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THOMSON'S VARIATIONAL MEASURE AND SOME CLASSICAL THEOREMS

Abstract

Using the conditions increasing* and decreasing*, and Thomson's variational measure, we give an easy proof of the Denjoy-Lusin-Saks Theorem [12, p. 230]. In Theorem 5.1 we extend (the function is not assumed to be continuous) Thomson's Theorems 44.1 and 44.2 of [13], that are closely related to the Denjoy-Lusin-Saks Theorem. From this extension we obtain another classical result: the Denjoy-Young-Saks Theorem [5]. As consequences of the Denjoy-Lusin-Saks Theorem we obtain two well-known results due to de la Vallée Poussin [12, p. 125, 127]. Then wee extend these results (the set E used there is not only Borel, but also Lebesgue measurable) and give in Theorem 8.1 a de la Vallée Poussin type theorem for VB^*G functions, that is in fact an extension of a result of Thomson [13, Theorem 46.3]. Finally, we give characterizations for Lebesgue measurable functions that are $VB^*G \cap (N)$, and for measurable functions that are $VB^*G \cap N^{+\infty}$ on a Lebesgue measurable set.

1 Introduction

Using the conditions increasing* and decreasing*, and Thomson's variational measure, we give (see Corollary 5.1, (i), (iii)) an easy proof of the following theorem of Saks:

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Theorem A. *Let $F : [a, b] \to \mathbb{R}$ and $E \subset [a, b]$. If F is VB^*G on E, then F is derivable a.e. on this set; and further if $N = \big\{x \in E : F'(x)$ does not exist (finite or infinite)$\big\}$, then $m\big(F(N)\big) = \Lambda\big(B(F; N)\big) = 0$.*

Since for continuous functions, this result has been proved independently by Denjoy and Lusin [12, p. 230], we call it the Denjoy-Lusin-Saks Theorem. In Theorem 5.1 (see also Corollary 5.1 and Remark 5.2) we extend (the function is not assumed to be continuous) Thomson's Theorems 44.1 and 44.2 of [13], that are closely related to Theorem A. From Theorem 5.1 we obtain another classical result: the Denjoy-Young-Saks Theorem [5].

Using Theorem A we obtain the following results of de la Vallée Poussin.

Theorem B. ([12, p. 125]) *For a function $F : [a, b] \to \mathbb{R}$ of bounded variation we have $\big|F^*(N)\big| = V_F^*(N) = m^*(N) = 0$ and $\Lambda\big(B(F; N)\big) = 0$, where V_F is the total variation of F and $N = \big\{x \in [a, b] : F$ is continuous at x, $F'(x)$ does not exist (finite or infinite)$\big\}$.*

Note that in the book of Saks [12], the proof of the Denjoy-Lusin-Saks Theorem is based on Theorem B.

Theorem C. ([12, p. 127]) *If $F : [a, b] \to \mathbb{R}$, $F \in VB$. Let $E_{+\infty} = \big\{x \in [a, b] : F'(x) = +\infty\big\}$, $E_{-\infty} = \big\{x \in [a, b] : F'(x) = -\infty\big\}$, and let V_F be the total variation of F.*

(i) *If X is a Borel measurable subset of $[a, b]$ and if F is continuous at each point of X, then*

$$F^*(X) = F^*\big(X \cap E_{+\infty}\big) + F^*\big(X \cap E_{-\infty}\big) + \int_X F'(x)\, dt\,,$$

and

$$V_F^*(X) = F^*\big(X \cap E_{+\infty}\big) + \big|F^*(X \cap E_{-\infty})\big| + \int_X \big|F'(x)\big|\, dx\,.$$

(ii) *Let $E = \big\{x \in [a, b] : F$ is continuous at x, F' and V_F' exist (finite or infinite), $V_F'(x) = |F'(x)|\big\}$. Then $V_F^*\big([a, b] \setminus E\big) = m^*\big([a, b] \setminus E\big) = 0$.*

In fact Theorem 7.2, (vii), (viii), (ix) is an extension of Theorem C (because in (vii) and (viii) the set E is not only Borel but also Lebesgue measurable). Note also that in order to prove Theorem C, Saks uses the Lebesgue Decomposition Theorem [12, p. 119], whereas our proof does not use this decomposition; it is instead essentially based on Theorem 8.2 of [4] (see Lemma 3.2).

In Theorem 8.1 we give a de la Vallée Poussin type theorem for VB^*G function, that is in fact an extension of a result of Thomson [13, Theorem 46.3].

Finally, as consequences of the previous results, we give characterizations: for Lebesgue measurable functions that are $VB^*G \cap (N)$, and for measurable functions that are $VB^*G \cap N^{+\infty}$ on a Lebesgue measurable set.

2 Preliminaries

Let $m^*(X)$ denote the outer measure of the set X and $m(E)$ the Lebesgue measure of E, whenever $E \subseteq \mathbb{R}$ is Lebesgue measurable. For the definitions of VB, VB^*, VB^*G and Lusin's condition (N), see [12]. We denote by $\mathcal{O}(F;[a,b])$ the oscillation of the function F on the closed interval $[a,b]$. Let $\mathrm{int}(E)$ denote the interior of the set E.

Definition 2.1. Let $F : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$. We denote by $V^*(F;E) = \big\{\sum_{k=1}^n \mathcal{O}(F;[a_k,b_k]) \ : \ \{[a_k,b_k]\}_{k=1}^n$ is a finite set of nonoverlapping closed intervals with $a_k, b_k \in E\big\}$.

Definition 2.2. [12, p 64.] Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function. For each set $E \subset \mathbb{R}$, let

$$F^*(E) = \inf\Big\{\sum_{i=1}^{\infty}\big(F(b_i) - F(a_i)\big) \ : \ E \subset \cup_{i=1}^{\infty}(a_i,b_i)\Big\}.$$

Lemma 2.1. [12] *Let F^* be defined as in Definition 2.2, and let $E \subset \mathbb{R}$.*

(i) *F^* is a metric outer measure (or with the notations of [12, p. 64], F^* is an outer measure in the sense of Caratheodory).*

(ii) *All Borel measurable sets of $\mathbb{R}$ are F^*-measurable; i.e.,*

$$F^*(X) = F^*(X \cap B) + F^*(X \setminus B)$$

whenever B is a Borel set and $X \subset \mathbb{R}$.

(iii) *For every $\epsilon > 0$, there is an open set G that contains E such that $F^*(G) \le F^*(E) + \epsilon$.*

(iv) *$F^*(E) = \inf\big\{F^*(G) \ : \ G$ is an open set that contains $E\big\}$.*

(v) *If F is continuous at each point of E, then $F^*(E) = m^*(F(E))$.*

(vi) *$F^*(A) = F(b-) - F(a+)$ for $A = (a,b)$, and $F^*(A) = F(b+) - F(a-)$ for $A = [a,b]$.*

Proof. (i) See [12, p. 64].
(ii) See Theorem 7.4 of [12, p. 52].
(iii) See Theorem 6.5, (i) of [12, p. 68].
(iv) See (iii).
(v) See [12, p. 100].
(vi) This is evident. □

Definition 2.3. Let $F : [a,b] \to \mathbb{R}$. For $x, y \in [a,b]$, $x < y$, let

$$\Delta F^{+}\big([x,y]\big) = \max\big\{F(y) - F(x),\ 0\big\} \text{ and}$$

$$\Delta F^{-}\big([x,y]\big) = \max\big\{F(x) - F(y),\ 0\big\}.$$

Clearly

$$\big|F(y) - F(x)\big| = \Delta F^{+}\big([x,y]\big) + \Delta F^{-}\big([x,y]\big).$$

Definition 2.4. [8, p. 51–52].
Let $F : [a,b] \to \mathbb{R}$. For each $x \in (a,b]$ let

$$V(F;[a,x]) = \sup\big\{\textstyle\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \,:\, a = x_0 < x_1 < \ldots < x_n = x\big\};$$

$$\overline{V}(F;[a,x]) = \sup\big\{\textstyle\sum_{i=1}^{n} \Delta F^{+}([x_{i-1}, x_i]) \,:\, a = x_0 < x_1 < \ldots < x_n = x\big\};$$

$$\underline{V}(F;[a,x]) = \sup\big\{\textstyle\sum_{i=1}^{n} \Delta F^{-}([x_{i-1}, x_i]) \,:\, a = x_0 < x_1 < \ldots < x_n = x\big\}.$$

Consider $F : \mathbb{R} \to \mathbb{R}$ where $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$. Let's put

$$V_F : \mathbb{R} \to \mathbb{R}, \quad V_F(x) = \begin{cases} 0 & \text{if } x \in (-\infty, a] \\ V(F;[a,x]) & \text{if } x \in (a,b] \\ V(F;[a,b]) & \text{if } x \in (b, +\infty) \end{cases}$$

$$\overline{V}_F : \mathbb{R} \to \mathbb{R}, \quad \overline{V}_F(x) = \begin{cases} 0 & \text{if } x \in (-\infty, a] \\ \overline{V}(F;[a,x]) & \text{if } x \in (a,b] \\ \overline{V}(F;[a,b]) & \text{if } x \in (b, +\infty) \end{cases}$$

$$\underline{V}_F : \mathbb{R} \to \mathbb{R}, \quad \underline{V}_F(x) = \begin{cases} 0 & \text{if } x \in (-\infty, a] \\ \underline{V}(F;[a,x]) & \text{if } x \in (a,b] \\ \underline{V}(F;[a,b]) & \text{if } x \in (b, +\infty) \end{cases}$$

Clearly $\underline{V}_F = \overline{V}_{-F}$.

Remark 2.1. Note that

$$\begin{aligned}&\overline{V}(F;[a,x]) = \overline{W}(F;[a,x]) = W_1([a,x])\,,\\&\underline{V}(F;[a,x]) = -\underline{W}(F;[a,x]) = -W_2([a,x]) \text{ and}\\&V(F;[a,x]) = W(F;[a,x]) = W([a,x])\end{aligned}$$

where the "W" variants are those defined in [12, p 61].

Theorem 2.1. [8, p. 52] *Let $F:[a,b]\to\mathbb{R}$, $F\in VB$. Then for $x\in[a,b]$ we have*

$$\begin{aligned}F(x)-F(a) &= \overline{V}(F;[a,x]) - \underline{V}(F;[a,x]) \text{ and}\\ V(F;[a,x]) &= \overline{V}(F;[a,x]) + \underline{V}(F;[a,x])\,.\end{aligned}$$

Thus, if one of the three numbers $V(F;[a,x])$, $\overline{V}(F;[a,x])$, $\underline{V}(F;[a,x])$ is finite, then the other two are also finite.

Definition 2.5. [12, p. 64]. Let $F:\mathbb{R}\to\mathbb{R}$, $F\in VB$ on $[a,b]$, F is constant on $(-\infty,a]$ and on $[b,+\infty)$. For each $E\subset\mathbb{R}$, let

$$F^*(E) = \overline{V}^*_F(E) - \underline{V}^*_F(E).$$

Lemma 2.2. *Let $F_1, F_2:\mathbb{R}\to\mathbb{R}$ be increasing functions, and let $E\subset\mathbb{R}$. Then*

$$(F_1+F_2)^*(E) = F_1^*(E) + F_2^*(E)\,.$$

In particular, we have $V^*_F(E) = \overline{V}^*_F(E) + \underline{V}^*_F(E)$.

PROOF. If $A=(a,b)$, then by Lemma 2.1, (vi) we have

$$(F_1+F_2)^*(A) = (F_1+F_2)(b-) + (F_1+F_2)(a+) =$$

$$F_1(b-) - F_1(a+) + F_2(b-) - F_2(a+) = F_1^*(A) + F_2^*(A)\,.$$

Now by Lemma 2.1, (ii), if B is an open set we have

$$(F_1+F_2)^*(B) = F_1^*(B) + F_2^*(B)\,.$$

Let G_1 and G_2 be open sets containing E, and let $G=G_1\cap G_2$. Then

$$(F_1+F_2)^*(E) \le (F_1+F_2)^*(G) = F_1^*(G) + F_2^*(G) \le F_1^*(G_1) + F_2^*(G_2)\,,$$

and by Lemma 2.1, (iv), it follows that $(F_1+F_2)^*(E)\le F_1^*(E)+F_2^*(E)$. Let D be an open set that contains E. Then

$$F_1^*(E) + F_2^*(E) \le F_1^*(D) + F_2^*(D) = (F_1+F_2)^*(D)\,.$$

Again by Lemma 2.1, (iv), we obtain that $F_1^*(E)+F_2^*(E)\le(F_1+F_2)^*(E)$. $\square$

3 Thomson's Variational Measure

Definition 3.1. Let $F : \mathbb{R} \to \mathbb{R}$, $E \subset \mathbb{R}$, $\delta : E \to (0, +\infty)$ and

$$\beta^*_\delta(E) = \Big\{ \big(\langle x, y\rangle, x\big) \,:\, x \in E\,,\; y \subset \big(x - \delta(x), x + \delta(x)\big) \Big\}.$$

A set $\pi = \big\{ \big(\langle x_i, y_i\rangle, x_i\big)\big\}_{i=1}^n$, with $\operatorname{int}\langle x_i, y_i\rangle \cap \operatorname{int}\langle x_j, y_j\rangle = \emptyset$ for $i \neq j$, is said to be a partition. Let

$$V^*_\delta(F; E) = \sup\Big\{ \sum_{i=1}^n |F(y_i) - F(x_i)| \,:\, \pi = \big\{ \big(\langle x_i, y_i\rangle, x_i\big)\big\}_{i=1}^n \text{ is a partition, } \pi \subset \beta^*_\delta(E) \Big\},$$

and let $\mu^*_F(E) = \inf_\delta V^*_\delta(F; E)$. Note that μ^*_F is in fact Thomson's variational measure $\mathcal{S}_o$-μ_F defined in [13].

Lemma 3.1. *Let $E \subset \mathbb{R}$. With the notations of Definition 3.1 we have:*

(i) *μ^*_F is a metric outer measure.*

(ii) *All Borel measurable sets of $\mathbb{R}$ are μ^*_F-measurable; i.e.*

$$\mu^*_F(X) = \mu^*_F(X \cap B) + \mu^*_F(X \setminus B)$$

whenever B is a Borel set and $X \subset \mathbb{R}$.

(iii) *If F is increasing on $\mathbb{R}$ and F is continuous at each point of E, then $\mu^*_F(E) = m^*(F(E))$.*

(iv) *For each $x \in E$ we have*

$$\mu^*_F\big(\{x\}\big) = \limsup_{t \to 0+} \big|F(x+t) - F(x)\big| + \limsup_{t \to 0-} \big|F(x+t) - F(x)\big|\,.$$

So, if F is increasing in a neighborhood of x, then

$$\mu^*_F\big(\{x\}\big) = F(x+) - F(x-)\,.$$

(v) *If F is VB on $[a, b]$ and constant on each of the intervals $(-\infty, a]$ and $[b, +\infty)$, then $\mu^*_F(E) = \mu^*_{V_F}(E)$.*

(vi) *$m^*(F(E)) \leq \mu^*_F(E)$.*

PROOF. (i) See [13, p. 40].

(ii) See Theorem 7.4 of [12, p. 52].

(iii) This follows easily.

(iv) See [13, p. 87].

(v) See [13, p. 92]

(vi) See [13, p. 101]. □

We denote by C_F the set of continuity points of the function F.

Lemma 3.2. [4, Theorem 8.2]. *Let $F : [a,b] \to \mathbb{R}$ and let E be a Lebesgue measurable subset of $[a,b]$. If $F \in VB^*G \cap (N)$ on E, then*

$$\mu_F(E \cap C_F) = (\mathcal{L}) \int_E \left|F'(t)\right| dt\,.$$

Lemma 3.3. [4, Corollary 6.1]. *Let $F, G : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$. If $F, G \in VB^*$ on E and $F = G$ on E, then*

$$\mu^*_F(E \cap C_F \cap C_G) = \mu^*_G(E \cap C_F \cap C_G)\,.$$

Lemma 3.4. *Let $F : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$. If F is increasing on $[a,b]$, then $\mu^*_F(E \cap C_F) = m^*(F(E \cap C_F))$.*

PROOF. This follows immediately by Lemma 3.1, (iii). □

4 The Conditions increasing*, decreasing* and VB*

Definition 4.1. ([7], [2, p. 47]) Let $F : [a,b] \to \mathbb{R}$, $E \subset [a,b]$, $c = \inf E$, $d = \sup E$. F is said to be increasing* (respectively decreasing*) on E if $F(x) \le F(y)$ (respectively $F(x) \ge F(y)$) whenever $c \le x < y \le d$ and $\{x,y\} \cap E \ne \emptyset$. F is said to be increasing$^*G$ (respectively decreasing*G) on E if there is a sequence of sets $\{E_n\}$ such that $E = \cup_n E_n$ and F is increasing* (respectively decreasing*) on each E_n. Note that the condition increasing* was introduced by Krzyzewski. See also the related condition "increasing around a set" of Thomson [13, p. 122].

Remark 4.1. Let $F : [a,b] \to \mathbb{R}$, $E \subset [a,b]$, $c = \inf E$, $d = \sup E$. Note that if F is increasing* on E, then $V^*(F;E) \le F(d) - F(c)$, so $F \in VB^*$ on E.

Lemma 4.1. *Let $F : \mathbb{R} \to \mathbb{R}$, and let E be a bounded set, $c = \inf E$, $d = \sup E$. The following assertions are equivalent.*

(i) *$F \in VB^*$ on E;*

(ii) $\sup\{\sum_{i=1}^{n} |F(d_i)-F(c_i)| : \{[c_i,d_i]\}_{i=1}^{n}$ *is a finite set of nonoverlapping closed intervals contained in* $[c,d]$, $\{c_i,d_i\}\cap E\neq\emptyset\} < +\infty$*;*

(iii) $\sup\{\sum_{i=1}^{n} \Delta F^{+}([c_i,d_i]) : \{[c_i,d_i]\}_{i=1}^{n}$ *is a finite set of nonoverlapping closed intervals contained in* $[c,d]$, $\{c_i,d_i\}\cap E\neq\emptyset\} < +\infty$*;*

(iv) $\sup\{\sum_{i=1}^{n} \Delta F^{-}([c_i,d_i]) : \{[c_i,d_i]\}_{i=1}^{n}$ *is a finite set of nonoverlapping closed intervals contained in* $[c,d]$, $\{c_i,d_i\}\cap E\neq\emptyset\} < +\infty$*;*

(v) *There exist* $F_1,F_2:[c,d]\to\mathbb{R}$ *increasing* on* E *such that* $F=F_1-F_2$.

PROOF. (i) $\Rightarrow$ (ii) Let $\{[c_i,d_i]\}_{i=1}^{n}$ be a finite set of nonoverlapping closed subintervals of $[c,d]$, with $\{c_i,d_i\}\cap E\neq\emptyset$. Let $\mathcal{A}_1=\{i : c_i\in E\}$ and $\mathcal{A}_2=\{i : c_i\notin E\}$. Suppose that $\mathcal{A}_1=\{i_1,i_2,\dots,i_p\}$, $p\le n$ and $c_{i_1}<c_{i_2}<\dots<c_{i_p}$. Then

$$\sum_{i\in\mathcal{A}_1}\left|F(d_i)-F(c_i)\right| \le \sum_{k=1}^{p-1}\mathcal{O}\big(F;[c_{i_k},c_{i_{k+1}}]\big)+\mathcal{O}\big(F;[c_{i_p},d]\big)\le V^*(F;\overline{E})\,.$$

Similarly $\sum_{i\in\mathcal{A}_2}\left|F(d_i)-F(c_i)\right| < V^*(F;\overline{E})$. Thus

$$\sum_{i=1}^{n}\left|F(d_i)-F(c_i)\right|\le 2V^*(F;\overline{E})\neq+\infty$$

(see [12, p. 229]), so we have (ii).

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are evident, because

$$\left|F(d_i)-F(c_i)\right| = \Delta F^{+}\big([c_i,d_i]\big)+\Delta F^{-}\big([c_i,d_i]\big)\,.$$

(iii) $\Rightarrow$ (v) Let $F_1:[c,d]\to\mathbb{R}$, $F_1(c)=0$, and for each $x\in(c,d]$, let

$$F_1(x)=\sup\Big\{\sum_{k=1}^{n}\Delta F^{+}([a_k,b_k]) : \{[a_k,b_k]\}_{k=1}^{n} \text{ is a finite set of nonoverlapping closed intervals with } \{a_k,b_k\}\cap E\neq\emptyset \text{ and } [a_k,b_k]\subset[c,x]\Big\}.$$

Let $F_2:[c,d]\to\mathbb{R}$, $F_2(x)=F_1(x)-F(x)$. Consider $x,y\in[c,d]$, $x<y$ with $\{x,y\}\cap E\neq\emptyset$. Then

$$F_1(y)-F_1(x)\ge\Delta F^{+}\big([x,y]\big)\ge F(y)-F(x)\,,$$

so $F_1(y)-F_1(x)\ge 0$ and $F_2(y)-F_2(x)\ge 0$. Therefore F_1 and F_2 are increasing* on E and $F=F_1-F_2$ on $[c,d]$.

(iv) $\Rightarrow$ (v) The proof is similar to that of (iii) $\Rightarrow$ (v).

(v) $\Rightarrow$ (i) By Remark 4.1, F_1 and F_2 are VB^* on E, so F is VB^* on E. □

Lemma 4.2. *Let $F : [a,b] \to \mathbb{R}$, $E \subset [a,b]$, $c = \inf E$, $d = \sup E$. Then F is increasing* on E if and only if there exist $H_1, H_2 : [c,d] \to \mathbb{R}$ increasing on $[c,d]$ such that $H_1(x) \le F(x) \le H_2(x)$ for each $x \in [c,d]$, and $H_1(x) = H_2(x) = F(x)$ for each $x \in E$.*

Moreover, let $[p,q] \subset [c,d]$:

- *If $p \in E$, then $H_1(q) - H_1(p) \le F(q) - F(p)$ and $H_2(q) - H_2(p) = \sup_{y\in[p,q]} F(y) - F(p)$.*
- *If $q \in E$, then $H_1(q) - H_1(p) = F(q) - \inf_{y\in[p,q]} F(y)$ and $H_2(q) - H_2(p) \le F(q) - F(p)$.*
- *If F is continuous and $x_o \in E$, then both, H_1 and H_2 are continuous at x_o.*

PROOF. "$\Rightarrow$" Let $H_1, H_2 : [c,d] \to \mathbb{R}$,

$$H_1(x) = \inf_{y\in[x,d]} F(y) \quad \text{and} \quad H_2(x) = \sup_{y\in[c,x]} F(y)\,.$$

Clearly H_1, H_2 are increasing on $[c,d]$ and $H_1(x) \le F(x) \le H_2(x)$ for each $x \in [c,d]$ and $H_1(x) = H_2(x) = F(x)$ for each $x \in E$.

"$\Leftarrow$" Let $x, y \in [c,d]$, $x < y$. If $x \in E$, then $F(x) = H_1(x) \le H_1(y) \le F(y)$. If $y \in E$, then $F(y) = H_2(y) \ge H_2(x) \ge F(x)$. Thus F is increasing* on E. $\square$

Corollary 4.1. [5, Proposition 2]. *Let $F : [a,b] \to \mathbb{R}$, $E \subset [a,b]$, F increasing* on E. Then F is derivable a.e. on E. Moreover, if F is VB^* on E, then F is derivable a.e. on E.*

Corollary 4.2. *Let $F : [a,b] \to \mathbb{R}$, $E \subset [a,b]$. If F is increasing* on E and F is continuous at each point of E, then*

$$\mu_F^*(E) = m^*\big(F(E)\big)\,.$$

PROOF. Let for example $H_1 : [a,b] \to \mathbb{R}$ be the function defined in Lemma 4.2. Then by Lemma 3.3 and Lemma 3.4 we obtain

$$\mu_F^*(E) = \mu_{H_1}^*(E) = m^*\big(H_1(E)\big) = m^*\big(F(E)\big)\,.$$

$\square$

Lemma 4.3. *Let $F : [a,b] \to \mathbb{R}$ and $E \subset [a,b]$ such that $\underline{D}F(x) > 0$ for each $x \in E$. Then F is increasing$^*G$ on E.*

PROOF. Let

$$E_n = \left\{x \in E \,:\, \frac{F(t)-F(x)}{t-x} > 0\,, \quad 0 < |t-x| \le \frac{1}{n}\right\}, \quad n = 1, 2, \ldots .$$

Let $E_{ni} = [\frac{i}{n}, \frac{i+1}{n}] \cap E_n$, $i = 0, \pm 1, \pm 2, \ldots$. Then $E = \cup E_{ni}$ and F is increasing* on each E_{ni}. □

5 The Denjoy-Lusin-Saks Theorem and an Extension of Two Theorems of Thomson

Definition 5.1. [5, p. 415] Let $\omega, F : [a,b] \to \mathbb{R}$, ω strictly increasing on $[a,b]$. We define the lower and upper derivatives of F with respect to ω at a point $x \in [a,b]$ as by

$$\underline{D}_\omega F(x) = \liminf_{y\to x} \frac{F(y)-F(x)}{\omega(y)-\omega(x)} \quad \text{and} \quad \overline{D}_\omega F(x) = \limsup_{y\to x} \frac{F(y)-F(x)}{\omega(y)-\omega(x)}\,.$$

F is said to be derivable with respect to ω at x if $\underline{D}_\omega F(x) = \overline{D}_\omega F(x) \in \mathbb{R}$. The derivative with respect to ω of F at x will be their common value and will be denoted by $F'_\omega(x)$.

Definition 5.2. [5, p. 416] Let $F : [a,b] \to \mathbb{R}$. A set $E \subset [a,b]$ is said to be F-null if $E = C \cup N$, with C an at most countable set and $\mu^*_F(N) = 0$. If F is the identity function, then the set E is said to be m-null.

Lemma 5.1. *Let $F : [a,b] \to \mathbb{R}$, $E \subset [a,b]$, $c = \inf E$, $d = supE$. If F is VB^* on E, then there exists a strictly increasing function $H : [c,d] \to \mathbb{R}$ such that $\mu^*_F(A) \le \mu^*_H(A)$, whenever $A \subset (c,d) \cap E$. Particularly, if $A \subseteq E$ is H-null, then A is F-null.*

PROOF. By Lemma 4.1 there exist $F_1, F_2 : [c,d] \to \mathbb{R}$ such that $F = F_1 - F_2$ and F_1, F_2 are increasing* on E. Let $G : [c,d] \to \mathbb{R}$, $G = F_1 + F_2$. Then G is increasing* on E and for $x, y \in [c,d]$ with $x < y$ and $\{x,y\} \cap E \ne \emptyset$ we have

$$\big|F(y)-F(x)\big| \le F_1(y) - F_1(x) + F_2(y) - F_2(x) = G(y) - G(x)\,.$$

By Lemma 4.2 there exist two increasing functions $H_1, H_2 : [c,d] \to \mathbb{R}$ such that $H_1(t) \le G(t) \le H_2(t)$ for $t \in [c,d]$ and $H_1(t) = H_2(t) = G(t)$ for $t \in E$. Let $H : [c,d] \to \mathbb{R}$, $H(t) = H_1(t) + H_2(t) + t$. If $x \in E$, then

$$\big|F(y)-F(x)\big| \le G(y) - G(x) \le H_2(y) - H_2(x) < H(y) - H(x)\,.$$

If $y \in E$, then

$$\left|F(y) - F(x)\right| \leq G(y) - G(x) \leq H_1(y) - H_1(x) < H(y) - H(x) .$$

Thus

$$\left|F(y) - F(x)\right| < H(y) - H(x) . \tag{1}$$

Let $A \subset (c, d) \cap E$. By (1) it follows immediately that $\mu_F^*(A) \leq \mu_H^*(A)$.

We show the second part. Let $D = \{x \in (c, d) \cap E : H$ is discontinuous at $x\}$. By (1), F is continuous on $E \setminus D$. Thus, if $A \subseteq E$ is H-null, then A is also F-null. □

Lemma 5.2. *Let $\omega, F : [a, b] \to \mathbb{R}$, $E \subset [a, b]$, ω strictly increasing on $[a, b]$ and $F \in VB^*$ on E. Then there exists a set $A \subset \overline{E}$ such that F is derivable with respect to ω on $\overline{E} \setminus A$, and A is an ω-null set.*

PROOF. Let $c = \inf E$, $d = supE$. Since $F \in VB^*$ on E, it follows that $F \in VB^*$ on $\overline{E}$ (see [12, p. 229]). We may suppose without loss of generality that F is increasing* on $\overline{E}$ (see Lemma 4.1). Then this is [5, Proposition 4]. □

Lemma 5.3 (Faure)**.** [5] *Let $\omega, F : [a, b] \to \mathbb{R}$, ω strictly increasing. If $F'_\omega(x) = 0$ on $A \subset [a, b]$, then $\mu_F^*(A) = 0$.*

Lemma 5.4. *Let $\omega, F : [a, b] \to \mathbb{R}$, ω strictly increasing, $E \subset [a, b]$. If $F \in VB^*$ on E, then the set $A = \{x \in E : \underline{D}_\omega F(x) \neq \overline{D}_\omega F(x)\}$ is F-null. Thus $F'_\omega(x)$ exists (finite or infinite) on $E \setminus A$.*

PROOF. By Lemma 5.1, for F there is a strictly increasing function $H : [c, d] \to \mathbb{R}$, $c = \inf E$, $d = \sup E$, such that if $B \subseteq E$ is H-null, then B is also F-null. Then the proof continues as in [5, Proposition 6]. □

Theorem 5.1. (An extension of Thomson's Theorems 44.1 and 44.2 of [13]). *Let $\omega, F : [a, b] \to \mathbb{R}$, ω strictly increasing, and let $E \subset [a, b]$. If $F \in VB^*G$ on E, then $F'_\omega(x)$ exists and is finite on E except an ω-null set A , and $F'_\omega(x)$ exists (finite or infinite) on E except a F-null subset B of A.*

PROOF. The first part follows by Lemma 5.2. The second part follows by Lemma 5.4 and the fact that the union of countable many ω-null sets is also an ω-null set. □

Lemma 5.5. *Let Z be a subset of $[a, b]$ such that $m^*(Z) = \mu_F^*(Z) = 0$. Then* $\Lambda(B(F; Z)) = 0$.

PROOF. Note that $m^*(Z) = \mu^*_\omega(Z)$, where ω is the identity function. Let $\epsilon > 0$. Since $m^*(Z) = \mu^*_F(Z) = 0$, there exists $\delta : Z \to (0, +\infty)$ such that $V^*_\delta(\omega, Z) < \frac{\epsilon}{4}$ and $V^*_\delta(F, Z) < \frac{\epsilon}{4}$. By the covering lemma of [9, p. 143], there exists a sequence $\{(\langle x_i, y_i\rangle, x_i)\}_i \subset \beta^*_\delta(Z)$ such that $\{(\langle x_i, y_i\rangle, x_i)\}_{i=1}^n$ is a partition for all n and $Z \subset \cup_i \langle x_i, y_i\rangle$. For each i let $c_i = \inf F\big(\langle x_i, y_i\rangle\big)$ and $d_i = \sup F\big(\langle x_i, y_i\rangle\big)$. Then we have

$$B(F, Z) \subset \cup_i \Big(\langle x_i, y_i\rangle \times [c_i, d_i]\Big).$$

For each i let $z_i \in \langle x_i, y_i\rangle$ such that $d_i - c_i < 3\big|F(z_i) - F(x_i)\big|$. Clearly

$$\operatorname{diam}\Big(\langle x_i, y_i\rangle \times [c_i, d_i]\Big) < |y_i - x_i| + 3\big|F(z_i) - F(x_i)\big|$$

and

$$\sum_i \operatorname{diam}\Big(\langle x_i, y_i\rangle \times [c_i, d_i]\Big) \le V^*_\delta(\omega, Z) + 3V^*_\delta(F, Z) < \epsilon\,.$$

It follows that $\Lambda\big(B(F, Z)\big) \le \epsilon$, and $\Lambda\big(B(F, Z)\big) = 0$ since ϵ is arbitrary. □

Remark 5.1. Lemma 5.5 is asserted by Faure in [5, p. 417] without proof.

Lemma 5.6. *Let $F : [a, b] \to \mathbb{R}$, and let Z be a subset of $[a, b]$ with $m^*(Z) = 0$, such that $F \in VB^*G$ on Z. Then the following assertions are equivalent.*

(i) *Z is F-null.*

(ii) $\Lambda\big(B(F; Z)\big) = 0$.

(iii) $m^*(F(Z)) = 0$.

PROOF. (i) $\Rightarrow$ (ii) See Lemma 5.5 and note that $\Lambda(B(F; A)) = 0$ whenever A is a countable set.

(ii) $\Rightarrow$ (iii) This is evident (see for example [12, p. 269] or [6, p. 31]).

(iii) $\Rightarrow$ (i) Let $D = \{x \in Z : F \text{ is discontinuous at } x\}$. By [3, Theorem 8], it follows that $\mu^*_F(Z \setminus D) = 0$. Thus Z is F-null. □

Corollary 5.1. *Let $F : [a, b] \to \mathbb{R}$, and let E be a subset of $[a, b]$ such that F is VB^*G on E. Let $Z = \{x \in E : F'(x)$ does not exist (finite or infinite)$\}$. Then:*

(i) *F is derivable a.e. on E;*

(ii) *Z is F-null;*

(iii) $\Lambda\big(B(F;Z)\big) = 0$;

(iv) $m^*(F(Z)) = 0$.

Moreover, (ii), (iii) and (iv) are equivalent.

Proof. (i), (ii) follow from Theorem 5.1. The other parts follow by Lemma 5.6. □

Remark 5.2.
Corollary 5.1, (i) is identic with Thomson's Theorem 44.1 of [13, p. 103].

Corollary 5.1, (ii) extends Thomson's Theorem 44.2 of [13, p. 104]. (Note that F is not assumed to be continuous.)

Corollary 5.1, (i), (iii) is in fact Theorem A. Since for continuous functions, this result has been proved independently by Denjoy and Lusin [12], we call it the Denjoy-Lusin-Saks Theorem.

6 The Denjoy-Young-Saks Theorem

Theorem 6.1 (Denjoy-Young-Saks)**.** ([5, Theorem 7] *Let* $\omega, F : [a,b] \to \mathbb{R}$, ω *strictly increasing. Let*

- $E_1 = \{x : F$ *is derivable with respect to* $\omega\}$;
- $E_2 = \{x : \underline{D}_\omega F(x) = -\infty$ *and* $\overline{D}_\omega F(x) = +\infty\}$;
- $E_3 = \{x : \underline{D}_\omega F(x) = \overline{D}_\omega F(x) = \pm\infty\}$;
- $E_4 = [a,b] \setminus (E_1 \cup E_2 \cup E_3)$.

Then

(i) $[a,b] \setminus (E_1 \cup E_2)$ *is* ω*-null and contains* E_3*, so* E_3 *is* ω*-null.*

(ii) E_4 *is both* ω*-null and* F*-null.*

Proof. The proof follows from Theorem 5.1 as in [5, p. 417]. □

Corollary 6.1. *Let* $F : [a,b] \to \mathbb{R}$*. Let*

- $E_1 = \{x : F$ *is derivable at* $x\}$;
- $E_2 = \{x : \underline{D}F(x) = -\infty$ *and* $\overline{D}F(x) = +\infty\}$;
- $E_3 = \{x : \underline{D}F(x) = \overline{D}F(x) = \pm\infty\}$;

- $E_4 = [a,b] \setminus (E_1 \cup E_2 \cup E_3)$.

Then

(i) $[a,b] \setminus (E_1 \cup E_2)$ *is* m*-null and contains* E_3*, so* E_3 *is* m*-null;*

(ii) E_4 *is both* m*-null and* F*-null.*

Moreover, (ii) may be replaced by "$\Lambda(B(F;E_4)) = 0$*", or by "*$m^*(F(E_4)) = 0$*".*

PROOF. (i) and (ii) follow by Theorem 6.1 with ω the identity function.

We show the second part. Since $E_4 \subset [a,b] \setminus E_2$, it follows that F is VB^*G on E_4 (see [12, p. 234]). Since E_4 is m-null, the assertion follows by Lemma 5.6. □

7 Extensions of Theorem B and Theorem C of de la Vallée Poussin

Theorem 7.1. *Let* $F : \mathbb{R} \to \mathbb{R}$*, and let* $E \subset \mathbb{R}$*. If* F *is increasing on* $[a,b]$ *and* F *is constant on* $(-\infty, a]$ *and on* $[b, +\infty)$*, then* $\mu_F^*(E) = F^*(E)$*.*

PROOF. Let $D = \{x \in E : F \text{ is discontinuous at } x\}$. Then D is countable. Suppose that $D = \{d_1, d_2, \ldots, d_i, \ldots\}$. By Lemma 2.1, (vi) and Lemma 3.1, (iv) we have

$$F^*(D) = \sum_i F^*\big(\{d_i\}\big) = \sum_i \mu_F^*\big(\{d_i\}\big) = \mu_F^*(D)\,.$$

The set D being Borel measurable, by Lemma 2.1, (ii), (vii) and Lemma 3.1, (ii), (iii), it follows that

$$F^*(E) = F^*(D) + F^*(E \setminus D) = \mu_F^*(D) + \mu_F^*(E \setminus D) = \mu_F^*(E)\,.$$

□

Corollary 7.1. *Let* $F : \mathbb{R} \to \mathbb{R}$*, and let* $E \subset \mathbb{R}$*. Suppose that* F *is* VB *on* $[a,b]$ *and* F *is constant on* $(-\infty, a]$ *and on* $[b, +\infty)$*.*

(i) $\mu_F^*(E) = \mu_{V_F}^*(E) = V_F^*(E) = \overline{V}_F^*(E) + \underline{V}_F^*(E)$ *;*

(ii) *If* $\mu_F^*(E) = 0$*, then* $V_F^*(E) = \overline{V}_F^*(E) = \underline{V}_F^*(E) = F^*(E) = \mu_{\overline{V}_F}^*(E) = \mu_{\underline{V}_F}^*(E) = 0$*.*

PROOF. (i) follows from Lemma 3.1, (v), Theorem 7.1 and Lemma 2.2, and (ii) is evident. □

Corollary 7.2 (Theorem B)**.** *Let $F : \mathbb{R} \to \mathbb{R}$, $F \in VB$ on $[a,b]$, F constant on $(-\infty, a]$ and on $[b, +\infty)$. Let $Z = \{x \in [a,b] : F$ is continuous at x and $F'(x)$ does not exist (finite or infinite)$\}$. Then we have*

$$F^*(Z) = V_F^*(Z) = \mu_F^*(Z) = m^*(Z) = 0 = \Lambda\big(B(F;Z)\big) = 0\,.$$

PROOF. For $m^*(Z) = \mu_F^*(Z) = \Lambda(B(F;Z)) = 0$ see Corollary 5.1, (i), (ii), (iii). That $V_F^*(Z) = F^*(Z) = 0$ follows now by Corollary 7.1. □

Lemma 7.1. *Let $F : \mathbb{R} \to \mathbb{R}$, $a \le c < d \le b$. Suppose that F is constant on $(-\infty, a]$ and on $[b, +\infty)$. Then:*

(i) $\overline{V}_F(d) - \overline{V}_F(c) \le \overline{V}\big(F;[c,d]\big) \le V\big(F;[c,d]\big) = V_F(d) - V_F(c)$;

(ii) *Let $E \subset [a,b]$ such that $[c,d] \subset [\inf E, \sup E]$. If $\{(c_i,d_i)\}_{i=1}^{\infty}$ are the intervals contiguous to $\big(\overline{E} \cap [c,d]\big) \cup \{c,d\}$ and F is decreasing* on E, then $\overline{V}\big(F;[c,d]\big) \le \sum_i V\big(F;[c_i,d_i]\big)$, so $\overline{V}_F(d) - \overline{V}_F(c) \le \sum_i V\big(F;[c_i,d_i]\big)$.*

PROOF. (i) Let $\{[\alpha_j,\beta_j]\}_{j=1}^{n}$ be a finite set of nonoverlapping closed intervals contained in $[a,d]$. Suppose that $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \ldots \le \alpha_n < \beta_n$ and $c \in (\alpha_{j_o}, \beta_{j_o})$ (the case $c \notin (\alpha_j,\beta_j)$, $j = 1,2,\ldots,n$ is easier). Then

$$\sum_{j=1}^{n}\big(F(\beta_j) - F(\alpha_j)\big) = \sum_{j=1}^{j_o-1}\big(F(\beta_j) - F(\alpha_j)\big) + F(c) - F(\alpha_{j_o}) +$$

$$+F(\beta_{j_o}) - F(c) + \sum_{j=j_o+1}^{n}\big(F(\beta_j) - F(\alpha_j)\big) \le \overline{V}_F(c) + \overline{V}\big(F;[c,d]\big)\,.$$

It follows that $\overline{V}_F(d) - \overline{V}_F(c) \le \overline{V}\big(F;[c,d]\big)$. The other parts are evident.

(ii) Let $\{[a_k,b_k]\}_{k=1}^{m}$ be a finite set of nonoverlapping closed intervals contained in $[c,d]$. Clearly if $[\alpha,\beta] \cap E \ne \emptyset$ and $[\alpha,\beta] \subset [c,d]$, then $F(\beta) - F(\alpha) \le 0$. Let

$$\mathcal{A} = \big\{k \in \{1,2,\ldots,m\} \,:\, F(b_k) - F(a_k) > 0\big\}\,.$$

Then for each $k \in \mathcal{A}$, $[a_k,b_k] \cap E = \emptyset$, so $[a_k,b_k] \subset [c_{i_k}, d_{i_k}]$ for some i_k. We also have that

$$\sum_{k=1}^{m}\big(F(b_k) - F(a_k)\big) \le \sum_{k \in \mathcal{A}}\big(F(b_k) - F(a_k)\big) \le \sum_{i} V\big(f;[c_i,d_i]\big)\,.$$

□

Lemma 7.2. *Let $F : [a,b] \to \mathbb{R}$, $F \in VB$ on $[a,b]$, and let $E \subset [a,b]$ such that F is continuous at each point of E. If F is decreasing* on E, then $\mu^*_{\overline{V}_F}(E) = 0$. Consequently, if F is decreasing$^*G$ on E, then $\mu^*_{\overline{V}_F}(E) = 0$, and if F is increasing$^*G$ on E, then $\mu^*_{\underline{V}_F}(E) = 0$.*

PROOF. Let $c = \inf E$, $d = \sup E$, and let $\{(c_i, d_i)\}_{i=1}^{\infty}$ be the intervals contiguous to $\overline{E}$ (for $i = 1, 2, \ldots, n$ the proof is easier). It is well known that V_F is continuous at each $x \in E$. Thus by Lemma 7.1, (i), $\overline{V}_F$ is continuous at such a x. It follows that

$$\mu^*_{\overline{V}_F}\Big(E \cap \big(\cup_{i=1}^{\infty}\{c_i, d_i\} \cup \{c, d\}\big)\Big) = 0\,,$$

so we may suppose without loss of generality that E contains neither c_i or d_i, nor c or d. Since $\sum_{i=1}^{\infty} V(F; [c_i, d_i]) < V(F; [a,b])$, for $\epsilon > 0$ there is an i_o such that

$$\sum_{i=i_o}^{\infty} V\big(F; [c_i, d_i]\big) < \epsilon\,.$$

Let $G = (c,d) \setminus \cup_{i=1}^{i_o-1}[c_i, d_i]$. Clearly $E \subset G$. Let $\delta : E \to (0, +\infty)$ be such that $\big(x - \delta(x), x + \delta(x)\big) \subset G$. Let $\pi = \big\{\big(\langle x_j, y_j\rangle, x_j\big)\big\}_{j=1}^{p} \subset \beta^*_{\delta}(E)$ be a partition. We may suppose without loss of generality that $x_j < y_j$ for each $j = 1, 2, \ldots, p$. By Lemma 7.1, we have that

$$\sum_{j=1}^{p}\big(\overline{V}_F(y_j) - \overline{V}_F(x_j)\big) \le \sum_{i=i_o}^{\infty} V\big(F; [c_i, d_i]\big) < \epsilon\,.$$

In general, it follows that $V^*_{\delta}(\overline{V}_F; E) \le 2\epsilon$; so $\mu^*_{\overline{V}_F}(E) \le 2\epsilon$. Since ϵ is arbitrary, we obtain that $\mu^*_{\overline{V}_F}(E) = 0$.

The second part follows from the fact that, if F is increasing$^*G$ on E, then $-F$ is decreasing$^*G$ on E and $\underline{V}_F(x) = \overline{V}_{-F}(x)$. $\square$

Corollary 7.3. *Let $F : [a,b] \to \mathbb{R}$, be a VB function, and let $E \subset [a,b]$ such that F is continuous at each point of E. If F is increasing$^*G$ on E, then*

$$F^*(E) = V^*_F(E) = \mu^*_F(E) = \overline{V}^*_F(E)\,.$$

Moreover, if F is decreasing$^*G$ on E, then

$$-F^*(E) = V^*_F(E) = \mu^*_F(E) = \underline{V}^*_F(E)\,.$$

PROOF. See Lemma 7.2 and Corollary 7.1, (i). $\square$

Theorem 7.2. *Let $F : [a,b] \to \mathbb{R}$ be a VB function. Let*

$$Z = \big\{x \in [a,b] \,:\, F'(x) \text{ does not exist (finite or infinite)}\big\};$$

$$E_{+\infty} = \big\{x \in [a,b] \,:\, F'(x) = +\infty\big\};$$

$$E_0 = \big\{x \in [a,b] \,:\, F'(x) = 0\big\};$$

$$E_{-\infty} = \big\{x \in [a,b] \,:\, F'(x) = -\infty\big\};$$

$$P = \big\{x \in [a,b] \,:\, F'(x) \in (0,+\infty)\big\};$$

$$N = \big\{x \in [a,b] \,:\, F'(x) \in (-\infty,0)\big\}.$$

Then we have:

(i) $\mu^*_{\overline{V}_F}(Z) = \mu^*_{\overline{V}_F}(E_0) = \mu^*_{\overline{V}_F}(E_{-\infty}) = \mu^*_{\overline{V}_F}(N) = 0$;

(ii) $\mu^*_{\underline{V}_F}(Z) = \mu^*_{\underline{V}_F}(E_0) = \mu^*_{\underline{V}_F}(E_{+\infty}) = \mu^*_{\underline{V}_F}(P) = 0$;

(iii) $\mu^*_{\overline{V}_F}(E\cap P) = \mu^*_{V_F}(E\cap P) = V^*_F(E\cap P) = \mu^*_F(E\cap P) = (\mathcal{L})\int_{E\cap P} F'(t)\,dt$, *whenever E is a Lebesgue measurable subset of $[a,b]$;*

(iv) $\mu^*_{\underline{V}_F}(E\cap N) = \mu^*_{V_F}(E\cap N) = V^*_F(E\cap N) = \mu^*_F(E\cap N) = -(\mathcal{L})\int_{E\cap N} F'(t)\,dt$, *whenever E is a Lebesgue measurable subset of $[a,b]$;*

(v) $\overline{V}^*_F(E) = \mu^*_{\overline{V}_F}(E) = \mu^*_{\overline{V}_F}(E\cap E_{+\infty}) + (\mathcal{L})\int_{E\cap P} F'(t)\,dt$, *whenever E is a Lebesgue measurable subset of $[a,b]$ and F is continuous at each point of E;*

(vi) $\underline{V}^*_F(E) = \mu^*_{\underline{V}_F}(E) = \mu^*_{\underline{V}_F}(E\cap E_{-\infty}) + (\mathcal{L})\int_{E\cap N} F'(t)\,dt$, *whenever E is a Lebesgue measurable subset of $[a,b]$ and F is continuous at each point of E;*

(vii) $F^*(E) = F^*(E\cap E_{+\infty}) + F^*(E\cap E_{-\infty}) + (\mathcal{L})\int_E F'(t)\,dt$, *whenever E is a Lebesgue measurable subset of $[a,b]$ and F is continuous at each point of E;*

(viii) $V^*_F(E) = F^*(E\cap E_{+\infty}) + \big|F^*(E\cap E_{-\infty})\big| + (\mathcal{L})\int_E |F'(t)|\,dt$, *whenever E is a Lebesgue measurable subset of $[a,b]$ and F is continuous at each point of E;*

(ix) $V^*_F([a,b]\setminus A) = m^*([a,b]\setminus A) = 0$, *where $A = \{x \in [a,b] \,:\, V'_F(x) = |F'(x)|$, F is continuous at $x\}$.*

PROOF. Note that F satisfies Lusin's condition (N) on $E_0 \cup P \cup N$ (see [12]).

(i) By Corollary 7.2, $\mu_F^*(Z) = 0$, and by Lemma 5.3, $\mu_F^*(E_0) = 0$. It follows that $\mu_{\overline{V}_F}^*(Z) = \mu_F^*(E_0) = 0$ (see Corollary 7.1, (ii)). By Lemma 4.3, F is decreasing$^*G$ on $E_{-\infty} \cup N$ so by Lemma 7.2 we have that $\mu_{\overline{V}_F}^*(E_{-\infty}) = \mu_{\overline{V}_F}^*(N) = 0$.

(ii) The proof follows by (i), because $\mu_{\underline{V}_F}^* = \mu_{\overline{V}_{-F}}^*$.

(iii) By Lemma 3.2 we have $\mu_F^*(E \cap P) = (\mathcal{L}) \int_{E\cap P} F'(t)\,dt$, and by Corollary 7.1, (i) it follows that

$$\mu_F^*(E \cap P) = \mu_{V_F}^*(E \cap P) = \mu_{\overline{V}_F}^*(E \cap P) + \mu_{\underline{V}_F}^*(E \cap P) = \mu_{\overline{V}_F}^*(E \cap P)$$

(see also (ii)).

(iv) The proof is similar to that of (iii).

(v) That $\overline{V}_F^*(E) = \mu_{\overline{V}_F}^*(E)$ follows by Theorem 7.1. Since $Z \cup E_{+\infty} \cup E_0 \cup E_{-\infty} \cup P \cup N = [a, b]$ and because Z, $E_{+\infty}$, E_0, $E_{-\infty}$, P and N are all Borel sets (so $\mu_{\overline{V}_F}^*$-measurable), Lemma 3.1, (ii) and by (i) and (iii) above, it follows that

$$\begin{aligned}
\mu_{\overline{V}_F}^*(E) = {} & \mu_{\overline{V}_F}^*(E \cap Z) + \mu_{\overline{V}_F}^*(E \cap E_{+\infty}) + \mu_{\overline{V}_F}^*(E \cap E_0) \\
& + \mu_{\overline{V}_F}^*(E \cap E_{-\infty}) + \mu_{\overline{V}_F}^*(E \cap P) + \mu_{\overline{V}_F}^*(E \cap N) \\
= {} & \mu_{\overline{V}_F}^*(E \cap E_{+\infty}) + \mu_{\overline{V}_F}^*(E \cap P) \\
= {} & \mu_{\overline{V}_F}^*(E \cap E_{+\infty}) + (\mathcal{L}) \int_{E\cap P} F'(t)\,dt\,.
\end{aligned}$$

(vi) The proof is similar to that of (v).

(vii) We have

$$\begin{aligned}
F^*(E) = {} & \overline{V}_F^*(E) - \underline{V}_F^*(E) = \overline{V}_F^*(E \cap E_{+\infty}) \\
& + (\mathcal{L}) \int_{E\cap P} F'(t)\,dt - \underline{V}_F^*(E \cap E_{-\infty}) + (\mathcal{L}) \int_{E\cap N} F'(t)\,dt \\
= {} & F^*(E \cap E_{+\infty}) + F^*(E \cap E_{-\infty}) + (\mathcal{L}) \int_{E\cap(P\cup N\cup E_0)} F'(t)\,dt \\
= {} & F^*(E \cap E_{+\infty}) + F^*(E \cap E_{-\infty}) + (\mathcal{L}) \int_{E} F'(t)\,dt
\end{aligned}$$

(see (ii), (i) and the facts that $(\mathcal{L}) \int_{E_0} F'(t)\,dt = 0$ and $m\big(E \setminus (P \cup N \cup E_0)\big) = 0$).

(viii) By Corollary 7.1, (i) we have:

$$\begin{aligned} V_F^*(E) = {} & \overline{V}_F^*(E) + \underline{V}_F^*(E) = \overline{V}_F^*(E \cap E_{+\infty}) \\ & + (\mathcal{L}) \int_{E \cap P} F'(t)\, dt + \underline{V}_F^*(E \cap E_{-\infty}) + (\mathcal{L}) \int_{E \cap N} |F'(t)|\, dt \\ = {} & F^*(E \cap E_{+\infty}) + \big|F^*(E \cap E_{-\infty})\big| + (\mathcal{L}) \int_E |F'(t)|\, dt \end{aligned}$$

(see (ii) and (i)).

(ix) By [11, Theorem, p. 15] it follows that $V_F'(x) = |F'(x)| \in [0, +\infty)$ *a.e.* on $[a,b]$, so $m^*\big([a,b] \setminus A\big) = 0$. By (viii), we have

$$V_F^*\big([a,b] \setminus A\big) = F^*\Big(\big([a,b] \setminus A\big) \cap E_{+\infty}\Big) + F^*\Big(\big([a,b] \setminus A\big) \cap E_{-\infty}\Big).$$

If $x \in E_{+\infty}$, then $F'(x) = +\infty$, so $V_F'(x) = +\infty$. Hence $x \in A$, and so $\big([a,b] \setminus A\big) \cap E_{+\infty} = \emptyset$. Similarly $\big([a,b] \setminus A\big) \cap E_{-\infty} = \emptyset$. It follows that $V_F^*\big([a,b] \setminus A\big) = 0$. □

Remark 7.1. Theorem 7.2, (vii), (viii), (ix) strictly contains Theorem C, because in (vii) and (viii) the set E is not only Borel but also Lebesgue measurable. Note also in order to prove Theorem C, Saks uses the Lebesgue Decomposition Theorem [12, p. 119], whereas our proof does not use this decomposition; it is instead essentially based on Theorem 8.2 of [4] (see Lemma 3.2).

8 A de la Vallée Poussin Type Theorem for VB*G Functions (An Extension of a Theorem of Thomson)

Lemma 8.1 (Thomson)**.** [13, Lemma 42.1]. *Let $F : [a,b] \to \mathbb{R}$, $E \subset [a,b]$. Then $\mu_F^*(E_o) = 0$, where $E_o = \{x \in [a,b] : F'(x) = 0\}$.*

Definition 8.1. With the notations of Definition 3.1, let:

- $\overline{V}_\delta^*(F;E) = \sup\big\{\sum_{i=1}^n \Delta F^+\big(\langle x_i, y_i\rangle\big) : \pi = \big\{\big(\langle x_i, y_i\rangle, x_i\big)\big\}_{i=1}^n$ is a partition, $\pi \subset \beta_\delta^*(E)\big\}$;
- $\underline{V}_\delta^*(F;E) = \sup\big\{\sum_{i=1}^n \Delta F^-\big(\langle x_i, y_i\rangle\big) : \pi = \big\{\big(\langle x_i, y_i\rangle, x_i\big)\big\}_{i=1}^n$ is a partition, $\pi \subset \beta_\delta^*(E)\big\}$;
- $\overline{\mu}_F^*(E) = \inf_\delta \overline{V}_\delta^*(F;E)$;
- $\underline{\mu}_F^*(E) = \inf_\delta \underline{V}_\delta^*(F;E)$;

Lemma 8.2. *Let $F : \mathbb{R} \to \mathbb{R}$, $E \subset \mathbb{R}$. Then we have:*

(i) $\overline{\mu}^*_F(E) \leq \mu^*_F(E)$;

(ii) $\underline{\mu}^*_F(E) \leq \mu^*_F(E)$;

(iii) $\mu^*_F(E) \leq \overline{\mu}^*_F(E) + \underline{\mu}^*_F(E)$.

PROOF. All assertions follow from the equality

$$\big|F(y) - F(x)\big| = \Delta F^+\big([x,y]\big) + \Delta F^-\big([x,y]\big)\,.$$

$\square$

Lemma 8.3. *Let $F : [a,b] \to \mathbb{R}$. Then $\mu^*_F(E) = \overline{\mu}^*_F(E)$ whenever $E \subset \{x \in [a,b] : F'(x) \in [0,+\infty]\}$.*

PROOF. We always have $\mu^*_F(E) \geq \overline{\mu}^*_F(E)$. We show the converse inequality. Let $P = \{x \in [a,b] : F'(x) \in (0,+\infty]\}$, $A \subset P$ and let $\eta : A \to (0,+\infty)$ such that

$$\frac{F(y) - F(x)}{y - x} > 0 \quad \text{whenever} \quad y \in \big(x - \eta(x), x + \eta(x)\big) \setminus \{x\}\,.$$

Let $\delta : A \to (0,+\infty)$, and let $\delta_1(x) = \min\{\delta(x), \eta(x)\}$ for each $x \in A$. If $\big([x,y],x\big)$ or $([x,y],y) \in \beta^*_{\delta_1}(A)$, then $0 < F(y) - F(x) = \Delta F^+([x,y])$. It follows that $(\langle x,y\rangle, x) \in \beta^*_{\delta_1}(A)$ and $\Delta F^+(\langle x,y\rangle) = |F(y) - F(x)|$. Hence

$$\mu^*_F(A) \leq V^*_{\delta_1}(F;A) = \overline{V}^*_{\delta_1}(F;A) \leq \overline{V}^*_{\delta}(F;A)\,.$$

Therefore $\mu^*_F(A) \leq \overline{\mu}^*_F(A)$. Now we obtain

$$\mu^*_F(E) \leq \mu^*_F(E \cap P) + \mu^*_F(E \cap E_0) = \mu^*_F(E \cap P) \leq \overline{\mu}^*_F(E \cap P) \leq \overline{\mu}^*_F(E)\,,$$

where $E_0 = \{x \in [a,b] : F'(x) = 0\}$. $\square$

Lemma 8.4. *Let $F : [a,b] \to \mathbb{R}$. Then $\overline{\mu}^*_F(E) = 0$ whenever $E \subset \{x \in [a,b] : F'(x) \in [-\infty, 0]\}$.*

PROOF. Let $N = \{x \in [a,b] : F'(x) \in [-\infty, 0)\}$, $A \subset N$, and let $\delta : A \to (0,+\infty)$ such that

$$\frac{F(y) - F(x)}{y - x} < 0 \quad \text{whenever} \quad y \in \big(x - \delta(x), x + \delta(x)\big) \setminus \{x\}\,.$$

If $([x,y],x)$ or $([x,y],y) \in \beta^*_\delta(A)$, then $F(y) - F(x) < 0$; so $\Delta F^+([x,y]) = 0$. It follows that $\overline{\mu}^*_F(A) \leq \overline{V}^*_\delta(F;A) = 0$, so $\overline{\mu}^*_F(A) = 0$. Now we obtain that

$$\overline{\mu}^*_F(E) \leq \overline{\mu}^*_F(E \cap N) + \overline{\mu}^*_F(E \cap E_0) \leq 0 + \mu^*_F(E \cap E_0) = 0 + 0 = 0\,,$$

where $E_0 = \{x \in [a,b] : F'(x) = 0\}$. $\square$

Theorem 8.1. (An extension of Theorem 46.3 of [13, p. 107]).
Let $F : \mathbb{R} \to \mathbb{R}$, and let $E \subset \mathbb{R}$ such that F is continuous at each point of E and $F \in VB^*G$ on E. Let $E_{+\infty} = \{x : F'(x) = +\infty\}$, $E_{-\infty} = \{x : F'(x) = -\infty\}$, $D = \{x : F'(x) \in (-\infty, +\infty)\}$, $P = \{x : F'(x) = (0, +\infty)\}$, $N = \{x : F'(x) = (-\infty, 0)\}$. Then we have:

(i) $\mu_F^*(E) = \mu_F^*(E \cap E_{+\infty}) + \mu_F^*(E \cap E_{-\infty}) + \mu_F^*(E \cap P) + \mu_F^*(E \cap N) = \mu_F^*(E \cap E_{+\infty}) + \mu_F^*(E \cap E_{-\infty}) + \mu_F^*(E \cap D)$;

(ii) $\overline{\mu}_F^*(E) = \mu_F^*(E \cap E_{+\infty}) + \mu_F^*(E \cap P)$;

(iii) $\underline{\mu}_F^*(E) = \mu_F^*(E \cap E_{-\infty}) + \mu_F^*(E \cap N)$.

Therefore $\mu_F^*(E) = \overline{\mu}_F^*(E) + \underline{\mu}_F^*(E)$.
Moreover, if E is Lebesgue measurable, then

(iv) $\mu_F^*(E) = \mu_F^*(E \cap E_{+\infty}) + \mu_F^*(E \cap E_{-\infty}) + (\mathcal{L}) \int_{E \cap D} |F'(t)|\, dt$;

(v) $\overline{\mu}_F^*(E) = \mu_F^*(E \cap E_{+\infty}) + (\mathcal{L}) \int_{E \cap P} F'(t)\, dt$;

(vi) $\underline{\mu}_F^*(E) = \mu_F^*(E \cap E_{-\infty}) - (\mathcal{L}) \int_{E \cap N} F'(t)\, dt$,

PROOF. Let $E_0 = \{x \in E : F'(x) = 0\}$ and $Z = \{x \in E : F'(x)$ does not exist (finite or infinite)$\}$. The sets Z, E_0, $E_{+\infty}$, $E_{-\infty}$, D, P, N are all Borel (see Hajek's Theorem of [1, p. 57]).

(i) Since $Z \cup E_{+\infty} \cup E_{+\infty} \cup D = \mathbb{R}$, we obtain

$$\begin{aligned} \mu_F^*(E) &= \mu_F^*(E \cap Z) + \mu_F^*(E \cap E_{+\infty}) + \mu_F^*(E \cap E_{-\infty}) + \mu_F^*(E \cap D) \\ &= \mu_F^*(E \cap E_{+\infty}) + \mu_F^*(E \cap E_{-\infty}) + \mu_F^*(E \cap D) \end{aligned}$$

by Lemma 3.1, (ii), and Corollary 5.1, (ii). Since $D = E_o \cup P \cup N$, we obtain

$$\begin{aligned} \mu_F^*(E \cap D) &= \mu_F^*(E \cap E_o) + \mu_F^*(E \cap P) + \mu_F^*(E \cap N) \\ &= \mu_F^*(E \cap P) + \mu_F^*(E \cap N) \end{aligned}$$

by Lemma 3.1, (ii), and Lemma 8.1.

(ii)[1] Since $Z \cup E_{+\infty} \cup P \cup (E_o \cup N \cup E_{-\infty}) = \mathbb{R}$, we obtain

$$\begin{aligned} \overline{\mu}_F^*(E) &= \overline{\mu}_F^*(E \cap Z) + \overline{\mu}_F^*(E \cap E_{+\infty}) + \overline{\mu}_F^*(E \cap P) + \overline{\mu}_F^*\big(E \cap (E_o \cup N \cup E_{-\infty})\big) \\ &= \overline{\mu}_F^*(E \cap Z) + \overline{\mu}_F^*(E \cap E_{+\infty}) + \overline{\mu}_F^*(E \cap P) \end{aligned}$$

by Lemma 3.1, (ii), Lemma 8.3 and Lemma 8.4. And we have

$$0 \le \overline{\mu}_F^*(E \cap Z) \le \mu_F^*(E \cap Z) = 0$$

[1] The proof of Theorem 8.1, (ii) uses that $\overline{\mu}_F^*$ is a metric outer measure.

by Lemma 8.2, (i), and Corollary 5.1, (ii).

(iii) The proof is similar to that of (ii).

(iv), (v) and (vi) follow by Lemma 3.2. □

9 Characterizations of $\mathbf{VB^*G} \cap (\mathbf{N})$ for Lebesgue Measurable Functions

Corollary 9.1. *Let $F : [a, b] \to \mathbb{R}$ and let E be a Lebesgue measurable subset of $[a, b]$. The following assertions are equivalent.*

(i) $F \in VB^*G \cap (N)$ *on* E.

(ii) $F \in VB^*G \cap (N)$ *on* Z, *whenever* Z *is a null subset of* E.

(iii) There exists a countable subset E_1 *of* E *such that* $\mu_F^*(Z) = 0$, *whenever* Z *is a null subset of* $E \setminus E_1$.

(iv) Z *is* F*-null whenever* Z *is a null subset of* E.

PROOF. For (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) see [3, Theorem 9] and (iii) $\Rightarrow$ (iv) is evident.

(iv) $\Rightarrow$ (ii) Let Z be a null subset of E. Then Z is F-null, so by Lemma 5.6, $m(F(Z)) = 0$. It follows that $F \in (N)$ on Z. For Z there is a countable set D such that $\mu_F^*(Z \setminus D) = 0$. By [13, Theorem 40.1], F is VB^*G on $Z \setminus D$, so on Z. □

10 A Characterization of $\mathbf{VB^*G} \cap \mathbf{N^{+\infty}}$ on a Lebesgue Measurable Set

Definition 10.1 (Saks)**.** [2, p. 79] Let $F : \mathbb{R} \to \mathbb{R}$. F is said to be $N^{+\infty}$ on a real set E if the set $\big(\{x \in E : (F_{|E})'(x) = +\infty\}\big)$ is of Lebesgue measure zero.

Lemma 10.1. *Let $F : \mathbb{R} \to \mathbb{R}$, and let $E \subset \mathbb{R}$ such that F is VB^*G on E. Let $E_{+\infty} = \{x : F'(x) = +\infty\}$. Then the following assertions are equivalent.*

(i) F *is* $N^{+\infty}$ *on* E.

(ii) $m^*\big(F(E \cap E_{+\infty})\big) = 0$.

PROOF. (i) $\Rightarrow$ (ii) Let $E_1 = \{x \in E : x \text{ is an accumulation point for } E\}$. Then $E \setminus E_1$ is at most countable and $E_1 \cap E_{+\infty} \subset \{x \in E : (F_{|E})'(x) = +\infty\}$.

(ii) $\Rightarrow$ (i) Let $Z = \{x \in E : F'(x) \text{ does not exist (finite or infinite)}\}$. Then we have $\{x \in E : (F_{|E})'(x) = +\infty\} \subset Z \cup E_{+\infty}$, and (i) follows because $m^*\big(F(E \cap Z)\big) = 0$ by Corollary 5.1, (iv). □

Lemma 10.2. *Let $F:\mathbb{R}\to\mathbb{R}$ and $E\subset\mathbb{R}$. If $\overline{\mu}^*_F(E)<+\infty$, then $F\in VB^*G$ on E.*

Proof. Suppose that $\overline{\mu}^*_F(E)=M<+\infty$. For $\epsilon=1$ there is a $\delta:E\to(0,+\infty)$ such that $\overline{V}^*_\delta(F;E)<M+1$. Let

$$E_n=\Big\{x\,:\,\delta(x)>\frac{1}{n}\Big\}\quad\text{and}\quad E_{ni}=E_n\cap\Big[\frac{i}{n},\frac{i+1}{n}\Big],\quad i=0,\pm1,\pm2,\dots.$$

If E_{ni} is countable, then F is VB^*G on this set. Fix some uncountable set E_{ni} and let $c_{ni}=\inf E_{ni}$, $d_{ni}=\sup E_{ni}$. We show that $F\in\overline{VB}(E_{ni};[c_{ni},d_{ni}])$ (for the definition see [2, Definition 2.7.1]). Let $\{[c_k,d_k]\}_{k=1}^p$ be a finite set of nonoverlapping closed intervals such that $\{c_k,d_k\}\cap E_{ni}\neq\emptyset$. Clearly, if $c_k\in E_{ni}$, then $\big([c_k,d_k],c_k\big)\in\beta^*_\delta(E)$, and if $d_k\in E_{ni}$, then $\big([c_k,d_k],d_k\big)\in\beta^*_\delta(E)$. It follows that

$$\sum_{k=1}^p\big(F(d_k)-F(c_k)\big)\le\sum_{k=1}^p\Delta F^+\big([c_k,d_k]\big)<\overline{V}^*_\delta(F;E)<M+1\,.$$

Thus $F\in\overline{VB}(E_{ni};[c_{ni},d_{ni}])$. By [2, Theorem 2.8.1, (xii), (i)], we obtain that $F\in VB^*$ on E_{ni}; so $F\in VB^*G$ on E. □

Theorem 10.1. *Let $F:\mathbb{R}\to\mathbb{R}$ and let E be a Lebesgue measurable subset of $\mathbb{R}$. Let $E_{+\infty}=\{x\,:\,F'(x)=+\infty\}$. Then the following assertions are equivalent.*

(i) *$F\in VB^*G\cap N^{+\infty}$ on E.*

(ii) *there exists a countable subset E_1 of E such that $\overline{\mu}^*_F(Z)=0$ whenever $Z\subset E\setminus E_1$ and $m^*(Z)=0$.*

Proof. (i) $\Rightarrow$ (ii) Since F is VB^*G on E, there exists a countable set E_1 such that F is continuous at each point of $E\setminus E_1$ (see [12]). Let $Z\subset E\setminus E_1$ with $m^*(Z)=0$. Then we have

$$\overline{\mu}^*_F(Z)=\mu^*_F(Z\cap E_{+\infty})=m^*(F(Z\cap E_{+\infty}))=0$$

by Theorem 8.1, (v), Lemma 5.6, (i), (iii) and Lemma 10.1.

(ii) $\Rightarrow$ (i) By Corollary 5.1, (i), $m^*(E_{+\infty})=0$, and by Lemma 8.3, we obtain that

$$\mu^*_F\big((E\cap E_{+\infty})\setminus E_1\big)=\overline{\mu}^*_F\big((E\cap E_{+\infty})\setminus E_1\big)=0\,.$$

It follows that $m^*\big(F(E\cap E_{+\infty})\big)=0$ (see Lemma 3.1, (vi)); so F is $N^{+\infty}$ on E (see Lemma 10.1). Let $Z\subset E\setminus E_1$ with $m^*(Z)=0$. Since $\overline{\mu}^*_F(Z)=0$, by Lemma 10.2, it follows that $F\in VB^*G$ on Z. Hence $F\in VB^*G$ on $E\setminus E_1$, so on E (see [3, Theorem 1]). □

Lemma 10.3. *Let $F, G : \mathbb{R} \to \mathbb{R}$, $E \subset \mathbb{R}$, $\alpha, \beta \geq 0$. Then*

$$\overline{\mu}^*_{\alpha F+\beta G}(E) \leq \alpha \cdot \overline{\mu}^*_F(E) + \beta \cdot \overline{\mu}^*_G(E)\,.$$

PROOF. From $\Delta(\alpha F + \beta G)^+\big([x,y]\big) \leq \alpha \cdot \Delta F^+\big([x,y]\big) + \beta \cdot \Delta G^+([x,y]\big)$ it follows immediately that $\overline{\mu}^*_{\alpha F+\beta G}(E) \leq \alpha \cdot \overline{\mu}^*_F(E) + \beta \cdot \overline{\mu}^*_G(E)\,.$ $\square$

Corollary 10.1. *Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Let*

$$\mathcal{A} = \Big\{F : \mathbb{R} \to \mathbb{R} \,:\, F \in VB^*G \cap N^{+\infty} \text{ on } E\Big\}\,.$$

Then $\mathcal{A}$ is a semi-linear subspace, i.e., $\alpha_1 F_1 + \alpha_2 F_2 \in \mathcal{A}$, whenever $\alpha_1, \alpha_2 \geq 0$ and $F_1, F_2 \in \mathcal{A}$.

PROOF. Let $\alpha_1, \alpha_2 \geq 0$ and $F_1, F_2 \in \mathcal{A}$. Clearly $\alpha_1 F_1 + \alpha_2 F_2 \in VB^*G$. By Theorem 10.1, there exist two countable subsets E_1, E_2 of E such that $\overline{\mu}^*_F(Z_1) = 0$ whenever $Z_1 = E \setminus E_1$ and $m^*(Z_1) = 0$, and $\overline{\mu}^*_{F_2}(Z_2) = 0$ whenever $Z_2 \subset E \setminus E_2$ and $m^*(Z_2) = 0$. Let $Z \subset E \setminus (E_1 \cup E_2)$ with $m^*(Z) = 0$. Then $\overline{\mu}^*_{F_1}(Z) = \overline{\mu}^*_{F_2}(Z) = 0$. By Lemma 10.3, $\overline{\mu}^*_{\alpha_1 F_1+\alpha_2 F_2}(Z) = 0$; so by Theorem 10.1 we obtain that $\alpha_1 F_1 + \alpha_2 F_2 \in \mathcal{A}$. $\square$

Corollary 10.2. *Let $E \subset \mathbb{R}$ be a Lebesgue measurable set and let*

$$\mathcal{A}_1 = \Big\{F : \mathbb{R} \to \mathbb{R} \,:\, F \in VB^*G \text{ on } E \text{ and } m\big(F(E \cap \{x : F'(x) = \pm\infty\})\big) = 0\Big\}\,.$$

Then $\mathcal{A}_1$ is a linear space.

PROOF. Let $\mathcal{A}$ be defined as in Corollary 10.1. If $F \in \mathcal{A}_1$, then F and $-F$ belong to $\mathcal{A}$. Applying Corollary 10.1 and Lemma 10.1, it follows that $\mathcal{A}_1$ is a linear space. $\square$

Remark 10.1. Note that $\mathcal{A}_1 = \{F : \mathbb{R} \to \mathbb{R} \,:\, F \in VB^*G \cap (N) \text{ on } E\}$. This follows by Lemma 5.6 and the well known fact that $F \in (N)$ on the set $\{x \in E \,:\, F'(x) \text{ exists and is finite}\}$. Therefore Corollary 10.2 is a special case of [3, Corollary 3].

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