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# GENERALIZING THE BLUMBERG THEOREM

**Abstract**

Given a collection of functions of some class defined on the real line, when can you find a large set upon which the restriction of every function is continuous? We consider this problem (and related problems) for various classes of functions and various notions of largeness. These problems can be considered in terms of finding the covering, uniformity(non), additivity, and cofinality numbers for some ideal-like collections of sets.

## 1 Preliminaries

We use standard notation as in [4]. In particular, for a set $X$ we denote its cardinality by $|X|$. Given sets $X$ and $Y$ we denote by $Y^X$ the set of all functions from $X$ into $Y$. We denote by $[X]^{<\kappa}$, $[X]^{\kappa}$, and $[X]^{\leq\kappa}$ the sets of all subsets of $X$ of cardinality less than $\kappa$, equal to $\kappa$, and less than or equal to $\kappa$, respectively. We let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Q}$ stand for the set of rational numbers. The cardinalities of $\mathbb{R}$ and $\mathbb{Q}$ will be denoted by $\mathfrak{c}$ and $\omega$, respectively. If $A \subseteq \mathbb{R}$ we let $\chi_A$ stand for the characteristic function of $A$.

Suppose $X$ is a topological space. For a set $A \subseteq X$ we write $\mathrm{cl}_X(A)$, $\mathrm{int}_X(A)$, $\mathrm{bd}_X(A)$ for the closure, interior, and boundary of $A$ in $X$, respectively. When it is understood what space we are referring to the subscript will be dropped. When $X$ is a metric space and $\epsilon > 0$ we denote the open

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epsilon ball around $x \in X$ by $\mathrm{B}_\epsilon(x)$. Let $\mathcal{M}$, and $\mathcal{G}$ denote the meager, and co-meager subsets of $X$, respectively. The dense subsets of $X$ will be denoted by $\mathcal{D}$. The collection of compact subsets of $X$ will be denoted by J. We say a nonempty set $S \subseteq X$ is perfect if $S$ has no isolated points and $S$ is compact. Let $\mathcal{K}$ be the collection of all perfect subsets of $X$. Finally, we denote by $\mathcal{T}$ the collection of all subsets $S$ of $\mathbb{R}$ with the property that for any $G \in \mathcal{G}$ and open set $U \subseteq X$ the set $S \cap G \cap U$ contains a perfect set.

If a function $f\colon X \to Y$ has the property that for every open set $U \subseteq Y$ the set $f^{-1}(U)$ is an $F_\sigma$ set of $X$, then we say $f$ is a Borel class one function. Let $\mathcal{B}_1$ denote the Borel class one real functions. We let $\mathcal{B}$ denote the real functions with the Baire property. For a function $f\colon X \to \mathbb{R}$ and $S \subseteq X$ we let $\operatorname{osc}(f, S) = \sup\{|f(x) - f(y)|\colon x, y \in S\}$.

We will also need some cardinals connected with the ideal of meager subsets of a space $X$:

$$\operatorname{cov}_X(\mathcal{M}) = \min\{|M|\colon M \subseteq \mathcal{M} \;\&\; \cup M = X\}$$

$$\operatorname{add}_X(\mathcal{M}) = \min\{|M|\colon M \subseteq \mathcal{M} \;\&\; \cup M \notin \mathcal{M}\}$$

$$\operatorname{non}_X(\mathcal{M}) = \min\{|S|\colon S \subseteq X \;\&\; S \notin \mathcal{M}\}$$

$$\operatorname{cf}_X(\mathcal{M}) = \min\{|M|\colon M \subseteq \mathcal{M} \;\&\; (\forall n \in \mathcal{M})(\exists m \in M)(n \subseteq m)\}.$$

These cardinals have been heavily studied, for example see [2]. In particular, it is known that if $X$ is a separable complete uncountable metric space, then $\operatorname{cov}_X(\mathcal{M}) = \operatorname{cov}_{\mathbb{R}}(\mathcal{M})$ and $\operatorname{non}_X(\mathcal{M}) = \operatorname{non}_{\mathbb{R}}(\mathcal{M})$. For this reason we will usually drop the subscript and write $\operatorname{cov}(\mathcal{M})$ and $\operatorname{non}(\mathcal{M})$ when dealing with separable complete uncountable metric spaces.

## 2 Introduction

In 1922 H. Blumberg proved the following theorem.

**Theorem 1.** [5] *For any $f\colon \mathbb{R} \to \mathbb{R}$, there is a dense set $D \subseteq \mathbb{R}$ such that $f|_D$ is continuous.*

In what follows we will consider the following cardinals which involve generalizing Blumberg's Theorem. Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ and $\mathcal{J}$ a family of subsets of $\mathbb{R}$. In interpreting the definitions of the following cardinal numbers it is helpful to mentally associate with each pair, $(\mathcal{F}, \mathcal{J})$, an "ideal-like" collection of sets of the form:

$$\{H \subseteq \mathcal{J} \colon \exists f \in \mathcal{F} \text{ such that } f|_S \text{ is not continuous for all } S \in H\}.$$

We now define the cardinal numbers associated with the pair $(\mathcal{F},\mathcal{J})$.

$\mathrm{cov}(\mathcal{F},\mathcal{J})$ is the smallest cardinality of a collection $F\subseteq\mathcal{F}$ such that for every $D\in\mathcal{J}$, there is an $f\in\mathcal{F}$ such that $f|D$ is not continuous.

$\mathrm{non}(\mathcal{F},\mathcal{J})$ is the smallest cardinality of a collection $H\subseteq\mathcal{J}$ such that for every $f\in\mathcal{F}$ there is an $S\in H$ such that $f|_S$ is continuous.

$\mathrm{cf}(\mathcal{F},\mathcal{J})$ is the minimum cardinality of a collection $F\subseteq\mathcal{F}$ such that for any $g\in\mathcal{F}$ there is an $f\in F$ such that

$$\{K\in\mathcal{J}\colon g|_K \text{ is discontinuous}\}\subseteq\{K\in\mathcal{J}\colon f|_K \text{ is discontinuous}\}.$$

$\mathrm{add}(\mathcal{F},\mathcal{J})$ is the smallest cardinality of a collection $F\subseteq\mathcal{F}$ such that for every $g\in\mathcal{F}$ there is an $f\in F$ and a $H\in\{K\in\mathcal{J}\colon f|_K \text{ is discontinuous}\}$ such that $g|_H$ is continuous.

One of these cardinals, $\mathrm{cov}(\mathbb{R}^{\mathbb{R}},\mathcal{D})$, is indirectly suggested in other places in questions about co-Blumberg spaces. Recall a space $X$ is called co-Blumberg provided that for any function $f\colon\mathbb{R}\to X$ there is a $D\in\mathcal{D}$ such that $f|_D$ is continuous. In particular, given the results and questions of author suggested [3] it is natural to ask for which cardinals $\kappa$ is $2^\kappa$ with the usual product topology a co-Blumberg space. The following easy observation shows the connection between the question above and $\mathrm{cov}(\mathbb{R}^{\mathbb{R}},\mathcal{D})$.

**Proposition 2.** *The following cardinals are equal:*

**(a)** $\mathrm{cov}(\mathbb{R}^{\mathbb{R}},\mathcal{D})$,

**(b)** $\mathrm{cov}(2^{\mathbb{R}},\mathcal{D})$,

**(c)** $\lambda=\min\{\kappa\colon 2^\kappa$ *is not co-Blumberg*$\}$.

Proof. Since $2^{\mathbb{R}}\subseteq\mathbb{R}^{\mathbb{R}}$, we have that $\mathrm{cov}(\mathbb{R}^{\mathbb{R}},\mathcal{D})\le\mathrm{cov}(2^{\mathbb{R}},\mathcal{D})$.

We show that $\mathrm{cov}(2^{\mathbb{R}},\mathcal{D})\le\lambda$. Let $\kappa<\mathrm{cov}(2^{\mathbb{R}},\mathcal{D})$ and $f\colon\mathbb{R}\to 2^\kappa$. For each $\alpha\in\kappa$ let $\pi_\alpha$ be the projection of $2^\kappa$ onto the $\alpha^{th}$ coordinate. Define $f_\alpha\in 2^{\mathbb{R}}$ by $f_\alpha(r)=\pi_\alpha(f(r))$. Since $|\{f_\alpha\colon\alpha\in\kappa\}|<\mathrm{cov}(2^{\mathbb{R}},\mathcal{D})$, there is a $D\in\mathcal{D}$ such that $f_\alpha|_D$ is continuous for every $\alpha\in\kappa$. It follows that $f|_D$ is continuous in each coordinate. Thus, $f|_D$ is continuous. So, $\mathrm{cov}(2^{\mathbb{R}},\mathcal{D})\le\lambda$.

We show that $\lambda\le\mathrm{cov}(\mathbb{R}^{\mathbb{R}},\mathcal{D})$. Let $F\subseteq\mathbb{R}^{\mathbb{R}}$ be such $|F|=\mathrm{cov}(\mathbb{R}^{\mathbb{R}},\mathcal{D})$ and $\bigcup_{f\in F}\{D\in\mathcal{D}\colon f|_D \text{ is discontinuous}\}=\mathcal{D}$. Let $\{B_n\}_{n\in\omega}$ be a countable base for $\mathbb{R}$. Let $F^*=\{\chi_{B_n}\circ f\in 2^{\mathbb{R}}\colon f\in F\ \&\ n\in\omega\}$. Note that $|F^*|\le\max\{\omega,\mathrm{cov}(\mathbb{R}^{\mathbb{R}},\mathcal{D})\}$. Notice also that for every $f\in F$

$$\{D\in\mathcal{D}\colon f|_D \text{ is discontinuous}\}\subseteq\bigcup_{n\in\omega}\{D\in\mathcal{D}\colon(\chi_{B_n}\circ f)|_D \text{ is discontinuous}\}.$$

So, $\mathcal{D} = \{D \in \mathcal{D}\colon f^*|_D$ is discontinuous for some $f^* \in F^*\}$. Define $g\colon \mathbb{R} \to 2^{F^*}$ by the formula $g(r) = \langle f^*(r)\rangle_{f^*\in F^*}$. We show that $g$ is continuous on no dense subset of $\mathbb{R}$. Let $D \in \mathcal{D}$. There is an $f \in F$ such that $f|_D$ is not continuous. So there is an $n \in \omega$ such that $(\chi_{B_n} \circ f)|_D$ is not continuous. Letting $f^* = \chi_{B_n} \circ f \in F^*$ and $\pi_{f^*}$ be the projection of $2^{F^*}$ onto the $(f^*)^{th}$ coordinate, we have that $(\pi_{f^*} \circ g)|_D$ is not continuous. Thus, $g|_D$ is not continuous. So, $\lambda \le \max\{\omega, \mathrm{cov}(\mathbb{R}^{\mathbb{R}}, \mathcal{D})\}$. However, $2^\omega$ is clearly a co-blumberg space since it embeds into $\mathbb{R}$, so $\omega < \lambda$. Therefore, $\lambda \le \mathrm{cov}(\mathbb{R}^{\mathbb{R}}, \mathcal{D})$ which completes the proof. □

The information we have on the pair $(\mathbb{R}^{\mathbb{R}}, \mathcal{D})$ is contained in the following two results.

**Theorem 3.** *If* $\mathfrak{c} = \omega_2$*, then* $\mathrm{cov}(\mathbb{R}^{\mathbb{R}}, \mathcal{D}) = \omega_1$.

**Theorem 4.** $\mathrm{add}(\mathbb{R}^{\mathbb{R}}, \mathcal{D}) = \omega_1$.

Aside from working with the pair $(\mathbb{R}^{\mathbb{R}}, \mathcal{D})$ we also consider some pairs involving more regular functions.

**Theorem 5.**

**(a)** $\mathrm{cov}(\mathcal{B}_1, \mathcal{D}) = \mathrm{cov}(\mathcal{B}_1, \mathcal{T}) = \mathrm{cov}(\mathcal{B}, \mathcal{T}) = \mathrm{cov}(\mathcal{B}, \mathcal{D}) = \mathrm{cov}(\mathcal{M})$.

**(b)** $\mathrm{non}(\mathcal{B}, \mathcal{D}) = \mathrm{non}(\mathcal{B}_1, \mathcal{D}) = \mathrm{non}(\mathcal{M})$.

**(c)** $\mathrm{add}(\mathcal{B}_1, \mathcal{E}) = \mathrm{add}(\mathcal{B}, \mathcal{E}) = \mathrm{add}(\mathcal{M})$ *for* $\mathcal{E} \in \{\mathcal{D}, \mathcal{G}, \mathcal{T}\}$.

**(d)** $\mathrm{cf}(\mathcal{B}_1, \mathcal{E}) = \mathrm{cf}(\mathcal{B}, \mathcal{E}) = \mathrm{cf}(\mathcal{M})$ *for* $\mathcal{E} \in \{\mathcal{D}, \mathcal{G}, \mathcal{T}\}$.

**(e)** $\mathrm{cov}(\mathcal{B}_1, \mathcal{G}) = \mathrm{cov}(\mathcal{B}, \mathcal{G}) = \mathrm{add}(\mathcal{M})$.

**(f)** $\mathrm{non}(\mathcal{B}_1, \mathcal{G}) = \mathrm{non}(\mathcal{B}, \mathcal{G}) = \mathrm{cf}(\mathcal{M})$.

Notice that the values of $\mathrm{non}(\mathcal{B}_1, \mathcal{T})$ and $\mathrm{non}(\mathcal{B}, \mathcal{T})$ are not mentioned in Theorem 5. The values of these cardinals remains an open question.

To state our results about the pair $(\mathcal{B}_1, \mathcal{K})$ we need some definitions. For each $F \subseteq \mathbb{R}$ let $\mathrm{N}(F) = \{P \in \mathcal{K}\colon F \cap P$ is not $P$-open$\}$. Let $\mathcal{Z}_p$ denote the set of all subsets $H$ of $\mathcal{K}$ such that there exist a countable collection of closed sets $\{C_n\colon n \in \omega\}$ such that $H \subseteq \bigcup_{n\in\omega} \mathrm{N}(C_n)$. Clearly, $\mathcal{Z}_p$ is a $\sigma$-ideal. The relationship between $\mathcal{Z}_p$ and the pair $(\mathcal{B}_1, \mathcal{K})$ is described in the following theorem

**Theorem 6.** $Z \in \mathcal{Z}_p$ *if and only if there is a* $\mathcal{B}_1$ *function* $f\colon \mathbb{R} \to \mathbb{R}$ *such that* $Z \subseteq \{P \in \mathcal{K}\colon f|_P$ *is discontinuous*$\}$. *In particular,* $\mathcal{Z}_p$ *is a proper* $\sigma$*-ideal.*

More facts about $\mathcal{Z}_p$ and its structure can be found in Section 5. Since $\mathcal{Z}_p$ is a $\sigma$-ideal, we can define the usual cardinals associated with a $\sigma$-ideal.

$$\begin{aligned}
\operatorname{cov}(\mathcal{Z}_p) &= \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathcal{Z}_p \ \& \ \bigcup F = \mathcal{K}\} \\
\operatorname{non}(\mathcal{Z}_p) &= \min\{|F| \colon F \subseteq \mathcal{K} \ \& \ (F \notin \mathcal{Z}_p)\} \\
\operatorname{add}(\mathcal{Z}_p) &= \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathcal{Z}_p \ \& \ \bigcup_{F \in \mathcal{F}} F \notin \mathcal{Z}_p\} \\
\operatorname{cf}(\mathcal{Z}_p) &= \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathcal{Z}_p \ \& \ (\forall G \in \mathcal{Z}_p)(\exists F \in \mathcal{F})(G \subseteq F)\}
\end{aligned}$$

**Theorem 7.**

**(a)** $\operatorname{cov}(\mathcal{B}_1, \mathcal{K}) = \operatorname{cov}(\mathcal{Z}_p) \geq \operatorname{cov}(\mathcal{M})$.

**(b)** $\operatorname{non}(\mathcal{B}_1, \mathcal{K}) = \operatorname{non}(\mathcal{Z}_p) \leq \operatorname{non}(\mathcal{M})$.

**(c)** $\operatorname{add}(\mathcal{B}_1, \mathcal{K}) = \operatorname{add}(\mathcal{Z}_p) = \omega_1$.

**(d)** $\operatorname{cf}(\mathcal{B}_1, \mathcal{K}) = \operatorname{cf}(\mathcal{Z}_p) = \mathfrak{c}$.

In the proof of Theorem 7 we will use the following notion which actually allows us to give a new characterization of the $\mathcal{B}_1$ functions. For a function $f \colon \mathbb{R} \to \mathbb{R}$ we call a collection of closed sets $\mathcal{C}$ an $f$-family provided that for any $P \in \mathcal{K}$ and $x \in P$ we have that, if $f|_P$ is discontinuous at $x$ then, there is a $C \in \mathcal{C}$ such that $x \in C$ and $x \notin \operatorname{int}_P(C \cap P)$.

**Theorem 8.** *If $f \colon \mathbb{R} \to \mathbb{R}$, then the following are equivalent:*

**(a)** $f \in \mathcal{B}_1$,

**(b)** *there is a countable $f$-family, and*

**(c)** *there is an $f$-family of cardinality less than* $\operatorname{cov}(\mathcal{M})$.

## 3 Proof of Theorem 5

We will use the following lemma often in what follows.

**Lemma 9.** *If $\{A_n \colon n \in \omega\}$ is a collection of closed sets in $\mathbb{R}$ then there is a $g \in \mathcal{B}_1$ such that for any set $X \subseteq \mathbb{R}$ if $X \cap A_n$ is not open in $X$ for some $n \in \omega$, then $g|_X$ is not continuous.*

PROOF. For every $n \in \omega$ let $g_n = \chi_{A_n}$. Considering $2^\omega$ as being embedded in $\mathbb{R}$, define $g\colon \mathbb{R} \to 2^\omega$ by the formula $g(r) = \langle g_n(r)\rangle_{n\in\omega}$. It is easily seen that $g \in \mathcal{B}_1$. Suppose $X \subseteq \mathbb{R}$ and $X \cap A_n$ is not open in $X$. Clearly, $g_n|_X$ is not continuous. Since $g_n$ is a coordinate function of $g$, we have that $g|_X$ is not continuous. So, $g$ is as desired. □

We first show (b), (c), (d), (e), and (f) of Theorem 5.

We show (c). Suppose $F \subseteq \mathcal{B}$ and $|F| < \mathrm{add}(\mathcal{M})$. For each $f \in F$ the restriction of $f$ to the complement of some meager $F_\sigma$-set $N_f$ is continuous. Since $|F| < \mathrm{add}(\mathcal{M})$, there is a meager $F_\sigma$-set $M$ such that $\bigcup_{f\in F} N_f \subseteq M$. Let $M = \bigcup_{n\in\omega} M_n$ where each $M_n$ is closed and nowhere dense in $\mathbb{R}$. By Lemma 9 there is a $g \in \mathcal{B}_1$ such that $g|_X$ is discontinuous for any $X \subseteq \mathbb{R}$ such that $X \cap M_k$ is not $X$-open. Let $f \in F$ and $D \in \mathcal{D}$ be such that $f|_D$ is not continuous. Since $D \cap N_f \neq \emptyset$ and $N_f \subseteq M$, there is a $k \in \omega$ such that $D \cap M_k$ is not empty. Since $M_k$ is nowhere dense, $M_k \cap D$ is not $D$-open. So, $g|_D$ is not continuous. Thus, $\bigcup_{f\in F}\{D \in \mathcal{D}\colon f|_D \text{ is discontinuous}\} \subseteq \{D \in \mathcal{D}\colon g|_D \text{ is discontinuous}\}$. Since $F \subseteq \mathcal{B}$ and $g \in \mathcal{B}_1$, we have

$$\mathrm{add}(\mathcal{M}) \leq \min\{\mathrm{add}(\mathcal{B}_1, \mathcal{D}), \mathrm{add}(\mathcal{B}, \mathcal{D})\}. \tag{1}$$

On the other hand, let $F = \{F_\alpha\}_{\alpha\in\kappa}$ be a collection of nowhere dense closed sets such that $\kappa = \mathrm{add}(\mathcal{M})$ and $\bigcup_{\alpha\in\kappa} F_\alpha$ is non-meager. Let $g_\alpha = \chi_{F_\alpha}$ note that $g_\alpha \in B_1$ for every $\alpha \in \kappa$. Let $f \in \mathcal{B}$. There is a dense $G_\delta$-set $G$ such that $f|_G$ is continuous. By our choice of $F$, there is is an $\alpha \in \kappa$ such that $F_\alpha \cap G \neq \emptyset$. Thus, $g_\alpha|_G$ is discontinuous. Since each $g_\alpha \in \mathcal{B}_1$ and $f$ was arbitrary chosen from $\mathcal{B}$, we have

$$\max\{\mathrm{add}(\mathcal{B}_1, \mathcal{G}), \mathrm{add}(\mathcal{B}, \mathcal{G})\} \leq \mathrm{add}(\mathcal{M}). \tag{2}$$

Since $\mathcal{G} \subseteq \mathcal{T} \subseteq \mathcal{D}$, (1) and (2) imply (c).

We now work for (d). Let $F = \{F_\alpha\}_{\alpha\in\kappa}$ be a collection of meager $F_\sigma$-sets such that $\kappa = \mathrm{cf}(\mathcal{M})$ and $F$ witnesses the definition of $\mathrm{cf}(\mathcal{M})$. For every $\alpha \in \kappa$ we may use Lemma 9, as before, to define a $g_\alpha \in \mathcal{B}_1$ so that for any $D \in \mathcal{D}$ if $D \cap F_\alpha \neq \emptyset$, then $g_\alpha|_D$ is not continuous. Let $f \in \mathcal{B}$. There is a dense $G_\delta$-set $G$ such that $f|_G$ is continuous. For any $D \in \mathcal{D}$ if $f|_D$ is discontinuous then $D \cap (\mathbb{R} \setminus G) \neq \emptyset$. By our choice of $F$, there is is an $\alpha \in \kappa$ such that $\mathbb{R} \setminus G \subseteq F_\alpha$. So, $D \cap F_\alpha \neq \emptyset$. Thus, for any $D \in \mathcal{D}$ if $f|_D$ is discontinuous then $g_\alpha|_D$ is also discontinuous. Since $f \in \mathcal{B}$ was arbitrary and $\{g_\alpha\colon \alpha \in \kappa\} \subseteq \mathcal{B}_1$, we have

$$\max\{\mathrm{cf}(\mathcal{B}_1, \mathcal{D}), \mathrm{cf}(\mathcal{B}, \mathcal{D})\} \leq \mathrm{cf}(\mathcal{M}). \tag{3}$$

Suppose we have a collection $F \subseteq \mathcal{B}$ such that $|F| < \mathrm{cf}(\mathcal{M})$. For each $f \in F$ there is a $G_f \in \mathcal{G}$ such that $f|_{G_f}$ is continuous. Since $|\{\mathbb{R} \setminus G_f\colon f \in F\}| <$

$\mathrm{cf}(\mathcal{M})$, there is a meager $F_\sigma$-set $M$ such that $M \cap G_f \neq \emptyset$ for all $f \in F$. Let $g \in \mathcal{B}_1$ be such that for any $D \in \mathcal{D}$ if $D \cap M \neq \emptyset$ then $g|_D$ is not continuous. It follows that $g|_{G_f}$ is discontinuous for every $f \in F$. So, for any collection $F$ of less than $\mathrm{cf}(\mathcal{M})$-many functions in $\mathcal{B}$ there is a $g \in \mathcal{B}_1$ such that $\{G \in \mathcal{G} \colon g|_G \text{ is discontinuous}\}$ is not contained in $\{G \in \mathcal{G} \colon f|_G \text{ is discontinuous}\}$ for any $f \in F$. Thus,

$$\min\{\mathrm{cf}(\mathcal{B}_1, \mathcal{G}), \mathrm{cf}(\mathcal{B}, \mathcal{G})\} \geq \mathrm{cf}(\mathcal{M}). \tag{4}$$

Since $\mathcal{G} \subseteq \mathcal{T} \subseteq \mathcal{D}$, (3) and (4) imply (d).

We now show (e). Clearly, $\mathrm{add}(\mathcal{M}) = \mathrm{add}(\mathcal{B}, \mathcal{G}) \leq \mathrm{cov}(\mathcal{B}, \mathcal{G}) \leq \mathrm{cov}(\mathcal{B}_1, \mathcal{G})$. It is enough show that $\mathrm{cov}(\mathcal{B}_1, \mathcal{G}) \leq \mathrm{add}(\mathcal{M})$. Let $\mathcal{F}$ be a collection of closed nowhere dense subsets of $\mathbb{R}$ such that $\bigcup \mathcal{F} \notin \mathcal{M}$ and $|\mathcal{F}| = \mathrm{add}(\mathcal{M})$. For each $F \in F$ let $g_F = \chi_F$. For any $D \in \mathcal{D}$ if $D \cap F \neq \emptyset$ then $g_F|_D$ is not continuous. For any $G \in \mathcal{G}$ there is an $F \in \mathcal{F}$ such that $F \cap G \neq \emptyset$. Thus, $\{g_F \colon F \in \mathcal{F}\}$ witnesses the definition of $\mathrm{cov}(\mathcal{B}_1, \mathcal{G})$. Thus, (e) holds.

The proof of (f) is of the same form as the proof of (e).

We now work for (b). We show that $\mathrm{non}(\mathcal{M}) \leq \mathrm{non}(\mathcal{B}_1, \mathcal{D})$. Let $D \subseteq \mathcal{D}$ and $|D| < \mathrm{non}(\mathcal{M})$. We may assume each element of $D$ is countable, so $E = \bigcup D$ has cardinality less than $\mathrm{non}(\mathcal{M})$. So, there a meager $F_\sigma$-set $M \subseteq \mathbb{R}$ such that $E \subseteq M$. Let $M = \bigcup_{k \in \omega} N_k$ where each $N_k$ is closed and nowhere dense. By Lemma 9 there is an $f \in \mathcal{B}_1$ such that $f|_X$ is discontinuous for any $X \subseteq \mathbb{R}$ such that $X \cap N_k$ is not $X$-open. Let $d \in D$ be arbitrary. There is an $N_k$ such that $d \cap N_k$ is nonempty. Clearly, $d \cap N_k$ is not d-open. So, $f|_d$ is not continuous for all $d \in D$. Thus, $\mathrm{non}(\mathcal{M}) \leq \mathrm{non}(\mathcal{B}_1, \mathcal{D})$.

We now show that $\mathrm{non}(\mathcal{B}, \mathcal{D}) \leq \mathrm{non}(\mathcal{M})$. Let $X \subseteq \mathbb{R}$ be such that $|X| = \mathrm{non}(\mathcal{M})$ and $X$ is non-meager. For each $x \in X$ define $D_x \in \mathcal{D}$ by $D_x = \{x + q \colon q \in \mathbb{Q}\}$. Clearly, $|\{D_x \colon x \in X\}| \leq \mathrm{non}(\mathcal{M})$. Let $f \in \mathcal{B}$. There is a $G \in \mathcal{G}$ such that $f|_G$ is continuous. Notice that $H = \bigcap_{q \in \mathbb{Q}} q + G$ is in $\mathcal{G}$. Since $X$ is non-meager there is an $x \in X$ such that $x \in H$. By definition of $H$ we have $D_x \subseteq H \subseteq G$. So, $f|_{D_x}$ is continuous. Thus, $\mathrm{non}(\mathcal{B}, \mathcal{D}) \leq \mathrm{non}(\mathcal{M})$.

The inequality $\mathrm{non}(\mathcal{B}_1, \mathcal{D}) \leq \mathrm{non}(\mathcal{B}, \mathcal{D})$ is obvious. So we have (b).

To get the item (a) of Theorem 5 we will have to consider the hyperspace $\mathrm{J}(\mathbb{R})$ of compact subsets of $\mathbb{R}$ with the Hausdorff metric which we will denote by $H_d$. Note that $\mathrm{J}(\mathbb{R})$ is an uncountable separable complete metric space. In particular, the meager covering numbers and non-meager numbers of $\mathbb{R}$ and $\mathrm{J}(\mathbb{R})$ are equal.

**Lemma 10.** *If $G \subseteq \mathbb{R}$ is a dense $G_\delta$-set then the set $G_1 = \{E \in \mathrm{J}(\mathbb{R}) \colon E \subseteq G$ and $E \in \mathcal{K}\}$ is a dense $G_\delta$-set of $\mathrm{J}(\mathbb{R})$.*

PROOF. Let $G = \bigcap_{n\in\omega} U_n$ where each $U_n$ is open and dense in $\mathbb{R}$. By definition of the Vietoris topology, $V_n = \{x \in \mathrm{J}(\mathbb{R})\colon x \subseteq U_n\}$ is open in $\mathrm{J}(\mathbb{R})$. Using the fact that the finite subsets of $\mathbb{R}$ form a dense subset of $\mathrm{J}(\mathbb{R})$ it is easy to check that $V_n$ is dense in $\mathrm{J}(\mathbb{R})$ for every $n \in \omega$. Thus, $H = \{x \in \mathrm{J}(\mathbb{R})\colon x \subseteq G\}$ is a dense $G_\delta$-set of $\mathrm{J}(\mathbb{R})$.

Since $G_1 = \mathcal{K} \cap H$ it is enough for us to check that $\mathcal{K}$ is a dense $G_\delta$-set. For each $n \in \omega$ let

$$F_n = \{x \in \mathrm{J}(\mathbb{R})\colon \text{ there exists } p \in x \text{ such that } \mathrm{dist}(p, x \setminus \{p\}) > 1/(n+1)\}.$$

We leave it to the reader to check that each $F_n$ is closed and nowhere dense in $\mathrm{J}(\mathbb{R})$. Notice now that $\mathcal{K} = \mathrm{J}(\mathbb{R}) \setminus \bigcup_{n\in\omega} F_n$. □

**Lemma 11.** *If $f \in \mathcal{B}$ then there is a dense $G_\delta$-set $G$ in $\mathrm{J}(\mathbb{R})$ such that $f|_{\bigcup G}$ is continuous and each $E \in G$ is perfect.*

PROOF. Since $f$ has the Baire Property there is a dense $G_\delta$-set $H \subseteq \mathbb{R}$ such that $f|_H$ is continuous. By Lemma 10 the set $G = \{E\colon E \subseteq H \text{ and } E \in \mathcal{K}\}$ is a dense $G_\delta$-set in $\mathrm{J}(\mathbb{R})$. Clearly, $f|_{\bigcup G}$ is continuous. □

We now work for (a). We first show that

$$\mathrm{cov}(\mathcal{B}_1, \mathcal{D}) \leq \mathrm{cov}(\mathcal{M}). \tag{5}$$

Let $\{N_\alpha\}_{\alpha<\mathrm{cov}(\mathcal{M})}$ be a collection of closed nowhere dense subsets of $\mathbb{R}$ which cover $\mathbb{R}$. For each $\alpha < \mathrm{cov}(\mathcal{M})$ let $f_\alpha = \chi_{N_\alpha} \in \mathcal{B}_1$. We claim that $F = \{f_\alpha\colon \alpha \in \mathrm{cov}(\mathcal{M})\}$ witnesses the definition of $\mathrm{cov}(\mathcal{B}_1, \mathcal{D})$. Let $D \in \mathcal{D}$ be arbitrary. There is an $\alpha < \mathrm{cov}(\mathcal{M})$ such that $D \cap N_\alpha \neq \emptyset$. Since $D \cap N_\alpha$ is not open in $D$ it follows that $f_\alpha|_D$ is not continuous. Therefore, we have (5).

Let $F \subseteq \mathcal{B}$ and $|F| < \mathrm{cov}(\mathcal{M})$. For each $f \in F$ let $G_f \subseteq \mathrm{J}(\mathbb{R})$ be as in Lemma 11. Since $|\{G_f\colon f \in F\}| < \mathrm{cov}(\mathcal{M})$, the set $G = \bigcap\{G_f\colon f \in F\}$ is non-empty and everywhere of second category. It follows that $H = \bigcup G \in \mathcal{T}$ and $f|_H$ is continuous for all $f \in F$. So, $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{B}, \mathcal{T})$. Since $\mathcal{T} \subseteq \mathcal{D}$ and $\mathcal{B}_1 \subseteq \mathcal{B}$, we have

$$\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{B}, \mathcal{T}) \leq \mathrm{cov}(\mathcal{B}_1, \mathcal{T}) \leq \mathrm{cov}(\mathcal{B}_1, \mathcal{D}) \tag{6}$$

and

$$\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{B}, \mathcal{T}) \leq \mathrm{cov}(\mathcal{B}, \mathcal{D}) \leq \mathrm{cov}(\mathcal{B}_1, \mathcal{D}) \tag{7}$$

(5) together with (6) and (7) yield (a). □

## 4 Proof of Theorem 8

**Lemma 12.** *If $f\colon \mathbb{R} \to \mathbb{R}$ is a $\mathcal{B}_1$ function, then there is a $f$-family $\mathcal{C}$ of size $\omega$.*

Proof. Let $\{B_n\}_{n\in\omega}$ be a base for $\mathbb{R}$. For $n \in \omega$ define $A_n = f^{-1}(B_n)$. Since $B_n$ is open, $A_n$ is an $F_\sigma$-set. So, there exist closed sets $\{E_n^k\colon k \in \omega\}$ such that $A_n = \bigcup_{k\in\omega} E_n^k$ for each $n \in \omega$. Let $\mathcal{C} = \{E_n^k\colon k, n \in \omega\}$. Clearly, $\mathcal{C}$ is a collection of closed sets of size $\omega$.

We now show that $\mathcal{C}$ is an $f$-family. Fix $P \in \mathcal{K}$ and $x \in P$. Suppose $f|_P$ is discontinuous at $x$. Pick a convergent sequence $x_l \to x$ contained in $P$ such that $\lim_{l\to\infty} f(x_l) \neq f(x)$. We may find, using a subsequence if necessary, a $B_n$ such that $f(x) \in B_n$ and $f(x_l) \notin B_n$ for all $l \in \omega$. Pick $k \in \omega$ so that $x \in E_n^k$. Since $x_l \notin E_n^k$ for all $l \in \omega$, it follows that $x \notin \mathrm{int}_P(E_n^k)$. □

**Lemma 13.** *If $X$ is a metric space and $f\colon X \to Y$ has an $f$-family of cardinality less than $\mathrm{cov}(\mathcal{M})$ then for any $P \in \mathcal{K}(X)$ there is an $x \in P$ such that$f|_P$ is continuous at $x$.*

Proof. Let $\kappa < \mathrm{cov}(\mathcal{M})$ and $\{\mathcal{C}_\alpha\}_{\alpha\in\kappa}$ be an $f$-family. Let $P \in \mathcal{K}(X)$. By way of contradiction, assume that $f|_P$ is nowhere continuous. Let $p \in P$. There is some $\alpha \in \kappa$ such that $p \in (P \cap C_\alpha) \setminus \mathrm{int}_P(C_\alpha \cap P)$. It follows that $P = \bigcup_{\alpha\in\kappa}((P \cap C_\alpha) \setminus \mathrm{int}_P(P \cap C_\alpha))$. However, each $(P \cap C_\alpha) \setminus \mathrm{int}_P(P \cap C_\alpha)$ is nowhere dense in $P$ and $\kappa < \mathrm{cov}(\mathcal{M})$ a contradiction to the fact that $P$ is an uncountable complete metric space. Thus, $f|_P$ is continuous at some point. □

Proof of Theorem 8. Lemma 12 shows that (a) implies (b). That (b) implies (c) is just the Baire Category Theorem (i.e. $\mathrm{cov}(\mathcal{M}) > \omega$). We show that (c) implies (a). If there is an $f$-family of size less than $\mathrm{cov}(\mathcal{M})$ then by Lemma 13 $f|_P$ has a continuity point for each $P \in \mathcal{K}$ which, by a well known theorem implies that $f \in \mathcal{B}_1$. □

## 5 Proof of Theorem 6 and Related Topics

Proof of Theorem 6. Suppose $Z \subseteq \{P \in \mathcal{K}\colon f|_P \text{ is discontinuous}\}$ and that $f \in \mathcal{B}_1$. By Lemma 12 there is a countable $f$-family $\mathcal{C}$. By, definition of $f$-family, we have that $Z \subseteq \{P \in \mathcal{K}\colon f|_P \text{ is discontinuous}\} \subseteq \bigcup_{C\in\mathcal{C}} \mathrm{N}(C)$. Thus, $Z \in \mathcal{Z}_p$.

Suppose now that $Z \in \mathcal{Z}_p$. Then, there exists a collection of closed sets $\{C_n\colon n \in \omega\}$ such that $Z \subseteq \bigcup_{n\in\omega} \mathrm{N}(C_n)$. By Lemma 9, there is an $f \in \mathcal{B}_1$

such that for any set $X \subseteq \mathbb{R}$ if $X \cap C_n$ is not open in $X$ for some $n \in \omega$ then $f|_X$ is not continuous. We now have that

$$\bigcup_{n\in\omega} \mathrm{N}(C_n) \subseteq \{P \in \mathcal{K} \colon f|_P \text{ is discontinuous}\} \tag{8}$$

which completes the proof of Theorem 6. $\square$

We will now take some time to investigate what the elements of $\mathcal{Z}_p$ look like.

**Proposition 14.** *If $C \subseteq \mathbb{R}$ is closed, then $\mathrm{N}(C)$ is a nowhere dense $G_\delta$-subset of $\mathcal{K}$.*

PROOF. It is easy to check that $\mathrm{N}(C)$ is exactly the collection of perfect sets for which the $\mathcal{B}_1$ function $f = \chi_C$ is not continuous. Since $V = \mathrm{int}(C) \cup (\mathbb{R} \setminus C)$ is dense and open in $\mathbb{R}$ and $f|_V$ is continuous, we have that $H = \{A \in \mathcal{K} \colon A \subseteq V\}$ is dense and open in $\mathcal{K}$ and that $H \cap \mathrm{N}(C) = \emptyset$. So, $\mathrm{N}(C)$ is nowhere dense in $\mathcal{K}$.

We now show that $\mathrm{N}(C)$ is a $G_\delta$-set of $\mathcal{K}$. Again, we let $f = \chi_C$. Pick a sequence of continuous functions $\langle f_n \rangle_{n\in\omega}$ so that $f_n \to f$ pointwise and,

**(i)** $f_0(x) = 1$ for all $x \in \mathbb{R}$,

**(ii)** $f_{n+1} \leq f_n$ for every $n \in \omega$,

**(iii)** for every $x \notin C$ there exists an $m_x \in \omega$ such that $1 > f_{m_x}(x) > 0$, and

**(iv)** for every $x \notin C$ there is an open set $U$ and a $n_x \in \omega$ such that $x \in U$ and $f_{n_x}[U] = \{0\}$.

Let $E_n = \{P \in \mathcal{K} \colon f_n[P] \subseteq \{0,1\}\}$ for every $n \in \omega$. Notice $E_n$ is closed in $\mathcal{K}$, since by continuity $f_n^{-1}(\{0,1\})$ is closed in $\mathbb{R}$ and $P \in E_n$ if and only if $P \subseteq f_n^{-1}(\{0,1\})$. Let $E = \bigcup_{n\in\omega} \bigcap_{m\geq n} E_m$. Clearly, $E$ is an $F_\sigma$-set. We will be done if we show that $P \in E$ if and only if $P \notin \mathrm{N}(C)$.

Suppose $P \in E$. There is an $n \in \omega$ such that $P \in \bigcap_{m\geq n} E_m$. Let $x \in P$. If $f(x) = 1$, then by (ii) $f_n(x) = 1$. If $f_n(x) = 1$, then by (iii) and the fact that $P \in \bigcap_{m\geq n} E_m$ we have that $f(x) = 1$. So, $f|_P = f_n|_P$. Since $f|_P$ is continuous, we must have that $P \notin \mathrm{N}(C)$.

Suppose $P \notin E$. For infinitely many $n \in \omega$ there is an $x_n \in P$ such that $0 < f_n(x_n) < 1$. Taking a subsequence if necessary, we may assume there is an $x \in P$ such that $\lim_{n\in\omega} x_n = x$. By (ii) and (i) we know that $x_n \in P \setminus C$ for every $n \in \omega$. We claim that $x \in C$. Suppose $x \notin C$. Then, there is by (iv) and (ii) an open set $U$ such that $f_m[U] = \{0\}$ for all $m \geq n_x$ and $x \in U$.

So there is a $p \in \omega$ such that $f_m(x_n) = 0$ for all $m \geq n_x$ and $n \geq p$, which contradicts that $1 > f_n(x_n) > 0$ for every $n \in \omega$. So $x \in C$. Thus, $P \in \mathrm{N}(C)$. Therefore, $P \in E$ if and only if $P \notin \mathrm{N}(C)$. $\square$

It follows immediately from Proposition 14 that $\mathcal{Z}_p$ is a $G_{\delta\sigma}$ supported $\sigma$-ideal. We now show that $\mathcal{Z}_p$ is neither $F_\sigma$ supported nor $G_\delta$ supported. I do not know if $\mathcal{Z}_p$ can be shown to be $F_{\sigma\delta}$ supported.

**Proposition 15.** *If $C$ is nowhere dense and perfect, then there is no $F_\sigma$-set in $\mathcal{Z}_p$ which contains $\mathrm{N}(C)$. In other words, the ideal $\mathcal{Z}_p$ is not $F_\sigma$ supported.*

PROOF. By way of contradiction, assume there is an $F_\sigma$-set $F \in \mathcal{Z}_p$ such that $\mathrm{N}(C) \subseteq F$. Let $L = \{P \in \mathcal{K}\colon P \cap C \neq \emptyset\}$ and notice $L$ is closed in $\mathcal{K}$. Since $C$ is nowhere dense, it is easy check that $\mathrm{N}(C)$ is dense in $L$. By Proposition 14, we have that $\mathrm{N}(C)$ is actually a dense $G_\delta$-set in $L$. Since $\mathrm{N}(C) \subseteq L \cap F$ and $F$ is an $F_\sigma$-set, there is an $L$-open set $U$ such that $U \subseteq F$. By Theorem 6, there is a $g \in \mathcal{B}_1$ such that $U \subseteq \{P \in \mathcal{K}\colon g|_P \text{ is discontinuous}\}$.

We claim that $D = \{P \in L\colon P \cap C \in \mathcal{K} \;\&\; P \setminus C \in \mathcal{K}\}$ is dense in $L$. Let $Q \in L$ and $\epsilon > 0$. Since $C$ is perfect and $Q \cap C \neq \emptyset$, there is a perfect set $J_0$ such that $Q \cap C \subseteq J_0 \subseteq C$ and $H_d(Q \cap C, J_0) < \epsilon$. Since $Q \setminus J_0$ is bounded, there is a finite set $x_1 \ldots x_n \in Q \setminus J_0$ such that $Q \subseteq \bigcup_{k \leq n} \mathrm{B}_\epsilon(x_k)$. For each $1 \leq k \leq n$ pick a perfect set $J_k \subseteq \mathrm{B}_\epsilon(x_k) \setminus C$ such that $x_k \in J_k$. It is now straight forward to check that $J = \bigcup_{0 \leq k \leq n} J_k$ is in $D$ and that $H_d(J, Q) < \epsilon$. Since $\epsilon$ was arbitrary, we have established the claim.

Pick $A \in D \cap U$. Since $g \in \mathcal{B}_1$, there is a $G_\delta$-set $H \subseteq A \cap C$ such that $H$ is dense in $A \cap C$ and $g|_H$ is continuous. Since $g \in \mathcal{B}_1$, there is a $G_\delta$-set $I \subseteq A \setminus C$ such that $I$ is dense in $A \setminus C$ and $g|_I$ is continuous. We can now pick perfect sets $P \subseteq I$ and $Q \subseteq H$ such that $H_d(P \cup Q, A)$ is as small as we want. Thus, we may assume $P \cup Q \in U$. However, it is clear that $g|_{P \cup Q}$ is continuous which contradicts our choice of $g$. $\square$

We show that $\mathcal{Z}_p$ is not $G_\delta$ supported. To see that let $Z = \bigcup_{n \in \omega} \mathrm{N}(\{q_n\})$, where $\{q_1, q_2, \ldots\}$ is an enumeration of the rationals. It is easy to show that $Z$ is dense in $\mathcal{K}$. Hence any $G_\delta$-set $H$ containing $Z$ would be a dense $G_\delta$ in $\mathcal{K}$. However, by Proposition 14 such an $H$ could not be in $\mathcal{Z}_p$.

While we are considering $\mathcal{Z}_p$ to be an ideal of small sets the next theorem says that these sets can be to some extent large.

**Theorem 16.** *Let $C \subseteq \mathcal{K}$ be compact. There is a $G_\delta$-set $G \subseteq C$ such that $G$ is dense in $C$ and $G \in \mathcal{Z}_p$.*

Before proving Theorem 16 we will need some definitions and a lemma. In what follows given an $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ we will write $\mathrm{dist}(x, A)$ to stand for $\inf\{|x - a|\colon a \in A\}$ and we write $\mathrm{diam}(A)$ to stand for $\sup\{|a - b|\colon a, b \in A\}$.

**Lemma 17.** *Let $C \subseteq \mathcal{K}$ be compact. For every $\epsilon > 0$ there is a $\delta > 0$ such that for any $P \in C$ and $x \in P$*

$$\operatorname{diam}(P \cap \mathrm{B}_\epsilon(x)) > \delta. \tag{9}$$

PROOF. Suppose the lemma is false. There is an $\epsilon > 0$ such that for every $n \in \omega$ there exists $P_n \in C$ and $x_n \in P_n$ such that $\operatorname{diam}(P_n \cap \mathrm{B}_\epsilon(x_n)) < 1/n$. Since $C$ is compact, there is, taking a subsequence if necessary, a $P \in C$ such that $\lim_{n\to\infty} P_n = P$. Taking a subsequence if necessary we may also find an $x \in P$ such that $\lim_{n\to\infty} x_n = x$. We will derive a contradiction by showing that $x$ is isolated in $P$. Let $\mathrm{B} = \mathrm{B}_\epsilon(x)$ and $D = \operatorname{diam}(P \cap \mathrm{B})$, it is enough to show that $D = 0$. Suppose $D > 0$ then there is a $w \in (P \cap B) \setminus \{x\}$. Pick $n \in \omega$ so large that

**(a)** $H_d(B_\epsilon(x_n), B) < \operatorname{dist}(w, \mathbb{R} \setminus B)/3$,

**(b)** $|x - x_n| < |x - w|/3$,

**(c)** $H_d(P_n, P) < \min\{|x - w|/3, \operatorname{dist}(w, \mathbb{R} \setminus B)/3\}$, and

**(d)** $\operatorname{diam}(P_n \cap \mathrm{B}_\epsilon(x_n)) < |x - w|/3$.

By (c), there is a point $z \in P_n$ so that

$$|z - w| < \min\{|x - w|/3, \operatorname{dist}(w, \mathbb{R} \setminus B)/3\}. \tag{10}$$

So, $\operatorname{dist}(z, \mathbb{R} \setminus B) > 2\operatorname{dist}(w, \mathbb{R} \setminus B)/3$. So by (a), we have that $z \in B_\epsilon(x_n)$. Since $z \in P_n \cap \mathrm{B}_\epsilon(x_n)$ we have, by (d) that

$$|z - x_n| < |x - w|/3. \tag{11}$$

Using (10) and (11) with (b) we get that

$$|x - w| < |x - x_n| + |x_n - z| + |z - w| < |x - w|$$

a contradiction. Thus, $D = 0$. □

PROOF OF THEOREM 16. Let $\{P_n : n \in \omega\}$ be a dense subset of $C$. We construct a closed set $Q$ such that $P_n \in \mathrm{N}(Q)$ for every $n$. Since $C$ is a compact subset of $\mathcal{K}$, there is an $D > 0$ such that $\operatorname{diam}(P) > D$ for every $P \in C$. Inductively we will define sequences $\{x_n \in \mathbb{R} : n \in \omega\}$ and $\{\delta_n > 0 : n \in \omega\}$ such that for every $n \in \omega$ we have

$(a_n)$ $x_n \in P_n$,

$(b_n)$ $\mathrm{B}_{\delta_k}(x_k) \cap \mathrm{B}_{\delta_l}(x_l) = \emptyset$ if $0 \le k < l \le n$,

$(c_n)$ for every $P \in C$ we have $P \setminus \mathrm{cl}(\bigcup_{k\le n} \mathrm{B}_{\delta_k}(x_k)) \neq \emptyset$

If we choose $x_0 \in P_0$ and $\delta_0 < D/2$, then $(a_0)$, $(b_0)$, and $(c_0)$ are satisfied. Suppose we have constructed $\{x_k : k \le n\}$ and $\{\delta_k : k \le n\}$ so that $(a_n)$, $(b_n)$, and $(c_n)$ are satisfied. We show how to pick $x_{n+1}$ and $\delta_{n+1}$. By $(c_n)$, we can pick $x_{n+1} \in P_{n+1} \setminus \mathrm{cl}(\bigcup_{k\le n} \mathrm{B}_{\delta_k}(x_k))$. Let $\epsilon = \mathrm{dist}(x_{n+1}, \mathrm{cl}(\bigcup_{k\le n} \mathrm{B}_{\delta_k}(x_k))$ and notice that $\epsilon > 0$. By Lemma 17, there is a $\delta > 0$ such that

$$\mathrm{diam}(P \cap \mathrm{B}_{\epsilon/3}(x)) > \delta \tag{12}$$

for every $P \in C$ and $x \in P$. We let $\delta_{n+1} < \min\{\epsilon/3, \delta\}$. We now show that $(a_{n+1})$, $(b_{n+1})$, and $(c_{n+1})$ are satisfied. It is easy to see that $(a_{n+1})$ is satisfied. To see that $(b_{n+1})$ is satisfied it enough to notice that $\delta_{n+1} < \epsilon/3$ by (12). We check $(c_{n+1})$. By way of contradiction, suppose $P \subseteq \mathrm{cl}(\bigcup_{k\le n+1} \mathrm{B}_{\delta_k}(x_k))$ for some $P \in C$. By $(c_n)$, $P \cap B_\epsilon(x_{n+1}) \neq \emptyset$. Also, $P \cap B_\epsilon(x_{n+1}) \subseteq \mathrm{cl}(B_{\delta_{n+1}}(x_{n+1}))$. Pick $w \in P \cap \mathrm{cl}(B_{\delta_{n+1}}(x_{n+1}))$. Since $|w - x_{n+1}| < \delta_{n+1} < \epsilon/3$, we have that $\mathrm{cl}(B_{\epsilon/3}(w)) \subseteq \mathrm{B}_\epsilon(x_{n+1})$. So we now have that $P \cap B_{\epsilon/3}(w) \subseteq P \cap \mathrm{B}_\epsilon(x_{n+1}) \subseteq \mathrm{cl}(B_{\delta_{n+1}}(x_{n+1}))$. Hence, $\mathrm{diam}(P \cap B_{\epsilon/3}(w)) \le \delta_{n+1} < \delta$. On the other hand, by (12), we have that $\mathrm{diam}(P \cap B_{\epsilon/3}(w)) > \delta$ a contradiction. So we have $(c_{n+1})$.

Let $Q = \mathrm{cl}\{x_n : n \in \omega\}$. Let $n \in \omega$. Clearly, $x_n \in P_n \cap Q$. Moreover, by the (b) conditions of the inductive construction we have that $x_n$ isolated in $P_n \cap Q$. Since $P_n$ has no isolated points, $P_n \cap Q$ is not open in $P_n$. Thus, $P_n \in \mathrm{N}(Q)$ for all $n \in \omega$. Since $\{P_n : n \in \omega\} \subseteq \mathrm{N}(Q)$, we have that $C \cap \mathrm{N}(Q)$ is dense in $C$. By Proposition 14, $\mathrm{N}(Q) \cap C$ is a $G_\delta$-set in $C$. Finally, by definition $\mathrm{N}(Q) \in \mathcal{Z}_p$. □

## 6 Proof of Theorem 7

PROOF OF THEOREM 7.

We show (a). We show $\mathrm{cov}(\mathcal{B}_1, \mathcal{K}) \le \mathrm{cov}(\mathcal{Z}_p)$. Suppose $\mathcal{F} \subseteq \mathcal{Z}_p$, $|\mathcal{F}| = \mathrm{cov}(\mathcal{Z}_p)$, and $\bigcup \mathcal{F} = \mathcal{K}$. For each $F \in \mathcal{F}$ there is, by Theorem 6, a $g_F \in \mathcal{B}_1$ such that $F \subseteq \{P \in \mathcal{K} : g_F|_P \text{ is discontinuous}\}$. Thus, $\mathcal{K} \subseteq \bigcup_{F\in\mathcal{F}}\{P \in \mathcal{K} : g_F|_P \text{ is discontinuous}\}$. So, $\mathrm{cov}(\mathcal{B}_1, \mathcal{K}) \le \mathrm{cov}(\mathcal{Z}_p)$.

We show $\mathrm{cov}(\mathcal{B}_1, \mathcal{K}) \ge \mathrm{cov}(\mathcal{Z}_p)$. Let $F \subseteq \mathcal{B}_1$ be such that $|F| = \mathrm{cov}(\mathcal{B}_1, \mathcal{K})$ and for every $P \in \mathcal{K}$ there exists an $f \in F$ such that $f|_P$ is not continuous. Then, $\mathcal{K} = \bigcup_{f\in F}\{P \in \mathcal{K} : f|_P \text{ is discontinuous}\}$. However, by Theorem 6, $\{P \in \mathcal{K} : f|_P \text{ is discontinuous}\} \in \mathcal{Z}_p$ for each $f \in F$. Thus, $\mathrm{cov}(\mathcal{B}_1, \mathcal{K}) \ge \mathrm{cov}(\mathcal{Z}_p)$.

We show (b). We show $\mathrm{non}(\mathcal{B}_1, \mathcal{K}) \le \mathrm{non}(\mathcal{Z}_p)$. Suppose $F \subseteq \mathcal{K}$, $|F| = \mathrm{non}(\mathcal{Z}_p)$, and $F$ is not contained in $\bigcup_{n\in\omega} \mathrm{N}(C_n)$ for any countable collection

$\{C_n : n \in \omega\}$ of closed sets. Let $f \in \mathcal{B}_1$ be arbitrary. By Theorem 6, we have that $\{P \in \mathcal{K} \colon f|_P \text{ is discontinuous}\} \in \mathcal{Z}_p$. By our choice of $F$ there is a perfect set $M \in F$ such that $M \notin \{P \in \mathcal{K} \colon f|_P \text{ is discontinuous}\}$ which implies that $f|_M$ is continuous. Thus, $\mathrm{non}(\mathcal{B}_1, \mathcal{K}) \leq \mathrm{non}(\mathcal{Z}_p)$.

We show $\mathrm{non}(\mathcal{B}_1, \mathcal{K}) \geq \mathrm{non}(\mathcal{Z}_p)$. Suppose $F \subseteq \mathcal{K}$ and $|F| < \mathrm{non}(\mathcal{Z}_p)$. Since $F \in \mathcal{Z}_p$, by Theorem 6 there is an $f \in \mathcal{B}_1$ such that $F \subseteq \{P \in \mathcal{K} \colon f|_P \text{ is discontinuous}\}$. Thus, $\mathrm{non}(\mathcal{B}_1, \mathcal{K}) \geq \mathrm{non}(\mathcal{Z}_p)$.

We show (c). We show $\mathrm{add}(\mathcal{B}_1, \mathcal{K}) \leq \omega_1$. Let $A \subseteq \mathbb{R}$ and $|A| = \omega_1$. Let $F = \{\chi_{\{a\}} \colon a \in A\}$ and note that $|F| = \omega_1$ and $F \subseteq \mathcal{B}_1$. Clearly,

$$\bigcup_{f \in F} \{P \in \mathcal{K} \colon \ f|_P \text{ is not continuous}\} = \{P \in \mathcal{K} \colon P \cap A \neq \emptyset\}.$$

Let $M = \{P \in \mathcal{K} \colon P \cap A \neq \emptyset\}$. Let $g \in \mathcal{B}_1$ be arbitrary. Since $|g|_A| = \omega_1$, there is a $B \subseteq A$ such that $|B| = \omega_1$ and for every $b \in B$ and $\epsilon > 0$ we have

$$|\mathrm{B}_\epsilon(\langle b, g(b)\rangle) \cap g|_A| = \omega_1. \tag{13}$$

Pick distinct $\{b_\alpha \colon \alpha \in \omega + 1\} \subseteq B$ so that $\lim_{n\to\infty} \langle b_n, g(b_n)\rangle = \langle b_\omega, g(b_\omega)\rangle$. We may assume that $\langle b_n\rangle_{n\in\omega}$ is a strictly decreasing sequence. By the fact that $g \in \mathcal{B}_1$ and (13), we can pick perfect sets $P_n$ such that $b_{n+1} \leq \inf(P_n) < \sup(P_n) < b_{n-1}$ and $g|_{P_n}$ is continuous and $\mathrm{osc}(g|_{P_n \cup \{b_n\}}) < 1/n$. Now $P = \{b_\omega\} \cup \bigcup_{n\in\omega} P_n$ is a perfect set such that $g|_P$ is continuous and $P \cap A \neq \emptyset$. Thus, $M$ is not contained in $\{P \in \mathcal{K} \colon g|_P \text{ is not continuous}\}$. Thus, $\mathrm{add}(\mathcal{B}_1, \mathcal{K}) \leq \omega_1$.

We show that $\mathrm{add}(\mathcal{Z}_p) \leq \mathrm{add}(\mathcal{B}_1, \mathcal{K})$. Suppose $F \subseteq \mathcal{B}_1$ and $|F| < \mathrm{add}(\mathcal{Z}_p)$. By Theorem 6, $\{P \in \mathcal{K} \colon f|_P \text{ is discontinuous}\} \in \mathcal{Z}_p$ for each $f \in F$. Since $|F| < \mathrm{add}(\mathcal{Z}_p)$,

$$\bigcup_{f \in F} \{P \in \mathcal{K} \colon f|_P \text{ is discontinuous}\} \in \mathcal{Z}_p.$$

By Theorem 6, there is a $g \in \mathcal{B}_1$ such that

$$\bigcup_{f \in F} \{P \in \mathcal{K} \colon f|_P \text{ is discontinuous}\} \subseteq \{P \in \mathcal{K} \colon g|_P \text{ is discontinuous}\}.$$

Thus, $\mathrm{add}(\mathcal{Z}_p) \leq \mathrm{add}(\mathcal{B}_1, \mathcal{K})$. Since $\mathcal{Z}_p$ is a $\sigma$-ideal, we have $\mathrm{add}(\mathcal{Z}_p) \geq \omega_1$ which completes the proof of (c).

We show (d). We show $\mathrm{cf}(\mathcal{B}_1, \mathcal{K}) \leq \mathrm{cf}(\mathcal{Z}_p)$. Suppose $F \subseteq \mathcal{Z}_p$ and $|F| < \mathrm{cf}(\mathcal{B}_1, \mathcal{K})$. By Theorem 6, for every $Z \in F$ there is a $f_Z \in \mathcal{B}_1$ such that

$Z \subseteq \{P \in \mathcal{K} \colon f_Z|_P$ is discontinuous$\}$. Since $|F| < \mathrm{cf}(\mathcal{B}_1, \mathcal{Z}_p)$, there is a $g \in \mathcal{B}_1$ such that

$$\{P \in \mathcal{K} \colon g|_P \text{ is discontinuous}\} \setminus \{P \in \mathcal{K} \colon f_Z|_P \text{ is discontinuous}\} \neq \emptyset. \quad (14)$$

for every $Z \in F$. By Theorem 6, $\{P \in \mathcal{K} \colon g|_P$ is discontinuous$\} \in \mathcal{Z}_p$. So, by (14), we have

$$\{P \in \mathcal{K} \colon g|_P \text{ is discontinuous}\} \setminus Z \neq \emptyset$$

for every $Z \in F$. Thus, $\mathrm{cf}(\mathcal{B}_1, \mathcal{K}) \leq \mathrm{cf}(\mathcal{Z}_p)$

We show that $\mathrm{cf}(\mathcal{B}_1, \mathcal{K}) \geq \mathfrak{c}$. Suppose $F \subseteq \mathcal{B}_1$ and $|F| < \mathfrak{c}$. For each $f \in F$ let $A_f = \{x \in \mathbb{R} \colon$ there is an $\epsilon > 0$ such that $|\mathrm{B}_\epsilon(\langle x, f(x)\rangle) \cap f| \leq \omega\}$. Notice that $A_f$ is countable for every $f \in F$. Pick $x \in \mathbb{R} \setminus \bigcup_{f \in F} A_f$. Let $g = \chi_{\{x\}}$. Let $f \in F$ be arbitrary. Since $f \in \mathcal{B}_1$ and $x \notin A_f$, we can find as, in the proof of (c) above, a perfect set $P$ such that $x \in P$ and $f|_P$ is continuous. Clearly, $g|_P$ is not continuous. Thus,

$$\{P \in \mathcal{K} \colon g|_P \text{ is discontinuous}\} \setminus \{P \in \mathcal{K} \colon f|_P \text{ is discontinuous}\} \neq \emptyset$$

for every $f \in F$. Therefore, $\mathrm{cf}(\mathcal{B}_1, \mathcal{K}) \geq \mathfrak{c}$. By Theorem 6, $M = \{\{P \in \mathcal{K} \colon f|_P$ is discontinuous$\} \colon f \in \mathcal{B}_1\}$ is a cofinal family in $\mathcal{Z}_p$. Since $|M| = \mathfrak{c}$, we have $\mathrm{cf}(\mathcal{Z}_p) \leq \mathfrak{c}$. □

## 7 Proofs of Theorem 4 and Theorem 3

PROOF OF THEOREM 4. We first show that $\mathrm{add}(\mathbb{R}^\mathbb{R}, \mathcal{D}) \geq \omega_1$. Let $F = \{f_n\}_{n \in \omega}$ be a subset of $\mathbb{R}^\mathbb{R}$. Define $h \colon \mathbb{R} \to \mathbb{R}^\omega$ by the formula $h(x) = (f_n(x))_{n \in \omega}$. Let $j \colon \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}^\omega$ be a continuous onto function. Let $S \subseteq \mathbb{R}$ be such that $j|_S \colon S \to h[\mathbb{R}]$ is a bijection. Define $g \in \mathbb{R}^\mathbb{R}$ by $g(x) = j|_S^{-1}(h(x))$. By Blumberg's Theorem there is a $D \in \mathcal{D}$ such that $g|_D$ is continuous. We claim that $f_n|_D$ is continuous for every $n \in \omega$. By way of contradiction, assume that there is an $n \in \omega$ such that $f_n|_D$ is not continuous. It follows that $h|_D$ is also not continuous. So, there is a convergent sequence $x_n \to x$ such that $\{x_n \colon n \in \omega\} \cup \{x\} \subseteq D$ and $\lim_{n \to \infty} h(x_n) \neq h(x)$. Since $j|_S$ is continuous, we have $\lim_{n \to \infty} j_S^{-1}(h(x_n)) \neq j_S^{-1}(h(x))$. Thus, $g|_D$ is not continuous contradicting our choice of $D$. So, we have the claim. Thus, $\mathrm{add}(\mathbb{R}^\mathbb{R}, \mathcal{D}) \geq \omega_1$

We show that $\mathrm{add}(\mathbb{R}^\mathbb{R}, \mathcal{D}) \leq \omega_1$. Let $\{P_\alpha\}_{\alpha \in \omega_1}$ be a partition of $\mathbb{R}$ into disjoint everywhere of second category subsets. For each $\alpha \in \omega_1$ let $f_\alpha = \chi_{P_\alpha}$. Notice that if $D \in \mathcal{D}$, and $f_\alpha|_D$ is continuous, then $D \cap P_\alpha$ is open in $D$. Let $g \in \mathbb{R}^\mathbb{R}$ be arbitrary. Since each $P_\alpha$ is everywhere of second category, there is for each $\alpha \in \omega_1$ a $D_\alpha \in \mathcal{D}$ such that $D_\alpha \subseteq P_\alpha$ and $g|_{D_\alpha}$ is continuous (to see that such a $D_\alpha$ may be found for an everywhere second category set see [1]).

For each $\alpha \in \omega_1$ pick $\langle x_\alpha, g(x_\alpha)\rangle \in g|_{D_\alpha}$. Since $\omega_1$ is uncountable, there exist distinct $\{\alpha_\xi \colon \xi \in \omega+1\}$ such that $\lim_{n\to\infty}\langle x_{\alpha_n}, g(x_{\alpha_n})\rangle = \langle x_{\alpha_\omega}, g(x_{\alpha_\omega})\rangle$. Without loss of generality we may assume that $\{x_{\alpha_n}\}_{n\in\omega}$ is an increasing sequence. For each $n \in \omega$ pick $\delta_n > 0$ such that $\operatorname{osc}(g|_{D_{\alpha_n}\cap \mathrm{B}_{\delta_n}(x_{\alpha_n})}) < 1/n$ and $\mathrm{B}_{\delta_n}(x_{\alpha_n}) \cap \mathrm{B}_{\delta_k}(x_{\alpha_k}) = \emptyset$ for any distinct $n, k \in \omega$. Let

$$E = \left[\bigcup_{n\in\omega}(D_{\alpha_n} \cap \mathrm{B}_{\delta_n}(x_{\alpha_n}))\right] \cup \left[D_{\alpha_\omega} \setminus \bigcup_{n\in\omega}\mathrm{cl}(\mathrm{B}_{\delta_n}(x_{\alpha_n})\right].$$

Notice that $g|_E$ is continuous and $E \cap P_{\alpha_\omega}$ is not open in $P_{\alpha_\omega}$. Thus, there is no $g \in \mathbb{R}^{\mathbb{R}}$ such that, $\bigcup_{\alpha\in\omega_1}\{D \in \mathcal{D} \colon f_\alpha|_D$ is discontinuous$\}$ is contained in $\{D \in \mathcal{D} \colon g|_D$ is discontinuous$\}$. Therefore, $\operatorname{add}(\mathbb{R}^{\mathbb{R}}, \mathcal{D}) \le \omega_1$. □

To prove Theorem 3 we will need the following proposition.

**Proposition 18.** (Thm.1.2 [6]) *There is a collection of sets $\{A_\alpha \subseteq \omega_1 \colon \alpha \in \omega_2\}$ such that $|A_\alpha| = \omega_1$ for every $\alpha \in \omega_2$ and $|A_\alpha \cap A_\beta| = \omega_1$ if and only if $\alpha = \beta$.*

Proof of Theorem 3. Let $\{A_\alpha \subseteq \omega_1 \colon \alpha \in \omega_2\}$ be as in Proposition 18. Let $\{r_\alpha \colon \alpha \in \omega_2\}$ be an enumeration of $\mathbb{R}$. For each $\beta \in \omega_1$ define $f_\beta \colon \mathbb{R} \to 2$

$$f_\beta(r_\alpha) = \begin{cases} 0 & \text{if } \beta \notin A_\alpha; \\ 1 & \text{if } \beta \in A_\alpha. \end{cases}$$

By way of contradiction, assume there is a $D \in \mathcal{D}$ such that $f_\beta|_D$ is continuous for every $\beta \in \omega_1$. Pick $x \in D$ and a strictly increasing sequence $\langle x_n\rangle_{n\in\omega}$ of points in $D$ which converge to $x$. By continuity we have that for every $\beta \in \omega_1$ there is an $n_\beta \in \omega$ such that $f_\beta(x_n) = f_\beta(x)$ for all $n \ge n_\beta$. There is an $\alpha \in \omega_2$ such that $x = r_\alpha$. Since $|A_\alpha| = \omega_1$, there are $\omega_1$-many $\beta \in \omega_1$ where $f_\beta(x) = 1$. So there are $\omega_1$-many $\beta \in \omega_1$ with the property that $f_\beta(x_{n_\beta}) = f_\beta(x) = 1$. Since $\omega < \omega_1$, there is an $N \in \omega$ and a $B \in [\omega_1]^{\omega_1}$ such that $n_\beta = N$ for every $\beta \in B$. Let $x_N = r_\zeta$. Now $\zeta \ne \alpha$ but $B \subseteq A_\zeta \cap A_\alpha$, a contradiction. □

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