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LINEAR INTEGRAL EQUATIONS OF VOLTERRA CONCERNING HENSTOCK INTEGRALS

Abstract

We establish conditions for the existence of solutions of the linear integral equation of Volterra

$$x\,(t) +^{*} \int_{[a,t]} \alpha(s)x(s)\,ds = f(t), \quad t \in [a,b], \qquad (V_{*})$$

where the functions are Banach space-valued and $^{*}\int$ denotes either the Bochner-Lebesgue or the Henstock integral. In some cases it is possible to calculate the solution of $(V)_{*}$ explicitly. We give several examples.

1 Introduction

In the literature, the study of Integral Equations deals mainly with the integrals of Riemann, Lebesgue or Dushnik, (the latter is also called the interior integral - see [9] or [10]). However these integrals have some deficiencies. The Riemann integral is weak, the classical definition of the Lebesgue integral may be difficult to deal with, and the vector integral of Dushnik, though more general than the Riemann-Stieltjes integral, may not coincide with the Kurzweil-Henstock vector integral which, in turn, is more general than both the Riemann-Stieltjes and the Lebesgue-Stieltjes integrals. On the other hand, when we consider integral equations in the sense of the Kurzweil-Henstock integrals, we benefit from its easy to handle Riemannian definition and well-known good properties.

The aim of this paper is to give conditions for the existence of a solution of the linear integral equation of Volterra

$$x(t) +^{*} \int_{[a,t]} \alpha(s)x(s)\,ds = f(t), \quad t \in [a,b], \qquad (V_{*})$$

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in the sense of the variational integral of Henstock which coincides with the integral of Kurzweil in the real-valued case. We also obtain special results for the equation $(V)_*$ when the integral is that of Bochner-Lebesgue.

We work in a general Banach space-valued context. Let $[a,b]$ be a compact interval of $\mathbb{R}$, X be a Banach space and $L(X) = L(X,X)$ be the space of linear continuous functions from X to X. Let x and f be functions from $[a,b]$ to X and let α be a function from $[a,b]$ to $L(X)$. As we intend that the kernel α of $(V)_*$ is weak enough so that discontinuities, singularities, infinite variation or nonabsolute integrability can be taken into account, we consider α Henstock integrable. We consider the functions x, $f : [a,b] \to X$ where X is a Banach space in order to use fixed point theorems and we prove that if either x is a continuous function or x is of bounded variation, then $\alpha x : [a,b] \to X$ is Henstock integrable (see Theorem 2.5 and [6]).

It is a well-known result that if α is Henstock integrable, then there exists a sequence of closed sets $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \uparrow [a,b]$, (i.e., $X_n \subset X_{n+1} \subset [a,b]$, for every $n \in \mathbb{N}$ and $\cup X_n = [a,b]$), and the restriction of α to X_n, (we write $\alpha|_{X_n}$), is Bochner-Lebesgue integrable for every $n \in \mathbb{N}$, (see, for instance, [16], Th. 2.10). However, it is a recent result, (see [7]), that

$$\lim_{n\to\circ} {}^L\!\int_{X_n\cap[a,t]} \alpha = {}^K\!\int_{[a,t]} \alpha \tag{1.1}$$

uniformly for every $t \in [a,b]$, where ${}^L\!\int$ and ${}^K\!\int$ denote respectively the integrals of Bochner-Lebesgue and of Henstock, (we use K for Kurzweil). From the Contraction Principle we can deduce the existence and uniqueness of a solution of $(V)_*$ in the sense of the Bochner-Lebesgue integral, provided ${}^L\!\int_{[a,b]} \|\alpha(\ )\| < 1$. Then we obtain conclusions about the existence of a solution of $(V)_*$ in the sense of the Henstock integral by applying a fixed point theorem for sequences of mappings corresponding to the sequence of equations $(V)_*$ in the sense of the Bochner-Lebesgue integral obtained through (1.1) and such that ${}^L\!\int_{X_n\cap[a,t]} \|\alpha(\ )\| < 1$, for every $n \in \mathbb{N}$.

Since the space of Bochner-Lebesgue integrable functions is a subspace of the space of Henstock integrable ones, then we obtain similar but stronger results when we suppose that α is Bochner-Lebesgue integrable.

In Section 1 we give the basic definitions and the introductory results. The main theorems lie on Section 2. Sections 3 and 4 consist of consequences and applications of the main results.

2 The Integrals of Kurzweil and Henstock

Let $[a,b]$ be a compact interval of $\mathbb{R}$. We say that $d = (\xi_i, t_i)$ is a tagged division of $[a,b]$ whenever (t_i) is a division of $[a,b]$ (i.e., $a = t_0 < t_1 < \ldots < t_n = b$) and $x_i \in [t_{i-1}, t_i]$, for every i. We denote by $TD_{[a,b]}$ the set of all tagged divisions of $[a,b]$. A gauge of $[a,b]$ is a function $\delta : [a,b] \to (0,\infty)$ and $d = (t_i) \in TD_{[a,b]}$ is δ-fine if for every i, $[t_{i-1}, t_i] \subset B_{\delta(\xi_i)}(\xi_i) = \{t \in [a,b]; |t - \xi_i| < \delta(\xi_i)\}$.

In what follows X and Y are Banach spaces, $L(X,Y)$ is the Banach space of linear continuous functions from X to Y, $L(X) = L(X,X)$ and $X' = L(X,\mathbb{R})$.

Given functions $f : [a,b] \to X$ and $\alpha : [a,b] \to L(X,Y)$, we say that f is Kurzweil α-integrable, (we write $f \in K^{\alpha}([a,b],X)$), and that $I \in Y$ is its integral, (we write $I = {}^K\int_{[a,b]} d\alpha\, f = {}^K\int_{[a,b]} d\alpha(t) f(t)$), if given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (t_i) \in TD_{[a,b]}$,

$$\left\| \sum_i [\alpha(t_i) - \alpha(t_{i-1})] f(\xi_i) - I \right\| < \varepsilon.$$

We say that f is Henstock α-integrable or variational α-integrable, (we write $f \in H^{\alpha}([a,b],X)$), if there is a function $F^{\alpha} : [a,b] \to Y$, (called the associated function of f with respect to α), such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (t_i) \in TD_{[a,b]}$,

$$\sum_i \|[\alpha(t_i) - \alpha(t_{i-1})] f(\xi_i) - [F^{\alpha}(t_i) - F^{\alpha}(t_{i-1})]\| < \varepsilon.$$

Clearly $H^{\alpha}([a,b],X) \subset K^{\alpha}([a,b],X)$ and if X is of finite dimension, then $H^{\alpha}([a,b],X) = K^{\alpha}([a,b],X)$. We denote by $\widetilde{f}^{\alpha}$ the indefinite integral of $f \in K^{\alpha}([a,b],X)$ that is, $\widetilde{f}^{\alpha}(t) = {}^K\int_{[a,t]} d\alpha\, f$ for every $t \in [a,b]$. When $\alpha(t) = t$, then we replace $K^{\alpha}([a,b],X)$, $H^{\alpha}([a,b],X)$ and $\widetilde{f}^{\alpha}$ respectively by $K([a,b],X)$, $H([a,b],X)$ and $\widetilde{f}$.

Given $A \subset [a,b]$, let χ_A denote the characteristic function of A and $f|_A$ denote the restriction of f to A. Let $\alpha : [a,b] \to L(X,Y)$ be a function and $[c,d] \subset [a,b]$. The following properties are not difficult to prove.

i) If $f \in K^{\alpha}([a,b],X)$ (resp. $f \in H^{\alpha}([a,b],X)$), then $f \in K^{\alpha}([c,b],X)$ (resp. $f \in H^{\alpha}([c,d],X)$).

ii) If $f \in K^{\alpha}([a,b],X)$, then ${}^K\int_{[a,b]} d\alpha(t)\, \chi_{[c,d]}(t)\, f(t) = {}^K\int_{[c,d]} d\alpha(t)\, f(t)$.

iii) Given $f \in K^{\alpha}([c,d],X)$, let $\widehat{f} : [a,b] \to X$ be such that $\widehat{f}\big|_{[c,d]} = f$ and $\widehat{f}(t) = 0$ otherwise. Then ${}^K\int_{[a,b]} d\alpha(t)\, \widehat{f}(t) = {}^K\int_{[c,d]} d\alpha(t)\, f(t)$.

Let $C([a,b],X)$ and $G([a,b],X)$ be respectively the Banach spaces of continuous and of regulated functions from $[a,b]$ to X endowed with the supremum norm which we denote by $\|\|_{\infty}$. We denote respectively by $f(\xi+)$ and by $f(\xi-)$ the right and left limits of $f:[a,b]\to X$ at $\xi\in[a,b]$ when they are defined and exist. Let $C^{\sigma}([a,b],L(X,Y))$ be the set of all functions $\alpha:[a,b]\to L(X,Y)$ that are weakly continuous, (i.e., for every $x\in X$, the function $t\in[a,b]\to\alpha(t)x\in Y$ is continuous), and let and $G^{\sigma}([a,b],L(X,Y))$ be the set of all weakly regulated functions $\alpha:[a,b]\to L(X,Y)$ (i.e., for every $x\in X$, the function $t\in[a,b]\to\alpha(t)x\in Y$ is regulated). Given $x\in X$, let $\alpha\left(\xi\hat{+}\right)x=\lim\limits_{\rho\downarrow 0}(\alpha x)(\xi+)$, for every $\xi\in[a,b)$, and let $\alpha\left(\xi\hat{-}\right)x=\lim\limits_{\rho\downarrow 0}(\alpha x)(\xi-)$, for every $\xi\in(a,b]$. By the Banach-Steinhauss Theorem, $\alpha\left(\xi\hat{+}\right)$ and $\alpha\left(\xi\hat{-}\right)$ exist and belong to $L(X,Y)$. Then by the Uniform Boundedness Principle it follows that $G^{\sigma}([a,b],L(X,Y))$ is a Banach space when equipped with the supremum norm. Let $SV([a,b],L(X,Y))$ be the space of all functions $\alpha:[a,b]\to L(X,Y)$ of bounded semivariation (also called of bounded B-variation - see [19]) with semivariation denoted by $SV(\alpha)$ and let $BV([a,b],X)$ be the space of all functions $f:[a,b]\to X$ of bounded variation with variation denoted by $V(f)$. Then $BV([a,b],L(X,Y))\subset SV([a,b],L(X,Y))$ and $SV([a,b],L(X,\mathbb{R}))=BV([a,b],X')$. Moreover, $SV([a,b],L(X))=BV([a,b],L(X))$ if X is of finite dimension. When endowed with the norm given by the variation, the space $BV_a([a,b],X)=\{f\in BV([a,b],X);f(a)=0\}$ is complete. For more information about the above spaces see [8], [9] or [10].

The following result is the analogous of Saks-Henstock Lemma for the Stieltjes case. Its proof follows the standard steps (see, for instance, [19])

Lemma 2.1. *(Saks-Henstock Lemma) Let $\alpha:[a,b]\to L(X,Y)$ and $f\in K^{\alpha}([a,b],X)$. If for $\varepsilon>0$, the gauge δ of $[a,b]$ is such that for every δ-fine $d=(t_i)\in TD_{[a,b]}$,*

$$\left\|\sum_{i=1}^{|d|}[\alpha(t_i)-\alpha(t_{i-1})]f(\xi_i)-{}^{K}\!\int_{[a,b]}d\alpha(t)\,f(t)\right\|<\varepsilon,$$

then for $a\le c_1\le\eta_1\le d_1\le c_2\le\eta_2\le d_2\le\ldots\le c_k\le\eta_k\le d_k\le b$ with $[c_j,d_j]\subset(\eta_j-\delta(\eta_j),\eta_j+\delta(\eta_j))$ for every j,

$$\left\|\sum_{j=1}^{k}\left\{[\alpha(d_j)-\alpha(c_j)]f(\eta_j)-{}^{K}\!\int_{[c_j,d_j]}d\alpha(t)\,f(t)\right\}\right\|\le\varepsilon.$$

Theorem 2.2. *([2], Th. 2.1.10) If* $\alpha \in G^{\sigma}([a,b], L(X,Y))$ *(respectively* $\alpha \in C^{\sigma}([a,b], L(X,Y))$*) and* $f \in K^{\alpha}([a,b], X)$*, then* $\widetilde{f}^{\alpha} \in G([a,b], X)$ *(resp.* $\widetilde{f}^{\alpha} \in C([a,b], X)$*).*

Proof. In what follows we prove that

$$\widetilde{f}^{\alpha}(\xi+) - \widetilde{f}^{\alpha}(\xi) = [\alpha(\xi\hat{+}) - \alpha(\xi)] f(\xi),$$

for every $\xi \in [a,b)$. The proof that $\widetilde{f}^{\alpha}(\xi) - \widetilde{f}^{\alpha}(\xi-) = [\alpha(\xi) - \alpha(\xi\hat{-})] f(\xi)$, for every $\xi \in (a,b]$, follows in an analogous way.

By hypothesis, $f \in K^{\alpha}([a,b], X)$. Hence, given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (t_i) \in TD_{[a,b]}$,

$$\| \sum_i [\alpha(t_i) - \alpha(t_{i-1})] f(\xi_i) - {}^{K}\!\!\int_{[a,b]} d\alpha f \| < \frac{\varepsilon}{2}.$$

Let $\xi \in [a,b)$. Since $\alpha \in G^{\sigma}([a,b], L(X,Y))$, there exists $(\alpha x)(\xi+)$, for every $x \in X$. In particular, there exists $\mu > 0$, such that for every $0 < \rho < \mu$,

$$\|[\alpha(\xi+\rho) - \alpha(\xi\hat{+})] f(\xi)\| < \frac{\varepsilon}{2}.$$

If $\delta(\xi) < \mu$ and $0 < \rho < \delta(\xi)$, then Lemma 2.1 implies that

$$\|[\alpha(\xi+\rho) - \alpha(\xi)] f(\xi) - {}^{K}\!\!\int_{[\xi,\xi+\rho]} d\alpha\, f\| \le \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned}
&\left\| \widetilde{f}^{\alpha}(\xi+) - \widetilde{f}^{\alpha}(\xi) - [\alpha(\xi\hat{+}) - \alpha(\xi)] f(\xi) \right\| \\
=&\left\| {}^{K}\!\!\int_{[\xi,\xi+\rho]} d\alpha\, f - [\alpha(\xi\hat{+}) - \alpha(\xi)] f(\xi) \right\| \\
\le&\left\| {}^{K}\!\!\int_{[\xi,\xi+\rho]} d\alpha\, f - [\alpha(\xi+\rho) - \alpha(\xi)] f(\xi) \right\| \\
&+ \left\| [\alpha(\xi+\rho) - \alpha(\xi)] f(\xi) - [\alpha(\xi\hat{+}) - \alpha(\xi)] f(\xi) \right\| < \varepsilon. \quad \square
\end{aligned}$$

Given a division $d = (t_i)$ of $[a,b]$, let $\Delta d = \max_i \{t_i - t_{i-1}\}$. Then the Riemann-Stieltjes integrals are given by

$$\int_{[a,b]} d\alpha\, f = \int_{[a,b]} d\alpha(t)\, f(t) = \lim_{\Delta d \to 0} \sum_i [\alpha(t_i) - \alpha(t_{i-1})\, f(\xi_i)]$$

and

$$\int_{[a,b]} \alpha\, df = \int_{[a,b]} \alpha(t)\, df(t) = \lim_{\Delta d \to 0} \sum_i \alpha(\xi_i)\, [f(t_i) - f(t_{i-1})].$$

The following assertion is well known.

Theorem 2.3. *([8], I.3.4 and I.4.5 or [9], I.4.6, I.4.12, I.4.19 and I.4.20)*

i) If $\alpha \in SV([a,b], L(X,Y))$ and $f \in C([a,b], X)$, then $\int_{[a,b]} d\alpha\, f$ exists and $\left\| \int_{[a,b]} d\alpha\, f \right\| \leq SV(\alpha)\, \|f\|_\infty$.

ii) If $\alpha \in C([a,b], L(X,Y))$ and $f \in BV([a,b], X)$, then $\int_{[a,b]} \alpha\, df$ exists and $\left\| \int_{[a,b]} \alpha\, df \right\| \leq \|\alpha\|_\infty\, V(f)$.

Theorem 2.4. *([2], Th. 1.4.1) $\int_{[a,b]} d\alpha\, f$ exists if and only if $\int_{[a,b]} \alpha\, df$ exists and, in this case, the Integration by Parts Formula $\int_{[a,b]} d\alpha\, f = \alpha(b) f(b) - \alpha(a) f(a) - \int_{[a,b]} \alpha\, df$ holds.*

PROOF. Suppose that the Riemann-Stieltjes integral, $\int_{[a,b]} d\alpha\, f$, exists. Then, for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every $d = (t_i) \in TD_{[a,b]}$ with $\Delta d = \max_i \{t_i - t_{i-1}\} < \delta$,

$$\left\| \sum_i [\alpha(t_i) - \alpha(t_{i-1})] f(\xi_i) - \int_{[a,b]} d\alpha\, f \right\| < \varepsilon.$$

Hence,

$$\begin{aligned} &\left\| \left\{ \alpha(b) f(b) - \alpha(a) f(a) - \int_{[a,b]} d\alpha\, f \right\} - \sum_i \alpha(\xi_i)\, [f(t_i) - f(t_{i-1})] \right\| \\ =& \left\| \sum_i [\alpha(t_i) f(t_i) - \alpha(t_{i-1}) f(t_{i-1})] - \int_{[a,b]} d\alpha\, f - \sum_i \alpha(\xi_i)\, [f(t_i) - f(t_{i-1})] \right\| \\ =& \left\| \sum_i [\alpha(t_i) - \alpha(\xi_i)] f(t_i) + \sum_i [\alpha(\xi_i) - \alpha(t_{i-1})] f(t_{i-1}) \right. \\ &\left. - \int_{[a,b]} d\alpha\, f - \sum_i \alpha(\xi_i)\, [f(t_i) - f(t_{i-1})] \right\| < \varepsilon. \end{aligned}$$

Analogously, if $\int_{[a,b]} \alpha\, df$ exists, then $\int_{[a,b]} d\alpha\, f$ exists with $\int_{[a,b]} d\alpha\, f = \alpha(b) f(b) - \alpha(a) f(a) - \int_{[a,b]} \alpha\, df$ and the proof is complete. □

The next assertion follows from Theorems 2.3 and 2.4.

Theorem 2.5. *The Riemann-Stieltjes integrals* $\int_{[a,b]} d\alpha\, f$ *and* $\int_{[a,b]} \alpha\, df$ *exists and the Integration by Parts Formula holds if one of the following conditions is satisfied:*

i) $\alpha \in SV\,([a,b], L(X,Y))$ *and* $f \in C([a,b], X)$*;*

ii) $\alpha \in C\,([a,b], L(X,Y))$ *and* $f \in BV([a,b], X)$*.*

Theorem 2.6. *([19], Th. 15) If* $\alpha \in SV\,([a,b], L(X,Y)) \cap G^{\sigma}\,([a,b], L(X,Y))$ *and* $f \in G([a,b], X)$*, then* $f \in K^{\alpha}([a,b], X)$*.*

The next result comes from the definitions.

Theorem 2.7. *Let* $\alpha \in H\,([a,b], L(X,Y))$ *and* $f \in K^{\widetilde{\alpha}}([a,b], X)$ *(respectively* $f \in H^{\widetilde{\alpha}}([a,b], X)$*). If* f *is bounded, then* $\alpha f \in K\,([a,b], Y)$ *(resp.* $\alpha f \in H\,([a,b], Y)$*) and* ${}^{K}\int_{[a,b]} \alpha\, f = {}^{K}\int_{[a,b]} d\widetilde{\alpha}\, f$*, where* $\widetilde{\alpha}$ *denotes the indefinite integral of* α*.*

Corollary 2.8. *Let* $\alpha \in H\,([a,b], L(X,Y))$ *with* $\widetilde{\alpha} \in SV\,([a,b], L(X,Y))$ *and* $f \in G([a,b], X)$ *(respectively* $f \in C([a,b], X)$*). Then* $\alpha f \in K\,([a,b], Y)$ *with* ${}^{K}\int_{[a,b]} \alpha\, f = {}^{K}\int_{[a,b]} d\widetilde{\alpha}\, f$ *(resp.* ${}^{K}\int_{[a,b]} \alpha\, f = \int_{[a,b]} d\widetilde{\alpha}\, f$ *).*

PROOF. Suppose that $f \in G([a,b], X)$. Then f is bounded. From Theorem 2.2, $\widetilde{\alpha} \in C\,([a,b], L(X,Y))$. Hence by Theorem 2.6, $f \in K^{\widetilde{\alpha}}([a,b], X)$. Suppose now that $f \in C([a,b], X)$. By Theorems 2.2 and 2.5, $f \in R^{\widetilde{\alpha}}([a,b], X)$. But $R^{\widetilde{\alpha}}([a,b], X) \subset K^{\widetilde{\alpha}}([a,b], X)$ (see the comments bellow) and the proof is complete. □

If we take constant gauges in the definition of $K^{\alpha}([a,b], X)$, then we obtain $R^{\alpha}([a,b], X)$ and this fact that the Riemann-Stieltjes integrals are particular cases of the Kurzweil vector integrals was essential in the previous proof. If X is of finite dimension, then the Riemann-Stieltjes integrals are also Henstock vector integrals, once in this case the spaces of vector integrable functions of Kurzweil and of Henstock coincide. However, when the dimension of X is infinite, then we may have $f \in R([a,b], X) \setminus H([a,b], X)$ as shown by the next example.

Example 2.1. Let $X = l_2\,([a,b])$ and $f : [a,b] \to X$ be defined by $f\,(t) = e_t$ (i.e., $e_t\,(s) = 1$ if $s = t$, and $e_t\,(s) = 0$ if $s \neq t$). Given $\varepsilon > 0$, there exists $\delta > 0$ with $\delta^{1/2} < \dfrac{\varepsilon}{(b-a)^{1/2}}$ such that for every $d = (t_i) \in TD_{[a,b]}$ with $\Delta d = \max_i \{t_i - t_{i-1}\} < \delta$,

$$\left\| \sum_i f\,(\xi_i)\,(t_i - t_{i-1}) - 0 \right\|_2 = \left\| \sum_i e_{\xi_i}\,(t_i - t_{i-1}) \right\|_2 = \left[\sum_i (t_i - t_{i-1})^2 \right]^{1/2}$$

where we applied the Bessel's equality. Hence,

$$\Big[\sum_i (t_i - t_{i-1})^2\Big]^{1/2} < \delta^{1/2} \sum_i (t_i - t_{i-1})^{1/2} = \big[\delta\,(b-a)\big]^{1/2} < \varepsilon$$

and it follows that $f \in R([a,b],X)$ with $\int_{[a,b]} f = 0$.

Now, suppose that $f \in H([a,b],X)$ and let F be the associated function of f. Then $F(t) - F(r) = \int_{[r,t]} f(s)\,ds = \widetilde{f}(t) - \widetilde{f}(r) = 0$, for every $[r,t] \subset [a,b]$. Hence F is constant and, therefore, for every $d = (t_i) \in TD_{[a,b]}$,

$$\begin{aligned}\sum_i \|F(t_i) - F(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})\| &= \sum_i \|e_{\xi_i}(t_i - t_{i-1})\| \\ &= \sum_i |t_i - t_{i-1}| = b - a\end{aligned}$$

and we have a contradiction. Thus, $f \notin H([a,b],X)$.

Theorem 2.9. *Let $\alpha \in H([a,b], L(X,Y))$ with $\widetilde{\alpha} \in BV([a,b], L(X,Y))$ and $f \in C([a,b],X)$. Then $\alpha f \in H([a,b],Y)$ with ${}^K\!\int_{[a,b]} \alpha f = \int_{[a,b]} d\widetilde{\alpha} f$.*

Proof. Since $\alpha \in H([a,b], L(X,Y))$, for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (t_i) \in TD_{[a,b]}$,

$$\sum_i \Big\| \alpha(\xi_i)(t_i - t_{i-1}) - {}^K\!\int_{[t_{i-1},t_i]} \alpha \Big\| < \varepsilon.$$

From the Corollary 2.8, $\alpha f \in K([a,b],Y)$ with ${}^K\!\int_{[a,b]} \alpha f = \int_{[a,b]} d\widetilde{\alpha} f$. Hence,

$$\begin{aligned}&\sum_i \Big\| \alpha(\xi_i) f(\xi_i)(t_i - t_{i-1}) - {}^K\!\int_{[t_{i-1},t_i]} \alpha f \Big\| \\ &< \sum_i \Big\| \alpha(\xi_i) f(\xi_i)(t_i - t_{i-1}) - {}^K\!\int_{[t_{i-1},t_i]} \alpha(t) f(\xi_i)\,dt \Big\| \\ &\quad + \sum_i \Big\| {}^K\!\int_{[t_{i-1},t_i]} \alpha f - {}^K\!\int_{[t_{i-1},t_i]} \alpha(t) f(\xi_i)\,dt \Big\| \\ &\le \Big\| f \Big\|_\infty \sum_i \Big\| \alpha(\xi_i)(t_i - t_{i-1}) - {}^K\!\int_{[t_{i-1},t_i]} \alpha \Big\| \\ &\quad + \sum_i \Big\| {}^K\!\int_{[t_{i-1},t_i]} \alpha(t)\,[f(t) - f(\xi_i)]dt \Big\| \\ &< \Big\| f \Big\|_\infty \varepsilon + \sum_i \Big\| \int_{[t_{i-1},t_i]} d\widetilde{\alpha}(t)\,[f(t) - f(\xi_i)]dt \Big\|.\end{aligned}$$

Now applying the Integration by Parts Formula and then the Fundamental Theorem of Calculus for the Riemann integral we have that

$$\begin{aligned}
&\sum_i \left\| \int_{[t_{i-1},t_i]} d\widetilde{\alpha}(t)[f(t)-f(\xi_i)] \right\| \\
=&\sum_i \Big\| \widetilde{\alpha}(t_i)[f(t_i)-f(\xi_i)] - \widetilde{\alpha}(t_{i-1})[f(t_{i-1})-f(\xi_i)] \\
&- \int_{[t_{i-1},t_i]} \widetilde{\alpha}(t)\, d[f(t)-f(\xi_i)] \Big\| \\
=&\sum_i \Big\| \widetilde{\alpha}(t_i)[f(t_i t)-f(\xi_i)] - \int_{[\xi_i,t_i]} \widetilde{\alpha}(t) df(t) - \widetilde{\alpha}(t_{i-1})[f(t_{i-1})-f(\xi_i)] \\
&- \int_{[t_{i-1},\xi_i]} \widetilde{\alpha}(t)\, df(t) \Big\| \\
=&\sum_i \Big\| \widetilde{\alpha}(t_i) \int_{[\xi_i,t_i]} df(t) - \int_{[\xi_i,t_i]} \widetilde{\alpha}(t)\, df(t) \\
&+ \widetilde{\alpha}(t_{i-1}) \int_{[t_{i-1},\xi_i]} df(t) - \int_{[t_{i-1},\xi_i]} \widetilde{\alpha}(t)\, df(t) \Big\| \\
=&\sum_i \left\| \int_{[\xi_i,t_i]} [\widetilde{\alpha}(t_i)-\widetilde{\alpha}(t)] df(t) + \int_{[t_{i-1},\xi_i]} [\widetilde{\alpha}(t_{i-1})-\widetilde{\alpha}(t)] df(t) \right\| \\
<&V(\widetilde{\alpha})
\end{aligned}$$

where $\omega(f)$ denotes the oscillation of f on $[a,b]$. Since f is continuous and $\widetilde{\alpha}$ is of bounded variation, the proof is complete. □

We denote by $\mathcal{L}_1([a,b],X)$ the space of all functions $f : [a,b] \to X$ which are Bochner-Lebesgue integrable with finite integral. The integral of $f \in \mathcal{L}_1([a,b],X)$ is denoted by ${}^L\int_{[a,b]} f = {}^L\int_{[a,b]} f(t)\,dt$ and we write $\|f\|_1 = {}^L\int_{[a,b]} \|f()\|$. From the Riemannian definition of $\mathcal{L}_1([a,b],X)$ (see [18] and [13]), it follows that $\mathcal{L}_1([a,b],X) \subset H([a,b],X)$. Besides, if $X = \mathbb{R}$, then the positive functions which are Kurzweil-Henstock integrable are also Lebesgue integrable. Thus, if $f \in H([a,b],\mathbb{R})$ is absolutely integrable (i.e., $\|f()\| \in \mathcal{L}_1([a,b],\mathbb{R})$), then f is Lebesgue integrable. From Example 1.1 before Theorem 2.9, we also observe that when the dimension of X is infinite, then it may happen that $f \in R([a,b],X) \setminus \mathcal{L}_1([a,b],X)$.

Theorem 2.10. *([13], 9) Let $f \in H([a,b],X)$. Then f is absolutely integrable if and only if $\widetilde{f} \in BV([a,b],X)$. In any case, $\|f\|_1 = V(\widetilde{f})$.*

Proof. Suppose that f is absolutely integrable. Since

$$V\left(\widetilde{f}\right)=\sup\left\{\sum_i\left\|\widetilde{f}\left(t_i\right)-\widetilde{f}\left(t_{i-1}\right)\right\|;\ a=t_0<t_1<\ldots<t_n=b\right\}$$

we have that

$$\sum_i\left\|\widetilde{f}\left(t_i\right)-\widetilde{f}\left(t_{i-1}\right)\right\|=\sum_i\left\|{}^K\int_{[t_{i-1},t_i]}f\right\|\le\sum_i{}^L\int_{[t_{i-1},t_i]}\|f\|=\|f\|_1\,.$$

Now, suppose that

$$\widetilde{f}\in BV([a,b],X).$$

We will prove that there exists ${}^K\int_{[a,b]}\|f\|={}^L\int_{[a,b]}\|f\|=V\left(\widetilde{f}\right)$. Given $\varepsilon>0$, we will find a gauge δ of $[a,b]$ such that for every δ-fine $d=(t_i)\in TD_{[a,b]}$,

$$\left|\sum_i\left\|f(\xi_i)\right\|(t_i-t_{i-1})-V(\widetilde{f})\right|<\varepsilon.$$

But if $d=(t_i)\in TD_{[a,b]}$ is δ-fine, then

$$\begin{aligned}&\left|\sum_i\left\|f(\xi_i t)\right\|(t_i-t_{i-1})-V(\widetilde{f})\right|\\&\le\sum_i\left|\left\|f(\xi_i)\right\|(t_i-t_{i-1})-\left\|{}^K\int_{[t_{i-1},t_i]}f\right\|\right|+\left|\sum_i\left\|{}^K\int_{[t_{i-1},t_i]}f\right\|-V\left(\widetilde{f}\right)\right|\\&\le\sum_i\left\|f\left(\xi_i\right)\left(t_i-t_{i-1}\right)-{}^K\int_{[t_{i-1},t_i]}f\right\|+\left|\sum_i\left\|\widetilde{f}\left(t_i\right)-\widetilde{f}\left(t_{i-1}\right)\right\|-V\left(\widetilde{f}\right)\right|\end{aligned}$$

By the definition of $V\left(\widetilde{f}\right)$, we may take the division of $[a,b]$ such that the last summand is smaller than $\varepsilon/2$. Since $f\in H([a,b],X)$, then we may take a gauge δ of $[a,b]$ such that for every δ-fine $d=(t_i)\in TD_{[a,b]}$, the first summand is also smaller than $\varepsilon/2$, and we may suppose that the points chosen for the second summand are points of the δ-fine tagged division $d=(t_i)$. The proof is then complete. $\square$

Theorem 2.11. *If* $\alpha\in\mathcal{L}_1\left([a,b],L(X,Y)\right)$ *and* $f\in G([a,b],X)$*, then* $\alpha f\in\mathcal{L}_1\left([a,b],Y\right)$ *and* ${}^L\int_{[a,b]}\alpha f={}^K\int_{[a,b]}d\widetilde{\alpha}\,f$.

Proof. Since f is bounded, then $\|f\|_\infty<\infty$. The function $\|\alpha\left(\right)\,f\left(\right)\|:[a,b]\to\mathbb{R}$ is m-measurable (m for the Lebesgue measure) and $\|\alpha\left(t\right)\,f\left(t\right)\|\le$

$\|f\|_\infty \|\alpha(t)\|$ for every $t \in [a,b]$. From the fact that $\mathcal{L}_1([a,b],\mathbb{R})$ is a vector lattice it follows that $\|\alpha()\, f()\| \in \mathcal{L}_1([a,b],\mathbb{R})$ and therefore $\alpha f \in \mathcal{L}_1([a,b],Y)$. Now we prove the equality. Let $\varepsilon > 0$. Since $\alpha \in \mathcal{L}_1([a,b],L(X,Y)) \subset H([a,b],L(X,Y))$, there is a gauge δ_1 of $[a,b]$ such that for every δ_1-fine $d = (\zeta_j, s_j) \in TD_{[a,b]}$,

$$\sum_j \|\alpha(\zeta_j)(s_j - s_{j-1}) - [\widetilde{\alpha}(s_j) - \widetilde{\alpha}(s_{j-1})]\| < \varepsilon.$$

By Theorem 2.10, $\widetilde{\alpha}$ is a function of bounded variation and therefore regulated and of bounded semivariation. Then by Theorem 2.6, there exists the Kurzweil integral ${}^K\int_{[a,b]} d\widetilde{\alpha}\ f$ which means that there is a gauge δ_2 of $[a,b]$ such that for every δ_2-fine $d = (\rho_k, r_k) \in TD_{[a,b]}$,

$$\Big\|\sum_k [\widetilde{\alpha}(r_k) - \widetilde{\alpha}(r_{k-1})]f(\rho_k) - {}^K\int_{[a,b]} d\widetilde{\alpha}\ f\Big\| < \varepsilon.$$

Let δ be a gauge of $[a,b]$ such that $\delta(\xi) \le \delta_l(\xi)$, for every $\xi \in [a,b]$ and $l = 1, 2$. Then for every δ-fine $d = (t_i) \in TD_{[a,b]}$, we have that

$$\begin{aligned}
&\Big\|\sum_i \alpha(\xi_i)f(\xi_i)(t_i - t_{i-1}) - {}^K\int_{[a,b]} d\widetilde{\alpha}\ f\Big\| \\
&\le \sum_i \|\alpha(\xi_i)\, f(\xi_i)\,(t_i - t_{i-1}) - [\widetilde{\alpha}(t_i) - \widetilde{\alpha}(t_{i-1})]f(\xi_i)\| \\
&\quad + \Big\|\sum_i [\widetilde{\alpha}(t_i) - \widetilde{\alpha}(t_{i-1})]f(\xi_i) - {}^K\int_{[a,b]} d\widetilde{\alpha}\ f\Big\| < \varepsilon\,\|f\|_\infty + \varepsilon. \quad \square
\end{aligned}$$

Corollary 2.12. *Let $I([a,b],X)$ denote one of the spaces $BV([a,b],X)$ or $C([a,b],X)$. If $\alpha \in \mathcal{L}_1([a,b],L(X,Y))$ and $f \in I([a,b],X)$, then $\alpha f \in \mathcal{L}_1([a,b],Y)$ with ${}^L\int_{[a,b]} \alpha f = \int_{[a,b]} d\widetilde{\alpha} f$.*

PROOF. In any of the cases the result comes from Theorem 2.5, since $\widetilde{\alpha} \in C([a,b],L(X,Y))$ (Theorem 2.2) and $\widetilde{\alpha} \in BV([a,b],L(X,Y))$ (Theorem 2.10). $\square$

3 The Volterra-Henstock Linear Integral Equation

The first result of this section gives a necessary condition for the existence of the Henstock integral.

Lemma 3.1. *If $f \in H([a,b],X)$, then there exists a sequence of closed sets $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \uparrow [a,b]$ (i.e., $X_n \subset X_{n+1} \subset [a,b]$, for every $n \in \mathbb{N}$ and $\cup X_n = [a,b]$) and $f \in \pounds_1(X_n,X)$, for every $n \in \mathbb{N}$. Furthermore, $\lim\limits_{n\to\infty} {}^L\int_{X_n\cap[a,t]} f = {}^K\int_{[a,t]} f$ uniformly for every $t \in [a,b]$.*

Proof. It suffices to adapt the proof given in [7] for the Banach space-valued case. □

Lemma 3.2. *([14], Th. 7.1.2) Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of mappings from X to X such that each mapping has a fixed point $x_n = T_n(x_n)$, let $T : X \to X$ be such that for some integer m, T^m is a contraction, where T^m is the composition of T m times, and suppose that $T_n \to T$ uniformly. Then $x_n \to x_0 = T(x_0)$.*

Theorem 3.3. *Given $\alpha \in H([a,b],L(X,Y))$ and a function $f : [a,b] \to X$, then $\alpha f \in H([a,b],Y)$ if one of the following conditions is satisfied:*

i) $f \in BV([a,b],X)$;

ii) $f \in C([a,b],X)$ and $\widetilde{\alpha} \in BV([a,b],L(X,Y))$.

In any case, ${}^K\int_{[a,b]} \alpha f = \int_{[a,b]} d\widetilde{\alpha} f$.

Proof. For i), see [2], Th. 2.2.8, or [6]; for ii) see Theorem 2.9. □

Remark 1. It is also true that if $\alpha \in BV([a,b],L(X,Y))$ and $f \in H([a,b],X)$, then $\alpha f \in H([a,b],Y)$ and ${}^K\int_{[a,b]} \alpha f = \int_{[a,b]} \alpha\, d\widetilde{f}$, (see [2], Th. 2.2.6 or [6]).

We consider the next Volterra-Henstock linear integral equation

$$x(t) + {}^K\int_{[a,t]} \alpha(s)xt(s)ds = f(t), \quad t \in [a,b], \qquad (V)_H$$

in the following two cases

a) $\alpha \in H([a,b],L(X))$ is bounded and $x,\ f \in BV_a([a,b],X)$;

b) $\alpha \in H([a,b],L(X))$ with $\widetilde{\alpha} \in BV([a,b],L(X))$ and $x,\ f \in C([a,b],X)$.

In view of Theorem 2.10, we can replace b) by

b') $\alpha \in H([a,b],L(X))$ is absolutely integrable and $x,\ f \in C([a,b],X)$.

In the real case when $X = \mathbb{R}$, it follows from the considerations before Theorem 2.10 that b') is equivalent to a case when $\alpha \in \pounds_1([a,b],\mathbb{R})$. But this will be treated in a more general context (the Bochner-Lebesgue integral) in Section 4.

By means of Lemmas 3.1 and 3.2 we will be able to obtain conclusions about equation $((V)_H)$ through the analysis of equations of $((V)_L)$ type in any of the cases a) and b').

Case a): Let $\alpha \in H([a,b], L(X))$ be bounded and $f \in BV_a([a,b],X)$. By Lemma 3.1 and the Corollary 2.12, given $n \in \mathbb{N}$, we can consider the mapping T_n given by

$$(T_n x)(t) = f(t) - {}^L\int_{[a,t]} \chi_{X_n}(s)\,\alpha(s)\,x(s)\,ds, \quad t \in [a,b],$$

where χ_{X_n} denotes the characteristic function of X_n. By Theorem 2.10, T_n takes elements from $BV_a([a,b],X)$ to $BV_a([a,b],X)$. Furthermore, each T_n is continuous since $\|\chi_{X_n}()\,\alpha()\|_1 < \infty$, and T_n is a contraction whenever $\|\chi_{X_n}()\,\alpha()\|_1 < 1$.

Consider the continuous mapping $T : BV_a([a,b],X) \to BV_a([a,b],X)$ defined by

$$(Tx)(t) = f(t) - {}^K\int_{[a,t]} \alpha(s)\,x(s)\,ds, \quad t \in [a,b].$$

Given $x \in BV_a([a,b],X)$, Tx really belongs to $BV_a([a,b],X)$. In fact, given any division (t_i) of $[a,b]$, then

$$\begin{aligned}\sum_i \|Tx(t_i) - Tx(t_{i-1})\| &\le \sum_i \left\| f(t_i) - f(t_{i-1}) \right\| \\ &+ \sum_i \left\| {}^K\int_{[t_{i-1},t_i]} \alpha(s)\,x(s)\,ds \right\| \\ &\le V(f) + \|\alpha\|_\infty \|x\|_\infty (b-a).\end{aligned}$$

With the notation and considerations above, the proof of the next theorem follows easily.

Theorem 3.4. *Given $\alpha \in H([a,b], L(X))$ bounded, consider the Volterra-Henstock linear integral equation*

$$x(t) + {}^K\int_{[a,t]} \alpha(s)\,x(s)\,ds = f(t), \quad t \in [a,b], \qquad (V)_H$$

the Volterra-Bochner-Lebesgue linear integral equations obtained from Lemma 3.1

$$x(t) + {}^L\int_{[a,t]} \chi_{X_n}(s)\alpha(s)x(s)ds = f(t), \quad t \in [a,b], \text{ and } n \in \mathbb{N} \qquad (V)_{L_n}$$

and the mapping $T : BV_a([a,b],X) \to BV_a([a,b],X)$ defined by

$$(Tx)(t) = f(t) - {}^K\int_{[a,t]} \alpha(s)\,x(s)\,ds, \quad t \in [a,b],$$

where in all cases x, $f \in BV_a([a,b],X)$. If $\|\chi_{X_n}()\alpha()\|_1 < 1$ for each $n \in \mathbb{N}$, then given $f \in BV_a([a,b],X)$, each equation $(V)_{L_n}$ admits one and only one solution $x_n \in BV_a([a,b],X)$. Consider also the following conditions:

i) $\{x_n\}_{n\in\mathbb{N}}$ *has a convergent subsequence* $x_{n_k} \to x_0$*;*

ii) T^m *is a contraction for some* $m > 1$*.*

If i) is satisfied, then $x_0 \in BV_a([a,b],X)$ is a solution of $(V)_H$. If ii) is satisfied, then there exists $x = \lim_n x_n$ and $x \in BV_a([a,b],X)$ satisfies $(V)_H$.

Proof. For each $n \in \mathbb{N}$, $\|\chi_{X_n}()\,\alpha()\|_1 < 1$. Hence each T_n given by

$$(T_n x)(t) = f(t) - {}^L\!\int_{[a,t]} \chi_{X_n}(s)\,\alpha(s)\,x(s)\,ds, \quad t \in [a,b],$$

is a contraction and therefore has a unique fixed point x_n by the Contraction Principle. Therefore $(V)_{L_n}$ has one and only one solution $x_n \in BV_a([a,b],X)$.

If i) holds, then given $n_k \in \mathbb{N}$,

$$\begin{aligned} &\|x_0 - Tx_0\| \\ &\leq \|x_0 - x_{n_k}\| + \|x_{n_k} - T_{n_k}x_{n_k}\| + \|T_{n_k}x_{n_k} - T_{n_k}x_0\| + \|T_{n_k}x_0 - Tx_0\|, \end{aligned}$$

where the first summand tends to zero as $k \to 0$ (by the convergence of the subsequence), the second summand is equal to zero (once x_{n_k} is a fixed point of T_{n_k}), the third summand is smaller than $\|\chi_{X_n}()\,\alpha()\|_1\,\|x_{n_k} - x_0\|$ which tends to zero as $k \to 0$ (because $\|\chi_{X_n}()\,\alpha()\|_1 < 1$ and by the convergence of the subsequence), and the fourth summand tends to zero as $k \to 0$, (by Lemma 3.1).

Suppose now that ii) holds. From Lemma 3.1, $T_n \to T$. As a matter of fact, $T_n \to T$ uniformly. Thus $x_n \to x = Tx$, (by Lemma 3.2), and we complete the proof. □

Lemma 3.5. *(see [12], 3.3; see [17] for the real-valued case) Suppose that $f \in H([a,b],X)$ and $g : [a,b] \to X$ is a function such that $g = f$ m-almost everywhere (m for the Lebesgue measure). Then $g \in H([a,b],X)$ and $\widetilde{g} = \widetilde{f}$.*

We say that two functions g and f of $H([a,b],X)$ are equivalent if $\widetilde{g} = \widetilde{f}$ and we write $H([a,b],X)_A$ to denote the space of all equivalence classes of functions of $H([a,b],X)$ endowed with the Alexiewicz norm $f \in H([a,b],X) \mapsto \|f\|_A = \|\widetilde{f}\|_\infty$. In what follows we will write $f \in H([a,b],X)_A$ to denote that we have picked up a function $f = f_\Phi \in \Phi$, where $\Phi \in H([a,b],X)_A$.

The next assertion is a consequence of [11], Th. 3.5.

Theorem 3.6. *Given $\alpha \in SV([a,b], L(X))$ with $\alpha(b) = 0$, consider the Volterra-Henstock linear integral equation*

$$x(t) + {}^K\!\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \qquad (V)_L$$

where $x, f \in H([a,b], X)_A$. Then for every $f \in H([a,b], X)_A$ there exists one and only one solution $x \in H([a,b], X)_A$ with

$$x(t) = f(t) - {}^K\!\int_{[a,t]} \rho(t,s)\, x(s)\, ds, \quad t \in [a,b],$$

where the kernel $\rho : [a,b] \times [a,b] \to L(X)$ is bounded and can be given by the Neumann series which converges in $L(H([a,b], X)_A)$.

Theorem 3.7. *Let $\alpha \in H([a,b], L(\mathbb{R})) \cong H([a,b], \mathbb{R})$ be bounded and suppose that there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ such that each X_n is a finite union of nonoverlapping closed intervals, $X_n \uparrow [a,b]$ and $\alpha \in BV(X_n, L(\mathbb{R})) \cong BV(X_n, \mathbb{R})$, for every $n \in \mathbb{N}$. Consider the Volterra-Kurzweil-Henstock linear integral equations*

$$x(t) + {}^K\!\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b]. \qquad (V)_H$$

and

$$x(t) + {}^K\!\int_{[a,t]} \chi_{X_n}(s)\, \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \text{ and } n \in \mathbb{N}, \quad (V)_{H_n}$$

where $x, f \in BV_a([a,b], \mathbb{R})$. Then given $n \in \mathbb{N}$ and $f \in BV_a([a,b], \mathbb{R})$, $(V)_{H_n}$ admits one and only one solution $x_n \in BV_a([a,b], \mathbb{R})$, there exists $x = \lim\limits_n x_n$ and $x \in BV_a([a,b], \mathbb{R})$ satisfies $(V)_H$.

PROOF. For every $n \in \mathbb{N}$, let $X_n = \bigcup\limits_{i=1}^{k_n} [a_n^i, b_n^i]$. By Lemma 3.5, we may suppose that $\alpha\left(b_n^i\right) = 0$ for every i and every n. Since $\alpha \in BV(X_n, L(\mathbb{R}))$ implies that $\alpha \in BV\left([a_n^i, b_n^i], L(\mathbb{R})\right)$ for every i, then given $n \in \mathbb{N}$ and $i \in \{1, \dots, k_n\}$, it follows from Theorem 3.6 that there exists a unique solution $x_n^i \in H\left([a_n^i, b_n^i], \mathbb{R}\right)_A$ of

$$x_n^i(t) + {}^K\!\int_{[a_n^i,t]} \alpha\big|_{[a_n^i,b_n^i]}(s)\, x(s)\, ds = f(t), \quad t \in [a_n^i, b_n^i], \qquad (V)_{H_{n,i}}$$

such that

$$x_n^i(t) = f\big|_{[a_n^i,b_n^i]}(t) - {}^K\!\int_{[a_n^i,t]} \rho_n^i(t,s)\, x_n^i(s)\, ds, \quad t \in [a_n^i, b_n^i],$$

with bounded kernel ρ_n^i:$[a_n^i,b_n^i] \times [a_n^i,b_n^i] \to L(\mathbb{R})$ given by the Neumann series which converges in $L\left(H\left([a_n^i,b_n^i],\mathbb{R}\right)_A\right)$. As a matter of fact, $x_n^i \in BV\left([a_n^i,b_n^i],\mathbb{R}\right)$ because $f \in BV\left([a_n^i,b_n^i],\mathbb{R}\right)$ and $\left\|\rho_n^i\right\|_\infty < \infty$ (see the considerations before Theorem 3.4). Now, for every $n \in \mathbb{N}$ and every $i \in \{1,\dots,k_n\}$, let $y_n^i : [a,b] \to \mathbb{R}$ be given by $y_n^i = x_n^i$ on $[a_n^i,b_n^i]$ and $y_n^i = 0$ otherwise, and let $\phi_n^i : [a,b]\times[a,b] \to L(\mathbb{R})$ be defined by $\phi_n^i = \rho_n^i$ on $[a_n^i,b_n^i]\times[a_n^i,b_n^i]$ and $\phi_n^i = 0$ otherwise. Then, $x_n = \sum_{i=1}^{k_n} y_n^i \in BV_a([a,b],\mathbb{R})$ is a (unique) solution of $(V)_{H_n}$. Moreover,

$$\begin{aligned}
x_n(t) =& \sum_{i=1}^{k_n} y_n^i(t) = \sum_{i=1}^{k_n}\Big(\chi_{[a_n^i,b_n^i]} f(t) - {}^K\!\int_{[a_n^i,b_n^i]\cap[a,t]} \phi_n^i(t,s)\, y_n^i(s)\; ds\Big)\\
=& f(t) - \sum_{i=1}^{k_n}\Big({}^K\!\int_{[a_n^i,b_n^i]\cap[a,t]} \phi_n^i(t,s)\, y_n^i(s)\; ds\Big)\\
=& f(t) - {}^K\!\int_{X_n\cap[a,t]} \sum_{i=1}^{k_n} \phi_n^i(t,s)\, y_n^i(s)\, ds\\
=& f(t) - {}^K\!\int_{X_n\cap[a,t]} \rho_n(t,s)\, x_n(s)\, ds, \qquad t\in[a,b],
\end{aligned}$$

where $\rho_n(t,s) = \sum_{i=1}^{k_n} \phi_n^i(t,s)$, for each $n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, let $T_n : BV_a([a,b],\mathbb{R}) \to BV_a([a,b],\mathbb{R})$ be defined by

$$(T_n x)(t) = f(t) - {}^K\!\int_{[a,t]} \chi_{X_n}(s)\, \alpha(s)\, x(s)\; ds, \quad t \in [a,b].$$

Since the Dominated Convergence Theorem holds for the real-valued Henstock integral and $X_n \uparrow [a,b]$, then $T_n \to T$, where $T : BV_a([a,b],\mathbb{R}) \to BV_a([a,b],\mathbb{R})$ is given by

$$(Tx)(t) = f(t) - {}^K\!\int_{[a,t]} \alpha(s)\, x(s)\; ds, \quad t \in [a,b],$$

with $x,\, f \in BV_a([a,b],\mathbb{R})$. From the fact that the functions x, f, x_n and $\chi_{X_n}(\cdot)\,\alpha(\cdot)$ are of bounded variation, then $T_n \to T$ uniformly. The rest of the proof follows the steps of the proof of Theorem 3.4, (see the Remark after Theorem 3.4). □

Remark 2. When we consider Banach space-valued functions, then neither the Dominated Convergence Theorem nor the Monotone Convergence Theorem hold for the Henstock integral. The next example of Birkhoff ([1]) shows us that fact.

Example 3.1. Consider $f : [0,1] \to X = l_2(\mathbb{N})$ defined by $f = \sum_{i=1}^{\infty} f_i$, where $f_i(t) = 2^i e_{i,j}$, if $\frac{j}{2^i} < t \leq \frac{j}{2^i} + \frac{1}{2^{2i}}$, $i = 1, 2, \ldots$, $j = 0, 1, \ldots, 2^i - 1$. We use $e_{i,j}$ to denote a doubly infinite set of orthonormal vectors of $l_2(\mathbb{N})$. Then $f^{\,i} = f_1 + \ldots + f_i$ is such that $f^{\,i} \in H([0,1], X)$ for every $i \in \mathbb{N}^*$, but $\widetilde{f}$ is nowhere differentiable and hence $f \notin H([0,1], X)$ (by the Fundamental Theorem of Calculus - see [12] or [5] for the Banach case or [16] for the real-valued case). We point out however that $\widetilde{f}(t) = {}^K\int_{[0,t]} f$ exists in the sense of the Kurzweil integral for every $t \in [0,1]$. Indeed, because the space $X = l_2(\mathbb{N})$ fulfills the required conditions which make the Monotone Convergence Theorem be valid for the Kurzweil integral (see [3]). □

The next result comes directly from the Contraction Principle.

Theorem 3.8. *Let $\alpha \in H([a,b], L(X))$ be bounded with $(b-a)\|\alpha\|_\infty < 1$, then given $f \in BV_a([a,b], X)$, there is one and only one $x \in BV_a([a,b], X)$ that satisfies the linear integral equation of Volterra-Henstock*

$$x(t) + {}^K\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b]. \qquad (V)_H$$

Case b'): Let $\alpha \in H([a,b], L(X))$ be absolutely integrable and suppose $f \in C([a,b], X)$. By Theorems 2.5 and 2.6 we can consider the mapping $T : C([a,b], X) \to C([a,b], X)$ defined by

$$(Tx)(t) = f(t) - {}^K\int_{[a,t]} \alpha(s)\, x(s)\; ds, \quad t \in [a,b].$$

In addition, by Lemma 3.1, for every $n \in \mathbb{N}$, we can consider the continuous mapping $T_n : C([a,b], X) \to C([a,b], X)$ given by

$$(T_n x)(t) = f(t) - {}^L\int_{[a,t]} \chi_{X_n}(s)\, \alpha(s)\, x(s)\; ds, \quad t \in [a,b].$$

Likewise case a), if $\|\chi_{X_n}()\, \alpha()\|_1 < 1$, then T_n is a contraction.

Theorem 3.9. *Given $\alpha \in H([a,b], L(X))$ absolutely integrable, consider the linear integral equation of Volterra-Henstock*

$$x(t) + {}^K\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \tag*{$(V)_H$}$$

the Volterra-Bochner-Lebesgue linear integral equations obtained through Lemma 3.1

$$x(t) + {}^L\int_{[a,t]} \chi_{X_n}(s)\, \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \text{ and } n \in \mathbb{N}, \tag*{$(V)_{L_n}$}$$

and the mapping $T : C([a,b], X) \to C([a,b], X)$ defined by

$$(Tx)(t) = f(t) - {}^K\int_{[a,t]} \alpha(s)\, x(s)\; ds, \quad t \in [a,b],$$

where $x,\, f \in C([a,b], X)$. If $\|\chi_{X_n}()\, \alpha()\|_1 < 1$ for each $n \in \mathbb{N}$, then given $f \in C([a,b], X)$, each equation $(V)_{L_n}$ admits one and only one solution $x_n \in C([a,b], X)$. Consider also the following conditions:

i) $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence $x_{n_k} \to x_0$;

ii) α is bounded and T^m is a contraction for some $m > 1$.

If i) is satisfied, then $x_0 \in C([a,b], X)$ is a solution of $(V)_H$. If ii) is satisfied, then there exists $x = \lim_n x_n$ and $x \in C([a,b], X)$ satisfies $(V)_H$.

Proof. The proof is analogous to the that of Theorem 3.4. □

Theorem 3.10. *Let $\alpha \in H([a,b], L(X))$ be absolutely integrable and bounded with $(b-a)\|\alpha\|_\infty < 1$. Then for every $f \in C([a,b], X)$, there is one and only one $x \in C([a,b], X)$ that satisfies the Volterra-Henstock linear integral equation*

$$x(t) + {}^K\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b]. \tag*{$(V)_H$}$$

Proof. The assertion follows directly from the Contraction Principle. □

4 The Volterra-Bochner-Lebesgue Linear Integral Equation

Let $I([a,b], X)$ denote one of the Banach spaces $G([a,b], X)$, $BV_a([a,b], X)$ or $C([a,b], X)$. From Theorem 2.11 and its Corollary, we can consider the

Volterra-Bochner-Lebesgue linear integral equation

$$x(t) + {}^{L}\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b] \qquad (V)_L$$

where $\alpha \in \mathcal{L}_1([a,b], L(X))$ and $x,\ f \in I([a,b],X)$. Then the Contraction Principle implies the following.

Theorem 4.1. *Suppose $\alpha \in \mathcal{L}_1([a,b], L(X))$ with $\|\alpha\|_1 < 1$. Then given $f \in I([a,b],X)$, there is one and only one $x \in I([a,b],X)$ that satisfies the linear integral equation of Volterra-Bochner-Lebesgue*

$$(t) + {}^{L}\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b]. \qquad (V)_L$$

Let $I \subset [a,b]$ be finite. We say that $f : [a,b] \to X$ is of bounded variation on $[a,b]\backslash I$ whenever f is of bounded variation on every closed interval contained in $[a,b]$. Let $f \in \mathcal{L}_1([a,b],X)$ be continuous or of bounded variation on $[a,b] \setminus I$. Then it is immediate that there exists a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ such that each X_n is the finite union of nonoverlapping closed intervals and $X_n \cap I = \emptyset$, $X_n \uparrow [a,b]$, and for every $t \in [a,b]$,

$$\lim_n {}^{L}\int_{X_n\cap[a,t]} f(s)\, ds = {}^{L}\int_{[a,t]} f(s)\, ds.$$

If $I \subset [a,b]$ is finite and $\alpha \in \mathcal{L}_1([a,b], L(X))$ is continuous or of bounded variation on $[a,b] \setminus I$, then there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ as above such that for every $t \in [a,b]$,

$$\lim_n {}^{L}\int_{X_n\cap[a,t]} \alpha(s)\, ds = {}^{L}\int_{[a,t]} \alpha(s)\, ds. \qquad (4.1)$$

Similarly, if $x \in G([a,b],X)$, then there exists the integral ${}^{L}\int_{[a,t]} \alpha\, x$ (Theorem 2.11) and we can find a sequence $\{Y_n\}_{n\in\mathbb{N}}$ such that each Y_n is the finite union of nonoverlapping closed intervals and $Y_n \cap I = \emptyset$, $Y_n \uparrow [a,b]$ and for every $t \in [a,b]$,

$$\lim_n {}^{L}\int_{Y_n\cap[a,t]} \alpha(s)\, x(s)\, ds = {}^{L}\int_{[a,t]} \alpha(s)\, x(s)\, ds.$$

We affirm however that in fact the same sequence for $\alpha \in \mathcal{L}_1([a,b], L(X))$ suits for $\alpha\, x \in \mathcal{L}_1([a,b],X)$, that is

$$\lim_n {}^{L}\int_{X_n\cap[a,t]} \alpha(s)\, x(s)\, ds = {}^{L}\int_{[a,t]} \alpha(s)\, x(s)\, ds,$$

for every $t \in [a,b]$. Indeed. Taking approximating Riemannian sums for the integrals ${}^L\int_{[a,t]} \alpha(s)\,x(s)\,ds$ and ${}^L\int_{X_n\cap[a,t]} \alpha(s)\,x(s)\,ds$ we have that

$$\begin{aligned}&\Big\|\sum_i \alpha(\xi_i)x(\xi_i)(t_i - t_{i-1}) - \sum_i \chi_{X_n}(\xi_i)\alpha(\xi_i)x(\xi_i)(t_i - t_{i-1})\Big\|\\ &\le \|x\|_\infty \sum_i \Big\|(1-\chi_{X_n})(\xi_i)\alpha(\xi_i)(t_i - t_{i-1})\Big\|\end{aligned}$$

which can be made sufficiently small by (4.1) and by the Riemannian definition of Bochner-Lebesgue integral, (see [13] and [18]).

Theorem 4.2. *Let $I([a,b],X)$ denote one of the two spaces $C([a,b],X)$ or $G([a,b],X)$, $BV_a([a,b],X)$. Given $\alpha \in \mathcal{L}_1\big([a,b],L(X)\big)$, consider the Volterra-Bochner-Lebesgue linear integral equations*

$$x(t) + {}^L\int_{[a,t]} \alpha(s)\,x(s)\,ds = f(t)\,, \quad t \in [a,b], \qquad (V)_L$$

and

$$x(t) + {}^L\int_{[a,t]} \chi_{X_n}(s)\,\alpha(s)\,x(s)\,ds = f(t)\,, \quad t \in [a,b],\ \text{and } n \in \mathbb{N}, \quad (V)_{L_n}$$

where x, $f \in I([a,b],X)$ and the sequence $\{X_n\}_{n\in\mathbb{N}}$ is obtained as in the previous paragraph for $\alpha \in \mathcal{L}_1([a,b],L(X))$. Suppose $f \in I([a,b],X)$, and that $\|\chi_{X_n}()\,\alpha()\|_1 < 1$ for every $n \in \mathbb{N}$. Then $(V)_{L_n}$ has one and only one solution $x_n \in I([a,b],X)$. Consider also the following conditions:

i) $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \to x_0$;

ii) α is bounded and T^m is a contraction for some $m > 1$.

If i) is satisfied, then $x_0 \in I([a,b],X)$ is a solution of $(V)_L$. If ii) is satisfied, then there exists $x = \lim_n x_n$ and $x \in I([a,b],X)$ satisfies $(V)_L$.

PROOF. For each $n \in \mathbb{N}$, let $T_n : I([a,b],X) \to I([a,b],X)$ be given by

$$T_n x(t) = f(t) - {}^L\int_{[a,t]} \chi_{X_n}(s)\,\alpha(s)\,x(s)\,ds, \quad t \in [a,b],$$

where $x \in I([a,b],X)$. Then T_n is continuous (because $\|\chi_{X_n}()\,\alpha()\|_1 < \infty$) and T_n is a contraction whenever $\|\chi_{X_n}()\,\alpha()\|_1 < 1$. The rest of the proof is analogous to the proof of Theorem 3.4 replacing the integral of Henstock by the integral of Bochner-Lebesgue. □

If either $I([a,b],X) = C([a,b],X)$ or $I([a,b],X) = BV_a([a,b],X)$ in the theorem above, we obtain the stronger results of Theorems 4.3 and 4.5 below.

Theorem 4.3. *Given $\alpha \in \mathcal{L}_1([a,b], L(X))$, and $I \subset [a,b]$ finite, suppose that α is continuous on $[a,b] \setminus I$ and consider the Volterra-Bochner-Lebesgue linear integral equations*

$$x(t) + {}^L\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \qquad (V)_L$$

and

$$x(t) + {}^L\int_{[a,t]} \chi_{X_n}(s)\, \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \text{ and } n \in \mathbb{N}, \quad (V)_{L_n}$$

where x, $f \in C([a,b], X)$ and the sequence $\{X_n\}_{n\in\mathbb{N}}$ satisfies the conditions of the paragraph before Theorem 4.2. Then given $f \in C([a,b], X)$ and $n \in \mathbb{N}$, $(V)_{L_n}$ has a solution $x_n \in C([a,b], X)$. Besides, there exists $x = \lim_n x_n$ and $x \in C([a,b], X)$ satisfies $(V)_L$.

PROOF. For each $n \in \mathbb{N}$, $(V)_{L_n}$ has a solution $x_n \in C([a,b], X)$. Indeed. For every $n \in \mathbb{N}$, let $X_n = \bigcup_{i=1}^{k_n} [a_n^i, b_n^i]$. Then $\alpha \in C\left([a_n^i, b_n^i], L(X)\right)$, for every $n \in \mathbb{N}$ and $i = 1, \ldots, k_n$ and by a well-known result from the Theory of Integral Equations (see, for instance, [15], p. 74), there exists a solution $x_n^i \in C\left([a_n^i, b_n^i], X\right)$ of

$$x_n^i(t) + {}^L\int_{[a_n^i,t]} \alpha\,\big|_{[a_n^i,b_n^i]}(s)\, x(s)\, ds = f(t), \quad t \in [a_n^i, b_n^i], \qquad (V)_{L_{n,i}}$$

such that

$$x_n^i(t) = f\,\big|_{[a_n^i,b_n^i]}(t) - {}^L\int_{[a_n^i,t]} \rho_n^i(t,s)\, x_n^i(s)\, ds, \quad t \in [a_n^i, b_n^i],$$

with continuous kernel $\rho_n^i : [a_n^i, b_n^i] \times [a_n^i, b_n^i] \to L(X)$ determined by the Neumann series method. For every $n \in \mathbb{N}$ and every $i \in \{1, \ldots, k_n\}$, let $y_n^i : [a,b] \to X$ be given by $y_n^i = x_n^i$ on $[a_n^i, b_n^i]$ and $y_n^i = 0$ otherwise, and let $\phi_n^i : [a,b] \times [a,b] \to L(X)$ be defined by $\phi_n^i = \rho_n^i$ on $[a_n^i, b_n^i] \times [a_n^i, b_n^i]$ and $\phi_n^i = 0$ otherwise. Then $x_n = \sum_{i=1}^{k_n} y_n^i$ is a solution of $(V)_{L_n}$ and

$$x_n(t) = f(t) - {}^L\int_{X_n \cap [a,t]} \rho_n(t,s)\, x_n(s)\, ds, \qquad t \in [a,b],$$

where $\rho_n(t,s) = \sum_{i=1}^{k_n} \phi_n^i(t,s)$, for each $n \in \mathbb{N}$. Moreover $x_n \in C([a,b],X)$, since f and the indefinite integral are continuous (Theorem 2.2). Hence, for every $n \in \mathbb{N}$, the mapping $T_n : C([a,b],X) \to C([a,b],X)$ given by

$$(T_n x)(t) = f(t) - {}^L\int_{[a,t]} \chi_{X_n}(s)\,\alpha(s)\,x(s)\;ds, \quad t \in [a,b],$$

has a fixed point. The rest of the demonstration follows the steps of Theorem 3.4 and the observation in the proof of Theorem 4.2. $\square$

Let $L_1([a,b],X)_A$ denote the space of all equivalence classes of functions of $\pounds_1([a,b],X)$ endowed with the Alexiewicz norm. When we write $f \in L_1([a,b],X)_A$ we mean that we have chosen a function $f = f_\Phi \in \Phi$, where $\Phi \in L_1([a,b],X)_A$. The next result is a consequence of [11], Th. 3.5 and the Remark that follows it.

Theorem 4.4. *Given $\alpha \in SV([a,b],L(X))$ with $\alpha(b) = 0$, consider the Volterra-Bochner-Lebesgue linear integral equation*

$$x(t) + {}^L\int_{[a,t]} \alpha(s)\,x(s)\,ds = f(t)\,, \quad t \in [a,b], \qquad (V)_L$$

where $x, f \in L_1([a,b],X)_A$. Then for every $f \in L_1([a,b],X)_A$ there exists one and only one solution $x \in L_1([a,b],X)_A$ with

$$x(t) = f(t) - {}^L\int_{[a,t]} \rho(t,s)\,x(s)\;ds, \quad t \in [a,b],$$

where the kernel $\rho:[a,b] \times [a,b] \to L(X)$ is bounded and can be given by the Neumann series which converges in $L(L_1([a,b],X)_A)$.

Theorem 4.5. *Given $\alpha \in \pounds_1([a,b],L(X))$ and $I \subset [a,b]$ finite, suppose that α is of bounded variation on $[a,b] \setminus I$ and consider the Volterra-Bochner-Lebesgue linear integral equations*

$$x(t) + {}^L\int_{[a,t]} \alpha(s)\,x(s)\,ds = f(t)\,, \quad t \in [a,b], \qquad (V)_L$$

and

$$x(t) + {}^L\int_{[a,t]} \chi_{X_n}(s)\,\alpha(s)\,x(s)\,ds = f(t)\,, \quad t \in [a,b], \text{ and } n \in \mathbb{N}, \quad (V)_{L_n}$$

where x, $f \in BV_a([a,b],X)$ and the sequence $\{X_n\}_{n\in\mathbb{N}}$ satisfies the conditions of the paragraph before Theorem 4.2. Then given $f \in BV_a([a,b],X)$ and $n \in \mathbb{N}$, $(V)_{L_n}$ has a unique solution $x_n \in BV_a([a,b],X)$. Moreover, there exists $x = \lim_n x_n$ and $x \in BV_a([a,b],X)$ satisfies $(V)_L$.

Proof. It follows from Theorem 4.4, each $(V)_{L_n}$ has a unique solution $x_n \in L_1([a,b],X)_A$ and

$$x(t) = f(t) - {}^L\int_{[a,t]} \rho(t,s)\, x(s)\, ds, \qquad t \in [a,b],$$

where the kernel $\rho{:}[a,b] \times [a,b] \to L(X)$ is bounded and can be given by the Neumann series. As a matter of fact, $x_n \in BV_a([a,b],X)$ for every $n \in \mathbb{N}$, since $f \in BV_a([a,b],X)$ and the indefinite integral of a Bochner-Lebesgue integrable function is of bounded variation (Theorem 2.10). This means that each T_n has a fixed point, where $T_n : BV_a([a,b],X) \to BV_a([a,b],X)$ is given by

$$(T_n x)(t) = f(t) - {}^L\int_{[a,t]} \chi_{X_n}(s)\, \alpha(s)\, x(s)\, ds, \qquad t \in [a,b].$$

The rest of the demonstration follows the steps of Theorem 3.4 and the observation in the proof of Theorem 4.2. □

5 Applications

5.1 Applications to Ordinary Differential Equations

Consider the equation

$$\dot{x} = f(t,x) \tag{5.1}$$

where $f : [a,b] \times B \to \mathbb{R}$ is a function and $B \subset \mathbb{R}$ is an open set. Let $J \subset [a,b]$ be a closed interval. We say that $x : J \to \mathbb{R}$ is a Carathéodory solution (respectively a Henstock solution) of equation (4.1) if and only if the following conditions are satisfied:

i) $x(t) \in B$ m-almost everywhere on J;

ii) $x(t) = x(c) + {}^*\int_{[c,t]} f(s, x(s))\, ds$, for every $t,\ c \in J$;

where m denotes Lebesgue measure and ${}^*\int = {}^L\int$ (resp. ${}^*\int = {}^K\int$), provided the integral exists. From the Fundamental Theorem of Calculus it is immediate that if x satisfies ii) for ${}^*\int = {}^L\int$ (resp. ${}^*\int = {}^K\int$), then x is absolutely continuous and differentiable (resp. x satisfies the Strong Lusin Condition and is differentiable m-almost everywhere) and $\dot{x} = f(t,x)$ m-almost everywhere.

Example 5.1. Let $F : [0,1] \to \mathbb{R}$ be given by $F(t) = t \sin \dfrac{1}{t}$ if $t \neq 0$, $F(0) = 0$, and $f = F'$, that is $f(t) = \sin \dfrac{1}{t} - \dfrac{1}{t} \cos \dfrac{1}{t}$ if $t \neq 0$ and $f(0) = 0$. Then $f \in H([0,1],\mathbb{R}) \setminus \mathcal{L}_1([0,1],\mathbb{R})$ and $F = \widetilde{f}$ (from the Fundamental Theorem of Calculus). Consider the initial value problem

$$\dot{x} + \alpha x = f, \qquad x(a) = 0 \tag{5.2}$$

where $x \in G([0,1],\mathbb{R})$ and $\alpha \in \mathcal{L}_1([0,1],\mathbb{R})$. Integrating (5.2) we obtain the Volterra-Lebesgue linear integral equation

$$x(t) + {}^{K}\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \tag{$(V)_L$}$$

By Theorem 4.1, if $\|\alpha\|_1 < 1$, then there exists a unique solution $x \in G([0,1],\mathbb{R})$ of equation $(V)_H$ (note that the indefinite integral F is continuous - Theorem 2.2). In other words, if $\|\alpha\|_1 < 1$, then there exists a unique Carathéodory solution x (which is in fact absolutely continuous by the Fundamental Theorem of Calculus) of equation (5.1).

Example 5.2. Consider the initial value problem

$$\dot{x} + \alpha x = \beta u, \quad x(a) = 0, \tag{5.3}$$

where $x \in C([a,b],\mathbb{R})$, $\alpha \in \mathcal{L}_1([a,b],\mathbb{R})$ and either $\beta \in H([a,b],\mathbb{R})$ and $u \in BV([a,b],\mathbb{R})$ or $\beta \in BV([a,b],\mathbb{R})$ and $u \in H([a,b],\mathbb{R})$. In any case there exists $F(t) = {}^{K}\int_{[a,t]} \beta(s)\, u(s)\, ds$, for every $t \in [a,b]$ (see [2] or [5]). Integrating (5.3) we obtain the linear integral equation of Volterra-Lebesgue

$$x(t) + {}^{K}\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \tag{$(V)_L$}$$

If $\|\alpha\|_1 < 1$, then there is a unique Carathéodory solution $x \in C([a,b],X)$ of equation (5.3) (Theorem 4.1).

Now we change the hypothesis. Let $x \in BV_a([a,b],\mathbb{R})$, $\alpha \in \mathcal{L}_1([a,b],\mathbb{R})$ and either $\beta \in \mathcal{L}_1([a,b],\mathbb{R})$ and $u \in BV([a,b],\mathbb{R})$ or $\beta \in BV([a,b],\mathbb{R})$ and $u \in \mathcal{L}_1([a,b],\mathbb{R})$, then there exists $F(t) = {}^{L}\int_{[a,t]} \beta(s)\, u(s)\, ds$, for every $t \in [a,b]$ (see [2] or [5]) and $F \in BV_a([a,b],\mathbb{R})$ (Theorem 2.10). By Theorem 4.1, there is a unique Carathéodory solution $x \in BV_a([a,b],X)$ of equation (5.3) whenever $\|\alpha\|_1 < 1$.

Example 5.3. The continuous-time linear dynamic system

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) \quad \dot{y}(t) = C(t)\, x(t) + D(t)\, u(t) \tag{5.4}$$

where A, B, C and D are continuous matrix 1×1, has solution given by

$$x(t) = e^{\int_{[a,t]} A(s)ds} x(a) + \int_{[a,t]} e^{\int_{[a,\tau]} A(s)ds} B(\tau)\, u(\tau)\, d\tau.$$

If $x \in C([a,b],\mathbb{R})$, $A \in \mathcal{L}_1([a,b],\mathbb{R})$ and either $B \in H([a,b],\mathbb{R})$ and $u \in BV([a,b],\mathbb{R})$ or $B \in BV([a,b],\mathbb{R})$ and $u \in H([a,b],\mathbb{R})$, then

$$x(t) = e^{L\int_{[a,t]} A(s)ds} x(a) + {}^K\int_{[a,t]} e^{L\int_{[a,\tau]} A(s)ds} B(\tau)\, u(\tau)\, d\tau \tag{5.5}$$

is a Henstock solution of the system. If moreover we have that $\|A\|_1 < 1$, then (5.5) is the unique (continuous) solution of (5.4) and, according to the first part of Example 5.2, it is in fact a Carathéodory solution.

Example 5.4. Consider the initial value problem

$$\dot{x} + \alpha x = f, \qquad x(a) = 0, \tag{5.6}$$

where $x \in BV_0([0,1],\mathbb{R})$, $f \in \mathcal{L}_1([0,1],\mathbb{R})$ and $\alpha : [0,1] \to \mathbb{R}$ is given by $\alpha(t) = \dfrac{\sin t}{t}$ if $t \neq 0$ and $\alpha(0) = 0$. Hence $\alpha \in H([0,1],\mathbb{R}) \setminus \mathcal{L}_1([0,1],\mathbb{R})$, $\alpha \in BV\left([\frac{1}{n},1],\mathbb{R}\right)$ for every $n \in \mathbb{N}^*$, and $F = \widetilde{f} \in BV_0([0,1],\mathbb{R})$ (by Theorem 2.10). Let us consider the Volterra-Henstock linear integral equation obtained from (5.6)

$$x(t) + {}^K\int_{[a,t]} \alpha(s) x t(s) ds = f(t), \quad t \in [a,b], \tag{$(V)_H$}$$

as well as the linear integral equations of Volterra-Henstock

$$x(t) + {}^K\int_{[a,t]} \chi_{X_n}(s)\, \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \text{ and } n \in \mathbb{N}, \tag{$(V)_{H_n}$}$$

By Theorem 3.6, for every $n \in \mathbb{N}^*$, there exists a unique solution $x_n \in H([0,1],\mathbb{R})_A$ of $(V)_{H_n}$ whose resolvent is given by the Neumann series:

$$\begin{aligned} x_n(t) &= F(t) + {}^K\int_{[0,t]\cap[\frac{1}{n},1]} \sum_{j=1}^{\infty} \frac{(-1)^{(j-1)} \sin(s)(-Si(t) + Si(s))^{(j-1)}}{(j-1)!\, s} F(s)\, ds \\ &= F(t) + {}^K\int_{[0,t]\cap[\frac{1}{n},1]} \frac{\sin(s) e^{(Si(t)-Si(s))}}{s} F(s)\, ds, \text{ for } t \in [0,1] \end{aligned}$$

where $Si(t) = \int_{[0,t]} \dfrac{\sin s}{s} ds$. However, since $F \in BV([0,1],\mathbb{R})$ and the kernel $\rho_n(t,s) = \dfrac{\sin(s)\, e^{(Si(t)-Si(s))}}{s}$ is bounded (Theorem 3.6), then it follows that x_n is of bounded variation. From Theorem 3.7, we have that

$$x(t) = \lim_n x_n(t) = F(t) + {}^K\int_{[0,t]} \frac{\sin(s)\, e^{(Si(t)-Si(s))}}{s} F(s)\, ds, \; t \in [0,1], \tag{5.7}$$

is of bounded variation and satisfies $(V)_H$. Thus x given by (5.7) is a Henstock solution of (5.6).

5.2 Applications to Singular Integral Equations

We call singular integral equations those integral equations whose kernel has a singularity. Such equations play an important role in the Theory of Integral Equations with applications in various areas.

Example 5.5. Let $x,\ f \in BV([0,1],\mathbb{R})$ and $\alpha : [0,1] \to \mathbb{R}$ be as in Example 4.4. Then according to Example 4.4, given $f \in BV_0([0,1],\mathbb{R})$, the singular integral equation

$$x(t) + {}^{K}\!\int_{[a,t]} \alpha(s)xt(s)ds = f(t), \quad t \in [a,b], \tag*{$(V)_H$}$$

admits a solution of bounded variation which is given by

$$x(t) = f(t) + {}^{K}\!\int_{[0,t]} \frac{\sin(s)\, e^{(Si(t)-Si(s))}}{s} f(s)\, ds, \quad t \in [0,1].$$

Example 5.6. Consider the unbounded function $\alpha(t) = \frac{1}{\sqrt{t}}$ for $t \in\ [0,1]$, and $\alpha(0) = 0$, and the singular integral equation

$$x(t) + {}^{K}\!\int_{[a,t]} \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \tag*{$(V)_L$}$$

where $x,\ f \in C([0,1],\mathbb{R})$. Since α is Lebesgue integrable on the interval $[0,1]$, then $\alpha \in\ \pounds_1\left([\frac{1}{n},1],\mathbb{R}\right)$ for each $n \in \mathbb{N}^*$, and we can consider the linear integral Volterra equations

$$x(t) + {}^{L}\!\int_{[a,t]} \chi_{X_n}(s)\, \alpha(s)\, x(s)\, ds = f(t), \quad t \in [a,b], \text{ and } n \in \mathbb{N}, \tag*{$(V)_{L_n}$}$$

where $x,\ f \in C([0,1],\mathbb{R})$ and $X_n = [\frac{1}{n},1]$ for each $n \in \mathbb{N}^*$. Given $f \in C([0,1],\mathbb{R})$, since $\alpha \in C\left([\frac{1}{n},1],\mathbb{R}\right)$ for every $n \in \mathbb{N}^*$, then there exists a continuous solution x_n of $(V)_{L_n}$ whose resolvent is given by the Neumann series:

$$\begin{aligned} x_n(t) &= f(t) + {}^{L}\!\int_{[0,t]\cap[\frac{1}{n},1]} \sum_{i=1}^{\infty} \frac{\left(\sqrt{t}-\sqrt{s}\right)^i}{\sqrt{s}} f(s)\ ds \\ &= f(t) + {}^{L}\!\int_{[0,t]\cap[\frac{1}{n},1]} \left(-\frac{1}{\sqrt{s}\left(\sqrt{t}-\sqrt{s}-1\right)}\right) f(s)\ ds, \quad t \in [0,1]. \end{aligned}$$

By Theorem 4.3 and the Monotone Convergence Theorem,

$$x(t) = \lim_n x_n(t) = f(t) + {}^L\int_{[0,t]} \left(-\frac{1}{\sqrt{s}\left(\sqrt{t}-\sqrt{s}-1\right)} \right) f(s)\, ds, \quad t \in [0,1],$$

is a continuous solution for the singular integral equation $(V)_L$.

Let us suppose now that $x, f \in BV_0([0,1],\mathbb{R})$ instead of $x, f \in C([0,1],\mathbb{R})$. From the fact that $\alpha \in BV\left([\frac{1}{n},1],\mathbb{R}\right)$ for every $n \in \mathbb{N}^*$, and given that $f \in BV_0([0,1],\mathbb{R}) \subset H([0,1],\mathbb{R})$, we can calculate the solution x_n of each $(V)_{L_n}$ by the Neumann series method (see Theorem 3.6). We have that

$$\begin{aligned} x_n(t) &= f(t) + {}^K\int_{[0,t]\cap[\frac{1}{n},t]} \sum_{i=1}^{\infty} \frac{\left(\sqrt{t}-\sqrt{s}\right)^i}{\sqrt{s}} f(s)\, ds \\ &= f(t) + {}^K\int_{[0,t]\cap[\frac{1}{n},t]} \left(-\frac{1}{\sqrt{s}\left(\sqrt{t}-\sqrt{s}-1\right)} \right) f(s)\,, ds, \quad t \in [0,1]. \end{aligned}$$

Then, by Theorem 4.4 and the Monotone Convergence Theorem,

$$x(t) = \lim_n x_n(t) = f(t) + {}^K\int_{[0,t]} \left(-\frac{1}{\sqrt{s}\left(\sqrt{t}-\sqrt{s}-1\right)} \right) f(s)\, ds, \quad t \in [0,1],$$

is a solution of bounded variation of the singular integral equation $(V)_L$.

The basis for this paper is [4].

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