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J. Bourgain, Institute for Advanced Study, Princeton, NJ 08540, USA.
e-mail: `bourgain@math.ias.edu`

S. Kostyukovsky, School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. e-mail: `sergey@post.tau.ac.il`

A. Olevskiĭ,* School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. e-mail: `olevskii@post.tau.ac.il`

A REMARK ON A MAXIMAL OPERATOR FOR FOURIER MULTIPLIERS

Abstract

For a finite set Λ on the circle we consider a family of the multiplier operators T_m in $l_2(\mathbb{Z})$ generated by the 2^{-m}-neighborhoods of Λ. We show that the norm of the corresponding maximal operator T can not be estimated by an absolute constant.

Let T be a circle group $\mathbb{R}/Z$ identified in a standard way with the interval $[0,1)$. For $g \in L^2(\mathbb{T})$ we denote by $\widehat{g}$ the Fourier transform:

$$\widehat{g}(k) = \int_{\mathbb{T}} g(t) e^{-2\pi i k t} dt \quad (k \in \mathbb{Z})$$

and by $f \mapsto \check{f}$ the inverse operator from $l_2(\mathbb{Z})$ onto $L^2(\mathbb{T})$.

Given a set $\Lambda \subset \mathbb{T}$ of N distinct points we denote

$$\Lambda(m) = \bigcup_{\lambda \in \Lambda} (\lambda - 2^{-m}, \lambda + 2^{-m}). \tag{1}$$

Consider the multiplier operator in $l_2(\mathbb{Z})$:

$$T_m : \quad f \mapsto \{\check{f} \chi_{\Lambda(m)}\}^{\wedge}$$

(χ_E is an indicator function of the set E)

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and the corresponding maximal operator

$$M_\Lambda : \quad f \mapsto \sup_{m \in Z} |(T_m f)(k)|$$

The following inequality was proved in [1] (and used there essentially for the "squares" ergodic theorems) :

$$||M_\Lambda f|| \le C \log^2 N ||f|| \tag{2}$$

(here and below we denote by C positive absolute constants).

It was asked in [1] whether the dependence on N might be removed from the inequality. Here we prove that it can not. Moreover, the log factor in (2) is essentially sharp .

Theorem. *For any N there exist a set $\Lambda \subset \mathbb{T}$, $\quad \operatorname{card} \Lambda = N$ and $f \in l_2(\mathbb{Z})$, s.t.*

$$||M_\Lambda f|| > C \log^\alpha N ||f||$$

($\alpha > 0$ is an absolute constant; one can take $\alpha = 1/4$).

The proof is based on the Kolmogorov "rearrangement" theorem: there exists an L^2 - Fourier series which diverges (unboundedly) almost everywhere after some permutation of its terms (see [2], ch. 3).

We use the following equivalent form of this theorem: given any $K > 0$ one can find a trigonometric polynomial

$$P(x) = \sum_{j=1}^{N} b_j e^{2\pi i n_j x}$$

such that the conditions below are fulfilled :

$$\{n_j\}_1^N \text{ is a rearrangement of } \{1, 2, \dots, N\}; \tag{3}$$

$$\sum_j |b_j|^2 = 1. \tag{4}$$

If denote

$$P^*(x) = \max_{1 \le l \le N} |\sum_{j=1}^{l} b_j e^{2\pi i n_j x}|$$

then

$$\operatorname{mes}\{x \in \mathbb{T}; \quad P^*(x) > K\} > 1/2. \tag{5}$$

The dependence $K \to N$ was studied by several authors. The best known (apparently) result [3, Lemma 4] means that one can choose

$$K = C \log^{1/4} N \quad (N = 2, 3, ...) \tag{6}$$

Remark. A simple measure theoretic argument shows that there is an appropriate translate of P, denoted here again by the same symbol, which satisfies properties (3), (4), (6) and

$$\operatorname{card}\{k \in \{1, 2, ..., N\} : \quad P^*(\frac{k}{N}) > K\} > N/2. \tag{7}$$

Now let N be an integer > 1, and the corresponding polynomial P is defined.

Set:

$$\Lambda = \{\lambda_j\}_1^N, \quad \lambda_j = \frac{n_j}{N} - \frac{9}{10} 2^{-2N+j-1}; \qquad \delta = \frac{1}{10} 2^{-2N}.$$

For a given l, $1 \le l \le N$ consider the set $\Lambda(2N - l + 1)$ according to (1). Obviously it contains δ-neighborhoods of the points $\{\frac{n_j}{N}\}$, $1 \le j \le l$ and does not intersects δ- neighborhoods of other points $\{\frac{k}{N}\}$.

Define for $x \in \mathbb{T}$:

$$\psi(x) = \frac{1}{\sqrt{\delta}} \chi_{(-\delta/2, \delta/2)}(x)$$

$$g(x) = \sum_{j=1}^{N} b_j \psi(x - \frac{n_j}{N}) \quad .$$

As supports of the summands are disjoint we get from (4):

$$||g|| = 1. \tag{8}$$

It follows that for $m = 2N - l + 1$

$$g(x)\chi_{\Lambda(m)}(x) = \sum_{j=1}^{l} b_j \psi(x - \frac{n_j}{N}),$$

so for $f = \widehat{g}$ we have:

$$(M_\Lambda f)(k) = \sup_m |(\widehat{g\chi_{\Lambda(m)}})(k)| \ge$$

$$\max_{1\le l\le N}|\sum_{j=1}^{l} b_j\widehat{\psi}(k)e^{2\pi i\frac{n_j}{N}k}| = |\widehat{\psi}(k)|P^*(\frac{k}{N}).$$

Using the elementary inequality

$$|\widehat{\psi}(k)| = \frac{1}{\sqrt{\delta}}\frac{sin\pi\delta k}{\pi k} > \frac{\sqrt{\delta}}{2}, \quad 1\le k < \frac{1}{2\delta}$$

we get:

$$||M_\Lambda f||^2 \ge \sum_{1\le k<1/(2\delta)} |\widehat{\psi}(k)|^2 P^*(\frac{k}{N})^2 > \frac{\delta}{4}\sum_{1\le k<1/(2\delta)} P^*(\frac{k}{N})^2.$$

Because of (7) the last sum contains at least C/δ members exceeding K^2, so we have:

$$||M_\Lambda|| > CK$$

and the result follows from (6) and (8).

References

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