INROADS

Real Analysis Exchange
Vol. 32(2), 2006/2007, pp. 569–582

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ABSOLUTELY CONTINUOUS FUNCTIONS WITH VALUES IN A METRIC SPACE

Abstract

We present a general theory of absolutely continuous paths with values in metric spaces using the notion of metric derivatives. Among other results, we prove analogues of the Banach-Zarecki and Vallée Poussin theorems.

1 Introduction.

In a nice expository article, Varberg [16] outlined an elegant approach towards the theory of real-valued absolutely continuous functions. In the present note, we will be interested in maps $f : [a, b] \to (M, \rho)$, where (M, ρ) is a metric space. We will see that a significant part of the theory carries over to this (very general) situation. As we have the following:

$$\begin{array}{l}\text{every metric space } (M,\rho) \text{ can be embedded} \\ \text{into a suitable Banach space } \ell_\infty(\Gamma) \text{ for some } \Gamma\end{array} \tag{1.1}$$

(see e.g. [4, Lemma 1.1]), we could without any loss of generality work with Banach spaces only.

The main obstacle in dealing with metric spaces (or arbitrary Banach spaces) is the absence of the Radon-Nikodým property and the resulting nonexistence of derivatives. Thus, instead of the "usual" derivative, we have to

Key Words: Absolutely continuous function with values in a metric space, metric differentials, Banach-Zarecki Theorem, Vallée Poussin Theorem.
Mathematical Reviews subject classification: Primary: 26A46; Secondary: 26E20
Received by the editors November 27, 2006
Communicated by: Bernd Kirchheim
*The author was supported in part by the Israel Science Foundation.

employ the notion of a "metric derivative" (which was introduced by Kirchheim in [10]; see also [1, 5, 11]). We will need some results about this notion from [6]. Recently, there has been a lot of progress in analysis in metric spaces using these notions of metric derivatives; see e.g. [3] for a survey about rectifiability in this context, [2] for analysis of currents in metric spaces, and others.

Let (M, ρ) be a metric space, and let $f : [a, b] \to (M, \rho)$. We say that f *is absolutely continuous*, provided for each $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $[a_1, b_1], \dots, [a_k, b_k]$ is a sequence of non-overlapping intervals in $[a, b]$ with $\sum_{i=1}^{k}(b_i - a_i) < \delta$, then

$$\sum_{i=1}^{k} \rho\big(f(b_i), f(a_i)\big) < \varepsilon.$$

It easily follows that absolutely continuous functions are continuous.

This paper is organized in the following way. In the second section, we present the basic definitions and establish some auxiliary results. In the third section, we present the theory of absolutely continuous functions with values in metric spaces. For example, we prove a version of the Banach-Zarecki theorem in this context; see Theorem 3.5 (which was recently proved by L. Zajíček and the author in [8]). The current proof is different from the one in [8]; it does not use the theorem of Luzin, but rather a generalization of ideas due to Varberg [16]. Among other results, we also show a version of Vallée Poussin's theorem (see Theorem 3.16) which characterizes the situation when a composition of two absolutely continuous functions is again absolutely continuous.

2 Preliminary Results.

By m we will denote the Lebesgue measure on $\mathbb{R}$. For each function $f : [a, b] \to (M, \rho)$, and for $x \in [a, b]$, we can define the variation

$$v_f(x) = \bigvee_a^x f = \sup_D \sum_{i=0}^{n(D)-1} \rho(f(x_i), f(x_{i+1})),$$

where the supremum is taken over all partitions D of $[a, x]$ (D is a *partition of* $[a, x]$ provided $D = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and $n = n(D) = \#D - 1$). We say that f *has bounded variation*, provided $\bigvee_a^b f < \infty$. It is easy to see that every absolutely continuous function has bounded variation.

We will need the notion of the "metric derivative." Let $f : [a, b] \to (M, \rho)$. For $x \in [a, b]$, we define

$$md(f, x) := \lim_{\substack{t \to 0 \\ x+t \in [a,b]}} \frac{\rho\big(f(x+t), f(x)\big)}{|t|}.$$

Following [10], we say that f *is metrically differentiable at* x, provided $md(f, x)$ exists and

$$\rho(f(y), f(z)) - md(f, x)|y - z| = o(|x - y| + |x - z|), \quad \text{when } (y, z) \to (x, x).$$

The following is an easy consequence of [6, Theorem 2.6]:

Theorem 2.1. *Let $f : [a, b] \to (M, \rho)$ be arbitrary. Then the following hold.*

(i) If $S(f) := \{x \in [a, b] : \limsup_{t\to 0} |t|^{-1}\rho(f(x+t), f(x)) < \infty\}$, then there is $N \subset [a, b]$ with $m(N) = 0$ such that f is metrically differentiable at all $x \in S(f) \setminus N$.

(ii) If f has bounded variation, then f is metrically differentiable at almost all $x \in [a, b]$.

PROOF. Part (i) is just a restatement of [6, Theorem 2.6]. To prove part (ii), note that v_f is differentiable almost everywhere in $[a, b]$. We easily see that at each such point we have $\limsup_{t\to 0} |t|^{-1}\rho(f(x + t), f(x)) < \infty$. Thus, part (i) implies that f is metrically differentiable at almost each $x \in [a, b]$. $\square$

We will need the following simple lemma.

Lemma 2.2. *Let (M, ρ) be a metric space, $f : [c, d] \to M$, $g : [a, b] \to [c, d]$, $x \in [a, b]$ be such that $g'(x) \neq 0$ and $md(f \circ g, x)$ exists. Then $md(f, g(x))$ exists.*

PROOF. Denote $\eta = g'(x)$. By the differentiability of g at x, we have

$$g(x + h) - g(x) - \eta\, h = o(h), \quad \text{when } h \to 0.$$

Thus, we can choose $\delta > 0$ such that $g(y) \neq g(x)$ for $|x - y| < \delta$, and for each $|h| < \delta$, there exists $h' \in \mathbb{R}$ such that $g(x + h') = g(x) + \eta\, h$. It is easy to see that $h \to 0$ if and only if $h' \to 0$. We have

$$g(x + h') = g(x) + \eta\, h' + o(h') = g(x) + \eta\, h,$$

and thus $h/h' \to 1$ when $h \to 0$. Now,

$$\frac{\rho\big(f(g(x)+\eta\, h), f(g(x))\big)}{\eta\, h} = \frac{\rho\big(f(g(x+h')), f(g(x))\big)}{\eta\, h'} \cdot \frac{h'}{h} \to \frac{md(f\circ g, x)}{\eta},$$

when $h \to 0$. Thus, $md(f, g(x))$ exists. $\square$

Let (M,ρ) be a metric space, and $A \subset M$. We define the *Hausdorff measure* $\mathcal{H}^1(A)$ as $\lim_{\delta\to 0}\mathcal{H}^1_\delta(A)$, where

$$\mathcal{H}^1_\delta(A) := \inf\Big\{\sum \operatorname{diam}(A_i) : A \subset \bigcup_i A_i \text{ with } \operatorname{diam}(A_i) < \delta\ \forall i\Big\}.$$

for $\delta > 0$. It is well known (see e.g. [9]) that $\mathcal{H}^1$ is a Borel measure on M.

The following is a "metric" version of Varberg's "Fundamental Lemma" (see [16, p. 832]).

Lemma 2.3. *Let $f : [a,b] \to (M,\rho)$ be a function, and let E be the set of all $x \in [a,b]$ where $md(f,x)$ exists and satisfies $md(f,x) \le K$. Then*

$$\mathcal{H}^1(f(E)) \le K\, m^*(E), \tag{2.1}$$

where m^* is the outer Lebesgue measure.

PROOF. If E is finite or denumerable, then the condition (2.1) follows trivially. Suppose that E is not denumerable. Let $\varepsilon > 0$ be given, and let A be an open subset of $[a,b]$ such that $E \subset A$ and $m(A) \le m^*(E) + \varepsilon$. Define inductively $E_0 := \emptyset$, and

$$\begin{aligned} E_i := \{&x \in A \setminus E_{i-1} : B(x, 1/i) \subset A \text{ and} \\ &\rho(f(x+t), f(x)) \le (K+\varepsilon)|t| \text{ for } |t| < 1/i\} \quad \text{for } i \in \mathbb{N}. \end{aligned}$$

Then each E_i is Borel (see (1.1) in conjunction with e.g. [6, Lemma 2.3]). Let E_{ij} be such that $\operatorname{diam}(E_{ij}) < 1/i$, $(E_{ij})_j$ is a pairwise-disjoint collection of Borel sets for each i, and $\bigcup_j E_{ij} = E_i$. Note that $E \subset \bigcup_i E_i$. We see that $f|_{E_{ij}}$ is $(K+\varepsilon)$-Lipschitz. It easily follows (see [9, Theorem 2.10.11]) that

$$\mathcal{H}^1(f(E_{ij})) \le (K+\varepsilon)\, m(E_{ij}),$$

and thus

$$\begin{aligned} \mathcal{H}^1(f(E)) &\le \mathcal{H}^1\bigg(f\Big(\bigcup_{i,j} E_{ij}\Big)\bigg) \le (K+\varepsilon)\sum_{i,j} m(E_{ij}) \\ &\le (K+\varepsilon)\, m(A) \le (K+\varepsilon)\,(m^*(E)+\varepsilon). \end{aligned}$$

To obtain (2.1), send $\varepsilon \to 0$. $\square$

We have the following metric analogue of [16, Theorem 1].

Theorem 2.4. *Let $f : [a, b] \to (M, \rho)$ be arbitrary, and let E be any measurable set on which $md(f, \cdot)$ is finite. Then*

$$\mathcal{H}^1(f(E)) \leq \int_E md(f, x)\, dx. \tag{2.2}$$

PROOF. Using Lemma 2.3, the proof is similar to the proof of [16, Theorem 1]. Here are the details. First suppose that $md(f, x) < B$ for some $B \in \mathbb{N}$ on E. Let

$$E_{nk} = \{x \in E : k - 1 \leq 2^n \cdot md(f, x) < k\},\ k = 1, \ldots, B2^n, n = 1, \ldots.$$

Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{H}^1(f(E)) &= \mathcal{H}^1\Big(f\Big(\bigcup_k E_{nk}\Big)\Big) = \mathcal{H}^1\Big(\bigcup_k f(E_{nk})\Big) \leq \sum_k \mathcal{H}^1(f(E_{nk})) \\ &\leq \sum_k \frac{k}{2^n} m(E_{nk}) = \sum_k \frac{k-1}{2^n} m(E_{nk}) + \frac{1}{2^n} \sum_k m(E_{nk}),\end{aligned}$$

where the second inequality follows from Lemma 2.3. Therefore,

$$\mathcal{H}^1(f(E)) \leq \lim_{n\to\infty} \Big[\sum_k \frac{k-1}{2^n} m(E_{nk}) + \frac{1}{2^n} \sum_k m(E_{nk})\Big] = \int_E md(f, x)\, dx.$$

Now, if $md(f, x)$ is not bounded on E, then let

$$A_k = \{x \in E : k - 1 \leq md(f, x) < k\}, \quad k = 1, \ldots,$$

$$\begin{aligned}\mathcal{H}^1(f(E)) &= \mathcal{H}^1\Big(f\Big(\bigcup_k A_k\Big)\Big) = \mathcal{H}^1\Big(\bigcup_k f(A_k)\Big) \leq \sum_k \mathcal{H}^1(f(A_k)) \\ &\leq \sum_k \int_{A_k} md(f, x)\, dx = \int_E md(f, x)\, dx.\end{aligned}$$

□

3 Absolutely Continuous Functions.

We say that $f : [a, b] \to (M, \rho)$ has (*Luzin's*) *property* (N) provided

$$\mathcal{H}^1(f(B)) = 0 \;\text{ whenever }\; B \subset [0, 1] \text{ with } m(B) = 0. \tag{3.1}$$

The proof of the following theorem is standard (see e.g. [15] and the proof of Theorem in [8]).

Theorem 3.1. *Let $f : [a, b] \to (M, \rho)$ be absolutely continuous. Then f has the property (N).*

The previous theorem has the following corollary.

Corollary 3.2. *An absolutely continuous function $f : [a, b] \to (M, \rho)$ maps measurable subsets of $[a, b]$ onto $\mathcal{H}^1$-measurable subsets of M.*

We will need the following theorem (see [16, Theorem 14] for the real-valued case).

Theorem 3.3. *Let $f : [a, b] \to (M, \rho)$ be continuous and has bounded variation. Then $md(f, \cdot)$ exists almost everywhere in $[a, b]$, is integrable, and*

$$\int_a^b md(f, x)\, dx \leq \bigvee_a^b f. \tag{3.2}$$

Further, if f has the property (N), then the equality holds.

PROOF. Denote $A = [a, b]$. Theorem 2.1 (ii) implies that $md(f, \cdot)$ exists for all $x \in A \setminus N$ with $m(N) = 0$. The area formula [6, Theorem 2.12] together with [9, Theorem 2.10.13] implies that

$$\bigvee_a^b f = \int N(f|_A, y)\, d\mathcal{H}^1 y \geq \int_{f(A\setminus N)} N(f|_A, y)\, d\mathcal{H}^1 y = \int_{A\setminus N} md(f, x)\, dx, \tag{3.3}$$

and thus (3.2) holds. Here, $N(f|_A, y)$ is the number of $x \in A$ such that $f(x) = y$.

If f has property (N), then clearly $\mathcal{H}^1(f(N)) = 0$, and we get equality instead of an inequality in (3.2). To see that, we have (again using [6, Theorem 2.12] together with [9, Theorem 2.10.13])

$$\begin{aligned}\bigvee_a^b f &= \int N(f|_A, y)\, d\mathcal{H}^1 y = \int_{f(A)\setminus f(N)} N(f|_A, y)\, d\mathcal{H}^1 y \\ &\leq \int N(f|_{A\setminus N}, y)\, d\mathcal{H}^1 y = \int_{A\setminus N} md(f, x)\, dx \leq \int_A md(f, x)\, dx. \qquad \square\end{aligned}$$

Remark 3.4. If f from the previous theorem is absolutely continuous, then we have equality in (3.2) (as f has bounded variation, and it also satisfies property (N) by Theorem 3.1). It is easy to see that f is absolutely continuous if and only if v_f is. If that is the case, then it follows that

$$\int_c^d md(f,x)\,dx = v_f(d) - v_f(c) = \int_c^d v_f'(x)\,dx,$$

for each interval $[c,d] \subset [a,b]$. It is easy to see that $md(f,x) \le v_f'(x)$ whenever $v_f'(x)$ and $md(f,x)$ exist. Thus, if f is absolutely continuous, then $md(f,x) = v_f'(x)$ almost everywhere.

The following version of the Banach-Zarecki theorem (see e.g. [13] or [16] for the real-valued statement) was proved by L. Zajíček and the author in [8] using a result of Luzin [12] and a theorem about the Banach indicatrix function from [9]. Here, we present a different proof, which is in the spirit of Varberg's approach (see [16, Theorem 3]).

Theorem 3.5. *Let $f : [a,b] \to (M,\rho)$ be a function. Then f is absolutely continuous if and only if f is continuous, has bounded variation, and has the property (N).*

PROOF. If f is absolutely continuous, then a standard argument shows that f is continuous, and f has bounded variation. Theorem 3.1 shows that f also has the property (N).

To prove the converse, let $[a_i,b_i], i = 1,\dots,k$, be non-overlapping intervals in $[a,b]$, and let $E_i = \{x \in [a_i,b_i] : md(f,x) \text{ exists}\}$. Since by Theorem 2.1 (ii), we have that $m([a_i,b_i] \setminus E_i) = 0$, and since f has the property (N), we obtain $\mathcal{H}^1\big(f(E_i)\big) = \mathcal{H}^1\big(f([a_i,b_i])\big)$. Therefore,

$$\begin{aligned}\sum_{i=1}^k \rho(f(b_i),f(a_i)) &\le \sum_{i=1}^k \mathcal{H}^1(f([a_i,b_i])) = \sum_{i=1}^k \mathcal{H}^1(f(E_i)) \\ &\le \sum_{i=1}^k \int_{E_i} md(f,x)\,dx = \sum_{i=1}^k \int_{a_i}^{b_i} md(f,x)\,dx,\end{aligned} \tag{3.4}$$

where the first inequality follows from [9, Corollary 2.10.12] and the second from Theorem 2.4. It is easy to see that the rightmost term in (3.4) goes to 0, as $\sum_{i=1}^k (b_i - a_i) \to 0$. This last property follows from the fact that $md(f,\cdot)$ is integrable by Theorem 3.3, and from a well-known property of the integral. ☐

The proofs of the next two theorems are analogous to the previous one (cf. [16, Theorems 4, 5]).

Theorem 3.6. *If $f : [a, b] \to (M, \rho)$ is continuous, $md(f, \cdot)$ exists for all but finite or denumerable set of points, and $md(f, \cdot)$ is integrable on $[a, b]$, then f is absolutely continuous on $[a, b]$.*

Theorem 3.7. *If $f : [a, b] \to (M, \rho)$ is continuous, $md(f, \cdot)$ exists almost everywhere and is integrable on $[a, b]$, and if f has the property (N), then f is absolutely continuous on $[a, b]$.*

The next theorem is a consequence of Theorem 3.6; see [13, p. 266] or [16, Theorem 6] for the real-valued version.

Theorem 3.8. *If $md(f, x)$ exists for all $x \in [a, b]$, and if $md(f, x)$ is integrable, then f is absolutely continuous on $[a, b]$.*

The following theorem is an analogue of [17, Theorem 30.12].

Theorem 3.9. *Let $f : [a, b] \to (M, \rho)$ be continuous. Assume that*

(i) there exists a closed and denumerable $E \subset [a, b]$ such that f is absolutely continuous on each closed interval in $[a, b] \setminus E$, and

(ii) $\int_a^b md(f, x)\, dx < \infty$.

Then f is absolutely continuous on $[a, b]$.

Proof. We will prove that f satisfies the assumptions of Theorem 3.7. By Theorem 2.1 (ii), we have that $md(f, x)$ exists almost everywhere in $[a, b]$, and the integrability of $md(f, \cdot)$ follows from (ii). Let (a_i, b_i), $(i \in \mathcal{I} \subset \mathbb{N})$ be the intervals contiguous to E in $[a, b]$. By Theorem 3.1 and condition (i), it follows that $f|_{[a_i,b_i]}$ has property (N) for each $i \in \mathcal{I}$. As E is denumerable, we easily obtain that f has property (N). Thus, Theorem 3.7 applies, and f is absolutely continuous on $[a, b]$. $\square$

We have the following (see also [13, p. 246] or [16, Theorem 9]):

Theorem 3.10. *Let $f : [a, b] \to (M, \rho)$ be an absolutely continuous function, and $md(f, x) = 0$ almost everywhere on $[a, b]$. Then f is a constant function.*

Proof. Theorem 3.3 implies that $\bigvee_a^b f = 0$. The only functions with zero variation are the constant ones. $\square$

The following theorem is an analogue of [16, Theorem 13].

Theorem 3.11. *Let $f : [a,b] \to (M,\rho)$ be one-to-one and have bounded variation, let A be any measurable set, and let E be the set of all $x \in A$ where $md(f,x)$ exists. Then*

$$\int_A md(f,x)\,dx = \mathcal{H}^1(f(E)) \le \mathcal{H}^1(f(A)). \tag{3.5}$$

The equality holds provided f is absolutely continuous.

PROOF. By (1.1), we can assume that M is a Banach space. First, assume that f is absolutely continuous. Then [6, Theorem 2.12] shows that

$$\begin{aligned}\int_A md(f,x)\,dx = \int_E md(f,x)\,dx &= \int N(f|_E, y)\,d\mathcal{H}^1 y\\ &= \mathcal{H}^1(f(E)) = \mathcal{H}^1(f(A)),\end{aligned}$$

as $m(A \setminus E) = 0$ by Theorem 2.1 (ii), and f has property (N) by Theorem 3.1.

Now, we will prove the equality from (3.5) for f, which are one-to-one with bounded variation (note that the inequality in (3.5) holds trivially). Define

$$A'_n := \{x \in E : \rho(f(x+t), f(x)) \le n|t| \text{ for } |t| < 1/n\},$$

and $A_n := A'_n \setminus_{j<n} A'_j$. Then each A_n is measurable (see e.g. [6, Lemma 2.3] together with (1.1)) and $A = \bigcup_n A_n$. Further, write $A_n = \bigcup_k A_{nk}$ so that $(A_{nk})_k$ is a pairwise-disjoint sequence of measurable sets with $\operatorname{diam}(A_{nk}) < 1/n$ for each k. Now extend each $f|_{A_{nk}}$ (which is n-Lipschitz by the definition of A_{nk}) to an n-Lipschitz function on $[a,b]$ (first extend $f|_{A_{nk}}$ to $\overline{A_{nk}}$ by continuity, and then linearly and continuously on the intervals contiguous to $\overline{A_{nk}}$; it is easy to see that the resulting function is n-Lipschitz); call the extensions f_{nk}. Then

$$\begin{aligned}\int_A md(f,x)\,dx = \sum_{n,k} \int_{A_{nk}} md(f_{nk},x)\,dx &= \sum_{n,k} \mathcal{H}^1(f_{nk}(A_{nk}))\\ &= \mathcal{H}^1\Big(\bigcup_{n,k} f(A_{nk})\Big) = \mathcal{H}^1(f(E)),\end{aligned}$$

where the first equality follows from the fact that almost all points of A_{nk} are points of density, and $md(f,x) = md(f_{nk},x)$ at all such points (see e.g. [6, Lemma 2.1]). The second equality follows by the previous paragraph. The third equality follows from the fact that f is one-to-one. $\square$

For the real-valued version of the following theorem, see [16, Theorem 15].

Theorem 3.12. *If $f : [a,b] \to (M,\rho)$ has bounded variation, and A is a measurable subset of $[a,b]$, then $m^*(v_f(A)) \geq \int_A md(f,x)\,dx$. The equality holds if f is absolutely continuous.*

PROOF. Let E be the subset of A where $md(f,x)$ exists. Thus,

$$\begin{aligned} m^*(v_f(A)) \geq m^*(v_f(E)) &= \int_E v_f'(x)\,dx \\ &\geq \int_E md(f,x)\,dx = \int_A md(f,x)\,dx, \end{aligned} \tag{3.6}$$

where the first equality follows from [16, Theorem 13], and the second inequality from the fact that $|v_f(y) - v_f(x)| \geq \rho(f(y), f(x))$ for all $x, y \in [a,b]$. Note that we can write equalities instead of inequalities in (3.6) provided f is absolutely continuous (as in that case Remark 3.4 implies that v_f is absolutely continuous, and that $md(f,x) = v_f'(x)$ for almost every $x \in [a,b]$). $\square$

Note that applying Lemma 2.3 with $K = 0$ yields the following version of Sard's theorem (cf. [7, Lemma 2.2]).

Theorem 3.13. *Let $f : [a,b] \to (M,\rho)$ and $E = \{x \in [a,b] : md(f,x) = 0\}$. Then $\mathcal{H}^1(f(E)) = 0$.*

The following is an analogue of [16, Theorem 18].

Theorem 3.14. *Let $f : [a,b] \to (M,\rho)$ be continuous and of bounded variation. Let N be any set such that $\mathcal{H}^1(f(N)) = 0$. Then, $m(v_f(N)) = 0$.*

PROOF. Denote $A = [a,b]$, and let $\varepsilon > 0$. Let K be a compact subset of $f(A)$ such that $K \cap f(N) = \emptyset$ and

$$\int_K N(f|_A, y)\,d\mathcal{H}^1 y \geq \int N(f|_A, y)\,d\mathcal{H}^1 y - \varepsilon;$$

existence of such a set K follows from the regularity of $\mathcal{H}^1$ (see e.g. [9, §2.10.48]). Then $H = f^{-1}(K)$ satisfies $H \cap N = \emptyset$, and $v_f(H) \cap v_f(N)$ is at most denumerable. Now by [9, Theorem 2.10.13], we have

$$\bigvee_a^b f = \int N(f|_A, y)\,d\mathcal{H}^1 y,$$

and thus

$$\begin{aligned} m^*(v_f(H)) &\geq \int_H md(f,x)\,dx = \int N(f|_H, y)\,d\mathcal{H}^1 y \\ &= \int_K N(f|_A, y)\,d\mathcal{H}^1 y \geq \int N(f|_A, y)\,d\mathcal{H}^1 y - \varepsilon \\ &= \bigvee_a^b f - \varepsilon, \end{aligned}$$

where the first inequality follows from Theorem 3.12, and the first equality from [6, Theorem 2.12]. It follows that $m^*(v_f(N)) \leq \varepsilon$, and as $\varepsilon > 0$ was arbitrary, we have that $m(v_f(N)) = 0$. $\square$

The following is an analogue of [16, Theorem 19].

Theorem 3.15. *Let $f : [a,b] \to (M,\rho)$ be continuous, have bounded variation, and let E be a measurable set for which $\mathcal{H}^1(f(E)) = 0$. Then, $md(f,x) = 0$ for almost all $x \in E$.*

Proof. We have

$$\int_E md(f,x) \leq m(v_f(E)) \leq 0,$$

where the first inequality follows from Theorem 3.12, and the second from Theorem 3.14. $\square$

The following theorem was established by Vallée Poussin [14] for real valued functions.

Theorem 3.16. *Let (M,ρ) be a metric space and $f : [c,d] \to M$, $g : [a,b] \to [c,d]$ be absolutely continuous functions. Then, $f \circ g$ is absolutely continuous if and only if $md(f,g(x)) \cdot g'(x)$ is integrable.*

Remark 3.17. The expression $h(x) = md(f,g(x)) \cdot g'(x)$ is interpreted in the following sense (usual in the measure-theory): $h(x) = 0$ provided $g'(x) = 0$ (even when $md(f,g(x))$ does not exist).

Proof. Suppose that $f \circ g$ is absolutely continuous. Then Theorem 3.3 implies that $md(f \circ g, x)$ is integrable. Let A be the set of all points x of $[a,b]$ where $g(x) \neq 0$ and $md(f \circ g, x)$ exists. Lemma 2.2 shows that for every $x \in A$, the metric derivative $md(f,g(x))$ exists. Thus, if $x \in A$, then we have that $md(f \circ g, x) = md(f,g(x)) \cdot g'(x)$ by a chain rule for metric derivatives (see e.g. [7, Lemma 2.4 (ii)]). Let $N := \{x \in [a,b] : g'(x) = 0\}$. Then $m([a,b] \setminus (A \cup N)) = 0$ (by Theorem 2.1 (ii)), and thus $md(f, g(x)) \cdot g'(x)$ is integrable on $[a,b]$.

Suppose that $md(f, g(x)) \cdot g'(x)$ is integrable. It is easily seen that $f \circ g$ has property (N) (as it is stable under compositions), and thus by Theorem 3.5, it is enough to show that $f \circ g$ has bounded variation. Let

$$A := \{x \in [a,b] : md(f, g(x)) \text{ exists and } g'(x) \neq 0\},$$

$B_1 := \{x \in [a,b] : g'(x) = 0\}$, and $B_2 := [a,b] \setminus (A \cup B_1)$. Note that for almost every $x \in [a,b]$, we have that either $g'(x) = 0$ or $g'(x) \neq 0$ and $md(f, g(x))$ exists (in the second case, we also have that $md(f \circ g, x)$ exists and is equal to $md(f, g(x)) \cdot g'(x)$ by the chain rule for metric derivatives [7, Lemma 2.4 (ii)]). Thus, it follows that $m(B_2) = 0$. Let $B := B_1 \cup B_2$. By Theorem 3.13, and because $f \circ g$ has property (N), we have that $\mathcal{H}^1((f \circ g)(B)) = 0$. We obtain

$$\begin{aligned} \bigvee_a^b (f \circ g) &= \int_M N(f \circ g, y)\, d\mathcal{H}^1 y = \int_{M \setminus (f \circ g)(B)} N(f \circ g, y)\, d\mathcal{H}^1 y \\ &= \int_{\{x \in [a,b] : f \circ g(x) \notin B\}} md(f \circ g, x)\, dx \\ &\leq \int_A md(f, g(x)) \cdot g'(x)\, dx < \infty, \end{aligned}$$

where the first equality follows from [9, Theorem 2.10.13], and the third by [6, Theorem 2.12]. We have that $f \circ g$ has finite variation, and thus we can apply Theorem 3.5. □

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