(t_i; \mathcal{U}_i) - (GH_k) \int_A U \right| \leq \bigvee_{i=1}^{\infty} c_{i, \varphi(i)}; \end{aligned}$$

and finally

$$\left| \sum_{\Pi} U - I \right| \leq \bigvee_{i=1}^{\infty} c_{i, \varphi(i)} + \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}.$$

This concludes the proof. \square

We now prove a version of the monotone convergence theorem. We begin with introducing the following concepts.

Definitions 4.3. • Let $(U_n : T \times \mathcal{F}^k \rightarrow R)_n$ be a sequence of GH_k integrable functions. We say that the U_n 's are GH_k *equiintegrable*, if there is a (D) -sequence $(b_{i,j})_{i,j}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there correspond a gauge ζ and an integer n_0 such that

$$\left| (GH_k) \int_T U_n - \sum_{\Pi} U_n \right| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for every ζ -fine partition Π and $n \geq n_0$.

- We say that the sequence $(U_n)_n$ is *equiconvergent* to $U : T \times \mathcal{F}^k \rightarrow R$ if there exist:

1) a function $h^* : T \times \mathcal{F}^k \rightarrow \mathbb{R}^+$, a gauge δ_0^* and a number $w \in \mathbb{R}^+$ such that for every δ_0^* -fine partition

$$\Pi^* := \{(t_i; F_{i,1}, \dots, F_{i,k}) : i = 1, \dots, q\} = \{(t_i; E_i) : i = 1, \dots, q\}$$

of T , with $\bigcup_{j=1}^k F_{i,j} = E_i$, $i = 1, \dots, q$, we get

$$\sum_{i=1}^q h^*(t_i; F_{i,1}, \dots, F_{i,k}) \leq w;$$

2) a (D) -sequence $(a_{i,j}^*)_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $t \in T$, a positive integer $p(t) \in \mathbb{N}$ can be found in such a way that, whenever $n \geq p(t)$ and $\Xi_1, \dots, \Xi_k \in \mathcal{F}$,

$$|U(t; \Xi_1, \dots, \Xi_k) - U_n(t; \Xi_1, \dots, \Xi_k)| \leq h^*(t; \Xi_1, \dots, \Xi_k) \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* \right). \quad (14)$$

Note that, when $k = 1$ and $T = [a, +\infty]$, $-\infty < a < +\infty$, $(U_n)_n$ is equiconvergent to U_0 as soon as it converges to U_0 pointwise "with respect to the same regulator" (in that case h^* can be defined (e.g.) by

$$h^*(t, \lambda) = \frac{\lambda}{1+t^2}, \quad t \in [a, +\infty]; \quad h^*(+\infty, \lambda) = 0, \quad (15)$$

see also [5, 9]).

We now are ready to prove the following result.

Theorem 4.4. *Let $(U_n : T \times \mathcal{F}^k \rightarrow R)_n$ be a sequence of GH_k equiintegrable functions, equiconvergent to U . Then U is (GH_k) integrable and*

$$\lim_n \int_T U_n = \int_T U.$$

PROOF. Let $(b_{i,j})_{i,j}$, ζ and n_0 be related with equiintegrability, and δ_0^* be associated with equiconvergence. By virtue of equiconvergence there is an element $w \in \mathbb{R}^+$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a gauge $\eta \subset \zeta \cap \delta_0^*$ such that, for each η -fine partition Π of T ,

$$\Pi = \{(t_i; F_{i,1}, \dots, F_{i,k}) : i = 1, \dots, q\} = \{(t_i; E_i) : i = 1, \dots, q\},$$

with $\bigcup_{j=1}^k F_{i,j} = E_i$ for all $i = 1, \dots, q$,

$$\begin{aligned} \left| \sum_{\Pi} U - \sum_{\Pi} U_n \right| &\leq \sum_{\Pi} |U(t_i; F_{i,1}, \dots, F_{i,k}) - U_n(t_i; F_{i,1}, \dots, F_{i,k})| \quad (16) \\ &\leq \sum_{i=1}^q h^*(t_i; F_{i,1}, \dots, F_{i,k}) \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* \right) \leq w \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* \right) \end{aligned}$$

whenever $n \geq \max\{p(t_i) : i = 1, \dots, q\}$. Put $a_{i,j} = w a_{i,j}^*$, $i, j \in \mathbb{N}$.

Without loss of generality, we can suppose that $p(t_i) \geq n_0$ for each $i = 1, \dots, q$. Then for a suitable (D) -sequence $(c_{i,j})_{i,j}$, for all η -fine partitions Π_1, Π_2 , we have definitely:

$$\begin{aligned} &\left| \sum_{\Pi_1} U - \sum_{\Pi_2} U \right| \leq \left| \sum_{\Pi_1} U - \sum_{\Pi_1} U_n \right| \\ &+ \left| \sum_{\Pi_1} U_n - (GH_k) \int_T U_n \right| + \left| (GH_k) \int_T U_n - \sum_{\Pi_2} U_n \right| \\ &+ \left| \sum_{\Pi_2} U_n - \sum_{\Pi_2} U \right| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \end{aligned}$$

GH_k integrability of U follows from this and the Cauchy criterion.

By equiconvergence, proceeding as in (16), we find a (D) -sequence $(c_{i,j})_{i,j}$ such that

$$\left| \sum_{\Pi} U - \sum_{\Pi} U_h \right| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

for definitely large h .

By GH_k integrability of U we obtain the existence of a (D) -sequence $(\bar{a}_{i,j})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a gauge η_1 , depending on φ , such that

$$\left| (GH_k) \int_T U - \sum_{\Pi} U \right| \leq \bigvee_{i=1}^{\infty} \bar{a}_{i,\varphi(i)}$$

for every η_1 -fine partition Π . By GH_k equiintegrability there is a (D) -sequence $(b_{i,j})_{i,j}$ such that

$$\left| \sum_{\Pi} U_h - (GH_k) \int_T U_h \right| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for h large enough and Π sufficiently fine.

Then for a suitable (D) -sequence $(d_{i,j})_{i,j}$ and for sufficiently fine partitions Π of T , we get:

$$\begin{aligned} & \left| (GH_k) \int_T U - (GH_k) \int_T U_h \right| \leq \left| (GH_k) \int_T U - \sum_{\Pi} U \right| \\ & + \left| \sum_{\Pi} U - \sum_{\Pi} U_h \right| + \left| \sum_{\Pi} U_h - (GH_k) \int_T U_h \right| \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)} \end{aligned}$$

for h large enough. Thus

$$(D) \lim_h \int_T U_h = \int_T U$$

with respect to the (D) -sequence $(d_{i,j})_{i,j}$. □

Before proving the monotone convergence theorem, let us introduce a further condition, similar to **H3**).

We say that a sequence $(U_n : T \times \mathcal{F}^k)_n$ satisfies **H3'**) if there exist $a \in \mathbb{R}$, $a \geq 0$, and a gauge $\hat{\gamma}$, such that, for every $\hat{\gamma}$ -fine partition Π of T ,

$$\left| \sum_{\Pi} U_n - (GH_k) \int_T U_n \right| \leq a \quad \text{for all } n \in \mathbb{N}.$$

We note that, by following the lines analogous to the ones explained in [6], Remark 1.2., pp. 56-57, it is possible to check that **H3'**) is satisfied when $k = 1$, $T = [a, b] \subset \mathbb{R}$, \mathcal{F} is the family of all finite unions of closed

subintervals of $[a, b]$; $R = \mathbb{R}$ (resp. $R = L^0(X, \mathcal{B}, \mu)$ with μ σ -additive and σ -finite), $U_n(x, [\alpha, \beta]) = f_n(x) \cdot [g(\beta) - g(\alpha)]$, $n \in \mathbb{N}$, where the f_n 's are taken Kurzweil-Henstock integrable, with increasing behaviour and convergent pointwise (resp. "w. r. to the same regulator"), and g is positive and increasing (see also [5, 12]).

We now demonstrate the Beppo-Levi monotone convergence theorem in our setting.

Theorem 4.5. *Let $(U_n : T \times \mathcal{F}^k \rightarrow R)_n$ be a sequence of GH_k integrable functions satisfying **H3'**, $U_n \leq U_{n+1}$ ($n \in \mathbb{N}$), equiconvergent to $U_0 : T \times \mathcal{F}^k \rightarrow R$, and let the sequence $\left((GH_k) \int_T U_n \right)_n$ be bounded. Then U is (GH_k) integrable on T and*

$$(GH_k) \int_T U = (D) \lim_n (GH_k) \int_T U_n.$$

PROOF. Since the sequence $\left((GH_k) \int_T U_n \right)_n$ is bounded and increasing, it admits the (D) -limit in R . Thus, there exists a (D) -sequence $(c_{i,j})_{i,j}$ in R such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a positive integer h_0 satisfying

$$\sup_{h,l > h_0} \left| (GH_k) \int_T U_h - (GH_k) \int_T U_l \right| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \tag{17}$$

Furthermore, from equiconvergence we get the existence of an element $w \in \mathbb{R}^+$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gauge γ^* such that, for every γ^* -fine partition Π of T , $\Pi = \{(t_i; F_{i,1}, \dots, F_{i,k}) : i = 1, \dots, q\} = \{(t_i; E_i) : i = 1, \dots, q\}$, with $\bigcup_{j=1}^k F_{i,j} = E_i$, $i = 1, \dots, q$, we have:

$$\begin{aligned} & \sum_{i=1}^q |U(t_i; F_{i,1}, \dots, F_{i,k}) - U_{p(t_i)}(t_i; F_{i,1}, \dots, F_{i,k})| \\ & \leq \sum_{i=0}^q h^*(t_i; F_{i,1}, \dots, F_{i,k}) \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* \right) \leq w \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* \right), \end{aligned} \tag{18}$$

as soon as the numbers $p(t_i)$ are sufficiently large (in particular greater than h_0). Since U_h is integrable for any $h \in \mathbb{N}$, then for each $h \in \mathbb{N}$ there exists a

(D)-sequence $(a_{i,j}^{(h)})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, a gauge γ_h can be found such that, for all γ_h -fine partitions Π of T ,

$$\left| \sum_{\Pi} U_h - (GH_k) \int_T U_h \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i+h+1)}^{(h)}. \quad (19)$$

For each $i, j \in \mathbb{N}$, put $b_{i,j}^{(1)} = 2w a_{i,j}^*$ and $b_{i,j}^{(m)} = a_{i,j}^{(m-1)} \vee a_{i,j}^{(m)}$ ($m = 2, 3, \dots$). Moreover, by the Fremlin Lemma 2.4 we can find a (D)-sequence $(b_{i,j})_{i,j}$ such that, for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $s \in \mathbb{N}$,

$$a \wedge \left(\sum_{m=1}^s \left(\bigvee_{i=1}^{\infty} b_{i,\varphi(i+m)}^{(m)} \right) \right) \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}, \quad (20)$$

where a is related to **H3'**). Let $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $h_0 = h_0(\varphi)$ be as in (17). Put

$$\gamma_0(t) = \gamma^*(t) \cap \hat{\gamma}(t) \cap \gamma_1(t) \cap \gamma_2(t) \cap \dots \cap \gamma_{p(t)}(t),$$

where the involved gauges are the ones associated with φ , as above. Choose any γ_0 -fine partition

$$\Pi = \{(t_i; F_{i,1}, \dots, F_{i,k}) : i = 1, \dots, q\} = \{(t_i; E_i) : i = 1, \dots, q\}.$$

Fix arbitrarily $h \geq h_0$. We have:

$$\begin{aligned} & \left| \sum_{\Pi} U_h - (GH_k) \int_T U_h \right| \\ & \leq \left| \sum_{p(t_i) \geq h} U_h(t_i; F_{i,1}, \dots, F_{i,k}) - \sum_{p(t_i) \geq h} (GH_k) \int_{E_i} U_h \right| \\ & + \left| \sum_{p(t_i) < h} U_h(t_i; F_{i,1}, \dots, F_{i,k}) - \sum_{p(t_i) < h} (GH_k) \int_{E_i} U_h \right|. \quad (21) \end{aligned}$$

Let $\tilde{\Pi} = \{(t_i; E_i) : h \leq p(t_i)\} \cup (\cup_{p(t_i) < h} \Pi_i)$, where Π_i is a sufficiently fine partition of E_i , in such a way that $\tilde{\Pi}$ is a γ_k -fine partition of T . Then

$$\left| \sum_{\tilde{\Pi}} U_h - (GH_k) \int_T U_h \right| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i+h+1)}^{(h)}.$$

Hence, by virtue of the Saks-Henstock lemma, we obtain

$$\left| \sum_{p(t_i) \geq h} U_h(t_i; F_{i,1}, \dots, F_{i,k}) - \sum_{p(t_i) \geq h} (GH_k) \int_{E_i} U_h \right| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i+h+1)}^{(h)}. \quad (22)$$

We now estimate the second part of the right side of (21). We have:

$$\begin{aligned} & \left| \sum_{p(t_i) < h} U_h(t_i; F_{i,1}, \dots, F_{i,k}) - \sum_{p(t_i) < h} (GH_k) \int_{E_i} U_h \right| \\ & \leq \left| \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} U_h(t_i; F_{i,1}, \dots, F_{i,k}) - \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} U_m(t_i; F_{i,1}, \dots, F_{i,k}) \right| + \\ & + \left| \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} U_m(t_i; F_{i,1}, \dots, F_{i,k}) - \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} (GH_k) \int_{E_i} U_m \right| \\ & + \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} (GH_k) \int_{E_i} (U_h - U_m) \\ & \leq \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} \{U_h(t_i; F_{i,1}, \dots, F_{i,k}) - U_m(t_i; F_{i,1}, \dots, F_{i,k})\} \\ & + \sum_{m=h_0}^{h-1} \left| \sum_{p(t_i)=m} U_m(t_i; F_{i,1}, \dots, F_{i,k}) - \sum_{p(t_i)=m} (GH_k) \int_{E_i} U_m \right| \quad (23) \\ & + \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} (GH_k) \int_{E_i} (U_h - U_m) \\ & \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i+1)}^{(1)} + \sum_{m=h_0}^{h-1} \bigvee_{i=1}^{\infty} b_{i,\varphi(i+m+1)}^{(m)} + (GH_k) \int_T (U_h - U_{h_0}) \\ & \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i+1)}^{(1)} + \sum_{m=1}^h \bigvee_{i=1}^{\infty} b_{i,\varphi(i+m)}^{(m)} + (GH_k) \int_T (U_h - U_{h_0}) \\ & = \sum_{m=1}^h \left(\bigvee_{i=1}^{\infty} b_{i,\varphi(i+m)}^{(m)} \right) + (GH_k) \int_T (U_h - U_{h_0}). \end{aligned}$$

Thus, from (17), (20) and (23) we find a (D) -sequence $(d_{i,j})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exist a gauge γ_0 and an integer h_0 such that, for each

γ_0 -fine partition Π and $h \geq h_0$,

$$\left| \sum_{\Pi} U_h - (GH_k) \int_T U_h \right| \leq \bigvee_{i=1}^{\infty} d_{i, \varphi(i)}. \quad (24)$$

The assertion follows from Lemma 4.4. \square

We now prove a version of the Lebesgue dominated convergence theorem.

Theorem 4.6. *Let $(U_n : T \times \mathcal{F}^k \rightarrow R)_n$ be a sequence of GH_k integrable functions equiconvergent to $U : T \times \mathcal{F}^k \rightarrow R$, such that $\bigvee_{n \in P_1, m \in P_2} |U_n - U_m|$ is GH_k integrable for every $P_1, P_2 \subset \mathbb{N}$. Then U is GH_k integrable and*

$$(GH_k) \int_T U = (D) \lim_n (GH_k) \int_T U_n.$$

PROOF. For all $s \in \mathbb{N}$ and $h \geq s$, put

$$g_{s,h} = \bigvee_{s \leq \min(n,m) \leq h} |U_n - U_m|;$$

moreover, for each $s \in \mathbb{N}$, set

$$g_s = \bigvee_{n,m \geq s} |U_n - U_m|.$$

We shall prove that, for each fixed $s \in \mathbb{N}$, the sequence $(g_{s,h})_{h \geq s}$ satisfies the hypothesis of Theorem 4.5.

First of all, we observe that the sequence

$$\left((GH_k) \int_T g_{s,h} \right)_h$$

is well-defined and bounded in R .

Fix now arbitrarily $s \in \mathbb{N}$. We have, for $h \geq s$:

$$\begin{aligned} g_s &= \bigvee_{n,m \geq s} |U_n - U_m| \\ &= \left(\bigvee_{s \leq \min(n,m) \leq h} |U_n - U_m| \right) \bigvee \left(\bigvee_{n,m \geq h} |U_n - U_m| \right) \\ &\leq \left(\bigvee_{s \leq \min(n,m) \leq h} |U_n - U_m| \right) + \left(\bigvee_{n,m \geq h} |U_n - U_m| \right), \end{aligned}$$

and hence

$$0 \leq g_s - g_{s,h} \leq \bigvee_{n,m \geq h} |U_n - U_m|.$$

From equiconvergence of the sequence $(U_n)_n$ it follows that the sequence $(g_{s,h})_h$ is equiconvergent too, where the rôle of the "limit function" is played by g_s .

We now turn to **H3'**). As $\bigvee_{n,m \geq s} |U_n - U_m|$ is GH_k integrable, there exist a gauge $\hat{\gamma}$ and a positive element $a^* \in R$ such that, for every $\hat{\gamma}$ -fine partition of T

$$\Pi := \{(t_i; F_{i,1}, \dots, F_{i,k}) : i = 1, \dots, q\} = \{(t_i; E_i) : i = 1, \dots, q\}$$

with $\bigcup_{j=1}^k F_{i,j} = E_i$, $i = 1, \dots, q$, for all $s \in \mathbb{N}$ and $h \geq s$, we get:

$$\sum_{i=1}^q \left[\bigvee_{s \leq \min(n,m) \leq h} |U_n(t_i; F_{i,1}, \dots, F_{i,k}) - U_m(t_i; F_{i,1}, \dots, F_{i,k})| \right] \leq a^*, \quad (25)$$

that is

$$\sum_{i=1}^q g_{s,h}(t_i; F_{i,1}, \dots, F_{i,k}) \leq a^*.$$

From this it follows that **H3'**) is satisfied. Thus g_s is GH_k integrable for every $s \in \mathbb{N}$ and

$$\int_T g_s = \bigvee_{h \geq s} (GH_k) \int_T g_{s,h}.$$

We now prove that the sequence $(-g_s)_s$ satisfies the hypotheses of Theorem 4.5.

First of all, it is easy to check that the sequence $\left((GH_k) \int_T g_s \right)_s$ is bounded. Moreover, since

$$g_s = |-g_s| = \bigvee_{n,m \geq s} |U_n - U_m|$$

and the sequence $(U_n)_n$ is equiconvergent, then the sequence $(-g_s)_s$ is too, where the rôle of the "limit function" is played by the null function.

Concerning **H3'**), it is enough to take suitable limits in (25). Thus,

$$\lim_s (GH_k) \int_T g_s = \bigwedge_{s \in \mathbb{N}} (GH_k) \int_T g_s = 0. \quad (26)$$

Proceeding analogously as in the proof of Theorem 4.5, it is possible to find (D) -sequences $(e_{i,j}^{(m)})_{i,j}$, $m \in \mathbb{N}$, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist a gauge γ' and $h' \in \mathbb{N}$ such that, for each γ' -fine partition

$$\Pi := \{(t_i; F_{i,1}, \dots, F_{i,k}) : i = 1, \dots, q\} = \{(t_i; E_i) : i = 1, \dots, q\}$$

and for all $h > h'$, we have:

$$\begin{aligned} & \left| \sum_{\Pi} U_h - (GH_k) \int_T U_h \right| \\ & \leq \sum_{m=1}^h \left(\bigvee_{i=1}^{\infty} e_{i,\varphi(i+m)}^{(m)} \right) + \sum_{m=h'}^{h-1} \sum_{p(t_i)=m} \left| (GH_k) \int_{E_i} (U_h - U_m) \right| \\ & \leq \sum_{m=1}^h \left(\bigvee_{i=1}^{\infty} e_{i,\varphi(i+m)}^{(m)} \right) + (GH_k) \int_T g_{h'}, \end{aligned}$$

and thus a (D) -sequence $(d'_{i,j})_{i,j}$ can be found, such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist a gauge γ' and $h' \in \mathbb{N}$ such that, for any γ' -fine partition Π and $h > h'$,

$$\left| \sum_{\Pi} U_h - (GH_k) \int_T U_h \right| \leq \bigvee_{i=1}^{\infty} d'_{i,\varphi(i)}.$$

This means that the functions U_n , $n \in \mathbb{N}$, are equiintegrable. Thus we get the assertion, thanks to Theorem 4.4. \square

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