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ADJOINT CLASSES OF LEBESGUE-STIELTJES INTEGRABLE FUNCTIONS

Abstract

This paper gives three pair of adjoint classes of the Lebesgue-Stieltjes integrable functions.

1 Introduction

Let a and b be real numbers with $a < b$. Let $\mathcal{B}[a,b]$ be the class of all Borel measurable functions defined on $[a,b]$, and $\mathcal{F}[a,b]$ be the class of all real-valued functions defined on $[a,b]$. Let $g \in \mathcal{F}[a,b]$ and $g_1(x)$, $g_2(x)$ be the positive, negative variations of g over $[a,x]$ with $a \le x \le b$, respectively. If $g_1(x) + g_2(x) < \infty$ for any $x \in [a,b)$ and either $g_1(b)$ or $g_2(b)$ is finite, then we say $g \in EBV[a,b]$, the class of functions of extended bounded variation on $[a,b]$ (cf. [8]). If $g \in EBV[a,b]$, we have

$$g(x) - g(a) = g_1(x) - g_2(x) \quad \text{for any} \quad x \in [a,b)\,.$$

Since, for $i = 1,2$, $g_i(x)$ is monotonically increasing on $[a,b)$, then there is a unique Baire measure μ_{g_i} such that

$$\mu_{g_i}(a_1,b_1] = g_i(b_1+) - g_i(a_1+) \quad \text{for all} \quad [a_1,b_1] \subset [a,b]$$

(define $g_i(b+) = g_i(b)$). Thus, in fact, a function $g \in EBV[a,b]$ gives rise to a σ-finite signed Baire measure $\mu_g = \mu_{g_1} - \mu_{g_2}$ on the class of all Borel sets in $[a,b]$ such that

$$\mu_g(a_1,b_2] = g(b_1+) - g(a_1+) \quad \text{for all} \quad [a_1,b_1] \subset [a,b]\,.$$

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Now, for $f \in \mathcal{B}[a,b]$ and $g \in EBV[a,b]$, we define the Lebesgue-Stieltjes integral of f with respect to g by

$$\text{(L}-\text{S)} \int_a^b f\, dg = \int_a^b f \; d\mu_g \, ,$$

where μ_g is the σ-finite signed Baire measure called the Lebesgue-Stieltjes measure corresponding to g.

In the next section we shall use the definition in [1, 5] (only change the (L$-$S) integral for the (R-S) integral) to discuss the adjoint classes of the Lebesgue-Stieltjes integrable functions.

2 Main Results

In the present paper, besides the following classes of functions defined on $[a,b]$:

- the class of functions of bounded variation $BV[a,b]$,
- the class of continuous functions of bounded variation $CBV[a,b]$, and
- the class of absolutely continuous functions $AC[a,b]$,

we shall also deal with the classes of functions as follows.

Definition 1. Let $g \in BV[a,b]$. Define $g^*(x) = g(x+)$ for $x \in [a,b)$ and $g^*(b) = g(b)$. If $g^* \in CBV[a,b]$ ($AC[a,b]$), then we say $g \in C_oBV[a,b]$ ($AC_o[a,b]$).

Definition 2. A function $f \in \mathcal{B}[a,b]$ is said to belong to the class $B[a,b]$ if it is bounded on $[a,b]$.

Definition 3. A function $f \in \mathcal{B}[a,b]$ is said to belong to the class $B_o[a,b]$ if there is a number $N_o > 0$ such that any closed subset of the set $E(x : |f(x)| > N_o)$ is at most countable.

In the following definitions we use $L^p[a,b]$ $(1 \le p < \infty)$ to denote the space of all Lebesgue measurable functions f on $[a,b]$ such that (L)$\int_a^b |f|^p < \infty$, and use $L^\infty[a,b]$ to denote the space of all Lebesgue measurable functions on $[a,b]$ which are bounded except possibly a subset of Lebesgue measure zero.

Definition 4. Let $1 \le q \le \infty$. A function $f \in \mathcal{B}[a,b]$ is said to belong to the class $B^q[a,b]$ of $f \in L^q[a,b]$.

Definition 5. Let $1 \le p \le \infty$. A function $g \in \mathcal{F}[a,b]$ is said to belong to the class $AC_o^p[a,b]$ if $g \in AC_o[a,b]$ and $g' \in L^p[a,b]$.

Let A and B be two classes of functions defined on $[a,b]$. If A and B are adjoint with respect to the Lebesgue-Stieltjes integral, then it will be denoted by $A * B(L\text{-}S)$. We will prove the following theorems in the next section.

Theorem 1. $B[a,b] * BV[a,b](L\text{-}S)$.

Theorem 2. *Let* $1/p + 1/q = 1$, $1 \le p \le \infty$. $B^q[a,b] * AC_o^p[a,b](L\text{-}S)$.

Theorem 3. $B_o[a,b] * C_oBV[a,b](L\text{-}S)$.

3 Proof of the Theorems

PROOF OF THEOREM 1.

(1) Suppose $f \in B[a,b]$ and $g \in BV[a,b]$. Let μ_g be the Lebesgue-Stieltjes measure corresponding to g. The condition $g \in BV[a,b]$ implies that $|\mu_g|$ is a finite measure on $[a,b]$, and so f is μ_g-integrable on $[a,b]$. Thus, $(\text{L}-\text{S})\int_a^b f\,dg = \int_a^b f\,d\mu_g$ exists.

(2) Suppose $g \in EBV[a,b]$ and $(\text{L}-\text{S})\int_a^b f\,dg$ exists for all $f \in B[a,b]$. By the Hahn Decomposition Theorem ([7, p. 273]), there is a function $f \in B[a,b]$ with $|f| \le 1$ such that

$$\int_a^b f\,d\mu_g = |\mu_g|[a,b]\,.$$

Hence $|\mu_g|$ is a finite measure on $[a,b]$. That is, $g \in BV[a,b]$.

(3) Suppose $f \in \mathcal{B}[a,b]$ and $(\text{L}-\text{S})\int_a^b f\,dg$ exists for all $g \in BV[a,b]$. Claim that $f \in B[a,b]$. Suppose $f \notin B[a,b]$. Then, there exists a sequence $\{a_n\} \subset [a,b]$ such that a_n monotonically converges to a point $c \in [a,b]$, and $|f(a_n)| \uparrow \infty$ as $n \to \infty$. Without loss of generality, we may assume $a_n \uparrow c$ with $a_0 = a$, $f(a) > 0$, and $f(a_n)+ \uparrow \infty$ as $n \to \infty$. Set

$$b_n = f(a_n)+; \quad d_n = 1/b_n - 1/b_{n+1} \quad \text{and} \quad D_{-1} = 0, \quad D_n = \sum_0^n d_i\,.$$

Then, define $g(x) = D_{n-1}$ for each $n \ge 0$ if $x \in [a_n, a_{n+1})$ and $g(x) = \lim D_n$ if $x \in [c,b]$. Since $\sum_0^\infty d_i < \infty$, so $g \in BV[a,b]$. But, since

$$(\text{L}-\text{S})\int_a^b (f+)\,dg = \sum_0^\infty \big(f(a_i)+\big)\mu_g\big(\{a_i\}\big) = \sum_0^\infty b_{i+1}d_i = \infty\,,$$

the integral $(\text{L}-\text{S})\int_a^b f\,dg$ does not exist, a contradiction. Consequently, $f \in B[a,b]$. □

PROOF OF THEOREM 2.

(1) Suppose $f \in B^q[a,b]$, $g \in AC_o^p[a,b]$ with $1/p + 1/q = 1$. Since

$$(\text{L--S})\int_a^b f\,dg = (\text{L--S})\int_a^b f\,dg^* = (\text{L})\int_a^b fg'\,dx$$

and so the fact that $f \in B^q[a,b]$ and $g' \in L^p[a,b]$ implies $(\text{L--S})\int_a^b f\,dg$ exists.

(2) Let $1 \le p < \infty$. Suppose $f \in \mathcal{B}[a,b]$ and $(\text{L--S})\int_a^b f\,dg$ exists for all $g \in AC_o^p[a,b]$. Whence, $(\text{L})\int_a^b fh\,dx$ exists for all $h \in L^p[a,b]$. Set $f_n(x) = f(x)$ if $|f(x)| \le n$ and $f_n(x) = 0$ otherwise. Now, for each f_n, $n = 1, 2, \dots$, define a linear functional:

$$F_n(h) = (\text{L})\int_a^b f_n h\,dx\,, \quad h \in L^p[a,b]\,.$$

From the Hölder Inequality, it follows that F_n is a bounded functional. Since $|f_n h| \le |fh|$ and $fh \in L[a,b]$, we have that

$$\lim F_n(h) = (\text{L})\int_a^b fh\,dx\,, \quad h \in L^p[a,b]$$

by the Lebesgue Convergence Theorem. By the Banach-Steinhaus Theorem ([3, p. 100]), $F(h) = \lim F_n(h)$ is a linear functional on $L^p[a,b]$. On the other hand, since

$$L^p[a,b]^* = L^q[a,b] \quad \text{with} \quad 1/p + 1/q = 1 \ \text{ and } \ 1 \le p < \infty\,,$$

where we denote the dual space of A by A^*, there exists a unique function $f_1 \in L^q[a,b]$ such that

$$F(h) = (\text{L})\int_a^b f_1 h\,dx\,, \quad h \in L^p[a,b]\,.$$

So, we have

$$(\text{L})\int_a^b (f - f_1)h\,dx = 0 \quad \text{for all} \ \ h \in L^p[a,b]\,.$$

Set $h = \chi[a,t] \in L^p[a,b]$. Then

$$(\text{L})\int_a^t (f - f_1)\,dx = 0 \quad \text{for} \ \ t \in [a,b]\,.$$

Thus, $f = f_1$ almost everywhere, and so $f \in L^q[a,b]$. Hence, $f \in B^q[a,b]$. Let $p = \infty$. If set $g \equiv x \in AC_o^\infty[a,b]$, then the fact that $(\mathrm{L{-}S})\int_a^b f\,dg =$ $(\mathrm{L})\int_a^b f\,dx$ exists implies $f \in L^1[a,b]$. Hence, $f \in B^1[a,b]$.

(3) Let $g \in EBV[a,b]$. Suppose $(\mathrm{L{-}S})\int_a^b f\,dg$ exists for all $f \in B^q[a,b]$, $1 \le q \le \infty$. We shall prove $g \in AC_o^p[a,b]$ with $1/p + 1/q = 1$. First of all, we are going to show it in the case $q = \infty$ ($p = 1$). In order to prove $g \in AC_o[a,b]$, it suffices to prove that $|\mu_g|(E) = 0$ for any Borel set $E \subset [a,b]$ with $m(E) = 0$. By the Hahn Decomposition Theorem, we can define a function $f \in B^\infty[a,b]$ such that

$$(\mathrm{L{-}S})\int_a^b f\,dg = (\mathrm{L{-}S})\int_E f\,dg = +\infty \cdot |\mu_g|(E) < \infty\,.$$

This means $|\mu_g|(E) = 0$, and so $g \in AC_o[a,b]$. Secondly, we are going to show $g \in AC_o^p[a,b]$ for $1 \le q < \infty$ with $1/p + 1/q = 1$. From the preceding proof for the case $q = \infty$ and $B^\infty[a,b] \subset B^q[a,b]$, it follows that $g \in AC_o[a,b]$. So,

$$(\mathrm{L})\int_a^b fg'\,dx = (\mathrm{L{-}S})\int_a^b f\,dg^* = (\mathrm{L{-}S})\int_a^b f\,dg$$

exists for all $f \in B^q[a,b]$, $1 \le q < \infty$. Hence, we can define a linear functional

$$F(f) = (\mathrm{L})\int_a^b fg'\,dx\,, \quad f \in B^q[a,b]\,.$$

Since $B^q[a,b]$ is dense in $L^q[a,b]$, and so it follows from the proof in (2) that $g' \in L^p[a,b]$ with $1/p + 1/q = 1$, thus $g \in AC_o^p[a,b]$. □

PROOF OF THEOREM 3.

(1) Let $f \in B_o[a,b]$ and $g \in C_oBV[a,b]$. Suppose any closed subset of the set $E(x : |f| > N_o)$ is countable. Since the Lebesgue-Stieltjes measure μ_g is regular, there exists a sequence $\{P_n\}$ of closed sets such that $P_n \subseteq E(x : |f| > N_o)$ for all $n \ge 1$ and

$$|\mu_g|(P_n) \to |\mu_g|E\big(x : |f| > N_o\big) \ \text{ as } \ n \to \infty\,.$$

Since $g \in C_oBV[a,b]$ and P_n is countable, and so $|\mu_g|(P_n) = 0$ for all $n \ge 1$. Hence, it follows that $|\mu_g|E(x : |f| > N_o) = 0$. Consequently, the integral $(\mathrm{L{-}S})\int_a^b f\,dg$ exists.

(2) Suppose $g \in EBV[a,b]$ and $(\mathrm{L{-}S})\int_a^b f\,dg$ exists for all $f \in B_o[a,b]$. Since $B[a,b] \subset B_o[a,b]$, and so $g \in BV[a,b]$ by Theorem A. Let $c \in [a,b]$.

Define a function f as follows: $f(x) = \infty$ if $x = c$, and 0 if $x \in [a, b] \sim \{c\}$. It is obvious that $f \in B_o[a, b]$. By hypothesis, the integral

$$(\text{L–S}) \int_a^b f\, dg = (\text{L–S}) \int_{\{c\}} f\, dg = f(c)\mu_g\{c\}$$

is finite. But, since$f(c) = \infty$, this implies $g^*(c) - g^*(c-) = \mu_g\{c\} = 0$. Hence, $g^*(x)$ is continuous at $x = c$. Therefore, $g \in C_oBV[a, b]$.

(3) Suppose $f \in \mathcal{B}[a, b]$ and $(\text{L–S})\int_a^b f\, dg$ exists for all $g \in C_oBV[a, b]$. We claim $f \in B_o[a, b]$. If $f \notin B_o[a, b]$, then for any $N > 0$ the set $E(x : |f| > N)$ contains a closed subset, which is uncountable and so must contain a perfect subset ([6, p. 130]). Hence, we construct a function $g \in C_oBV[a, b]$ such that the integral $(\text{L–S})\int_a^b f\, dg$ does not exist. First of all, since $AC_o[a, b] \subset C_oBV[a, b]$, so $f \in B^\infty[a, b]$ by Theorem B. Thus, there exists a number $N_o > 0$ such that for each $n > N_o$ the set $E(x : |f| > n)$ contains a Cantor set S_n with $m(S_n) = 0$. Set $x_n = \max(S_n)$ for each $n > N_o$. If necessary, we can modify those Cantor sets so that $x_n \neq x_m$ if $n \neq m$. Let η be a cluster point of the sequence $\{x_n\}$. Without loss of generality we may assume there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \uparrow \eta$ $(k \to \infty)$. Now, we construct a function g as follows. For $k = 1$, let $y_{n_1} = \min(S_{n_1})$. In the same way as in the proof of Theorem 2.2 in [2] we define $g(x)$ as a Cantor function on $[y_{n_1}, x_{n_1}]$, which is locally constant on $[y_{n_1}, x_{n_1}] \sim S_{n_1}$ with the range $[0, 1 - 1/n_1]$, and $g(x) = 0$, if $x \in [a, y_{n_1})$. In general, for each $k > 1$ we define $g(x)$ as follows. Noting that $S_{n_k} \cap [x_{n_{k-1}}, x_{n_k}]$ is also a Cantor set with measure zero, let $y_{n_k} = \min(S_{n_k} \cap [x_{n_{k-1}}, x_{n_k}])$ and define $g(x)$ as a Cantor function on $[y_{n_k}, x_{n_k}]$, which is locally constant on $[y_{n_k}, x_{n_k}] \sim S_{n_k}$ with the range $[1 - 1/n_{k-1}, 1 - 1/n_k]$, and $g(x) = 1 - 1/n_{k-1}$, if $x \in [x_{n_{k-1}}, y_{n_k})$. Obviously, through this way we can define $g(x)$ for any $x \in [a, \eta)$. If we define $g(x) = 1$ on $[\eta, b]$, we have $g \in C_oBV[a, b]$. Since $|f(x)| > n_k$ for $x \in S_{n_k} \cap [x_{n_{k-1}}, x_{n_k}]$ and

$$\mu_g\big(S_{n_k} \cap [x_{n_{k-1}}, x_{n_k}]\big) = 1/n_{k-1} - 1/n_k\,,$$

we have

$$(\text{L–S}) \int_a^b |f(x)|\, dg \geq (\text{L–S}) \int_{x_{n_1}}^{\eta} |f(x)|\, dg \geq \sum_{k=2}^{\infty} (\text{L–S}) \int_{x_{n_{k-1}}}^{x_{n_k}} |f(x)|\, dg$$

$$\geq \sum_{k=2}^{\infty} (\text{L–S}) \int_{S_{n_k} \cap [x_{n_{k-1}}, x_{n_k}]} |f(x)|\, dg \geq \sum_{k=2}^{\infty} n_k\big(1/n_{k-1} - 1/n_k\big)$$

$$= \sum_{k=2}^{\infty} \big(n_k - n_{k-1}\big)/n_{k-1} = \infty\,.$$

Consequently, the integral $(\mathrm{L{-}S})\int_a^b |f|\,dg$ does not exist, and neither does the integral $(\mathrm{L{-}S})\int_a^b f\,dg$. But, this contradicts the hypothesis, hence we must have $f \in B_o[a,b]$. □

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