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POINTS OF WEAK SYMMETRY

Abstract

We show that every set of reals is a set of points of weak symmetry for some function $f\colon \mathbb{R} \to \mathbb{N}$.

1 Preface.

The weakest notion of continuity is probably the following definition:

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is *weakly* (or *peripherally*) *continuous at* x if $\lim_{n\to\infty} f(x_n) = f(x)$ for <u>some</u> sequence $x_n \to x$.

The following theorem characterizes the sets of points of weak continuity:

Theorem 1.2 (Chapter 2 of [4] and Theorem 4 of [3])**.** *Any function has only countably many points of weak discontinuity, and any countable set is the set of points of weak discontinuity for some function.*

2 Notation and Definitions.

Basic notion for our investigations is the following definition (see, for example, [1], [2] or [3]):

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Definition 2.1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *weakly symmetrically continuous* at a point x if there exists a sequence $\langle h_n \rangle$ of positive numbers converging to 0 such that

$$\lim_{n\to\infty} f(x+h_n) - f(x-h_n) = 0.$$

For $f\colon \mathbb{R} \to \mathbb{R}$, let $S(f)$ denote the set of all points at which f is not weakly symmetrically continuous.

Definition 2.2. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *weakly symmetric* (also called *Schwartz symmetric*) at a point x if there exists a sequence $\langle h_n \rangle$ of positive numbers converging to 0 such that

$$\lim_{n\to\infty} f(x+h_n) + f(x-h_n) - 2\cdot f(x) = 0.$$

Analogously, for $f\colon \mathbb{R} \to \mathbb{R}$, let $T(f)$ be the set of all $x \in \mathbb{R}$ such that f is not weakly symmetric at x, that is,

$$T(f) = \{x \in \mathbb{R}\colon f \text{ is not weakly symmetric at } x\} =$$

$$\{x \in \mathbb{R}\colon \text{there is no sequence } h_n \searrow 0 \text{ with} \lim_{n\to\infty} f(x+h_n)+f(x-h_n)-2f(x) = 0\}.$$

Let us formulate a "template" of general problems for characterizing the set $S(f)$, namely:

Problem 2.3. Suppose that $Y \subseteq \mathbb{R}$ is a fixed set. Find a characterization of the collection of all possible sets of the form $S(f)$ for any $f\colon \mathbb{R} \to Y$.

Notice that for some special cases of Y the answer is known, while for some others it is still an open problem. For example, we have:

Theorem 2.4 (M. Szyszkowski, [3])**.** *Any set $A \subset \mathbb{R}$ is the set of points of weak symmetry for some function $f : \mathbb{R} \to \mathbb{N}$.*

However, such a problem is still open for the case $Y = n$, where $n \geq 4$. Let us formulate a little stronger definition than Definition 2.2:

Definition 2.5. A function $f : \mathbb{R} \to \mathbb{R}$ is **-weakly symmetric* at $x \in \mathbb{R}$ if there exists a sequence $\langle h_n \rangle$ of positive numbers converging to 0 such that $\forall_n\ f(x-h_n) = f(x+h_n) = f(x)$.

Analogously, for $f\colon \mathbb{R} \to \mathbb{R}$, let $T^*(f)$ be the set of all $x \in \mathbb{R}$ such that f is not *-weakly symmetric at x. Obviously, we have $T(f) \subseteq T^*(f)$. Notice that there is no provable inclusion between $T(f)$ and $S(f)$. The aim of this paper is to prove an analogous result to Theorem 2.4 in case of $T(f)$ instead of $S(f)$. Let us notice that the case of $T^*(f)$ is still open; i.e., we are unable to solve the following:

Problem 2.6. Is every set $A \subset \mathbb{R}$ of the form $T^*(f)$ for some function from $\mathbb{R}$ to $\mathbb{N}$?

3 Main Result.

Let us formulate the main theorem which solves Problem 6 from [2]:

Theorem 3.1. *For every set $A \subseteq \mathbb{R}$, there exists a function $f \colon \mathbb{R} \to \omega$ such that $T(f) = A$.*

We will need the following notion:

Definition 3.2. By a *four-points-block* with a center $r \in \mathbb{R}$, we mean a set $\mathcal{B}(r, \eta, \delta) = \{r - \eta, r - \delta, r + \delta, r + \eta\}$, where $\eta, \delta > 0$ are arbitrary and $\eta \neq \delta$.

Notice that a *center* of a fixed four-points-block is determined uniquely, that is, $\mathcal{B}(r_1, \eta_1, \delta_1) = \mathcal{B}(r_2, \eta_2, \delta_2) \Rightarrow r_1 = r_2$. Let us start with a lemma which is a strengthening of Lemma 8 from [3].

Lemma 3.3. *Suppose that $A \subseteq \mathbb{R}$ is any set, and let $\{a_\mu \colon \mu < \kappa\}$ be an enumeration without repetitions of its elements, where $\kappa = |A|$. Then for each $\mu < \kappa$, there exist a system of sequences:*

$$(x^n_\mu)_{n\in\omega}; \quad (y^n_\mu)_{n\in\omega};$$

such that:

1. $\forall_{\mu<\xi<\kappa}\ \{x^n_\mu, y^n_\mu \colon n \in \omega\} \cap \{x^n_\xi, y^n_\xi \colon n \in \omega\} = \emptyset$.
2. $\forall_{\mu<\kappa}\forall_{n\in\omega}\ \frac{x^n_\mu + y^n_\mu}{2} = a_\mu$ *and* $x^n_\mu \nearrow a_\mu$.
3. *The set* $X = \{x^n_\mu, y^n_\mu \colon\ n \in \omega \wedge \mu < \kappa\}$ *contains no four-points-block with a center from* $\mathbb{R} \setminus A$.
4. $\forall_{\mu<\kappa}\ \{x^n_\mu, y^n_\mu \colon n \in \omega\} \cap \{a_\xi \colon \xi < \mu\} = \emptyset$.

PROOF. We will construct the sequences

$$(x^n_\mu)_{n\in\omega}; \quad (y^n_\mu)_{n\in\omega};$$

by a transfinite induction. So, suppose that we are in the stage $\zeta < \kappa$, and we have constructed sequences:

$$(x^n_\mu)_{n\in\omega}; \quad (y^n_\mu)_{n\in\omega};$$

for $\mu < \zeta$. Denote $X^*_\zeta = \{x^n_\mu, y^n_\mu \colon \mu < \zeta \wedge n \in \omega\}$. We are looking for a sequence $(x^n_\zeta)_{n\in\omega}$; $(y^n_\zeta)_{n\in\omega}$ of the form:

$$x_\zeta^n = a_\zeta - \tfrac{d_\zeta}{2^n} \text{ and } y_\zeta^n = a_\zeta + \tfrac{d_\zeta}{2^n},$$

where $d_\zeta > 0$ is a positive real number which fulfills the following conditions.

1. $d_\zeta \neq \pm 2^n \cdot (x - y - z + a_\zeta)$ for $x, y, z \in X_\zeta^*$ and $n \in \omega$.
2. $d_\zeta \neq \frac{x-y}{\pm\frac{1}{2^n} \pm \frac{1}{2^m}}$ for $x, y \in X_\zeta^*$ and $n, m \in \omega$, $n \neq m$.
3. $d_\zeta \neq \frac{x-y}{\pm\frac{2}{2^n}}$ for $x, y \in X_\zeta^*$ and $n \in \omega$.
4. $d_\zeta \neq \frac{x+y-2a_\zeta}{\pm\frac{1}{2^n} \pm \frac{1}{2^m}}$ for $x, y \in X_\zeta^*$ and $n, m \in \omega$, $n \neq m$.
5. $d_\zeta \neq \frac{a_\zeta - x}{\pm\frac{1}{2^n} \pm \frac{1}{2^m} \pm \frac{1}{2^l}}$ for $x \in X_\zeta^*$ and $m, n, l \in \omega$ and $\pm\frac{1}{2^n} \pm \frac{1}{2^m} \pm \frac{1}{2^l} \neq 0$.
6. $d_\zeta \neq \pm 2^n \cdot (x - a_\zeta)$ for $x \in X_\zeta^*$ and $n \in \omega$.
7. $d_\zeta \neq \pm 2^n \cdot (a_\zeta - a_\mu)$ for $\mu < \zeta$ and $n \in \omega$.

Notice that all these conditions are fulfilled if we simply assume that $d_\zeta \notin \operatorname{span}_{\mathbb{Q}}(X_\zeta^* \cup \{a_\mu : \mu \leq \zeta\})$ (linear space spanned on $X_\zeta^* \cup \{a_\mu : \mu \leq \zeta\}$). Define $X = \{x_\mu^n, y_\mu^n : \mu < \kappa \wedge n \in \omega\}$.

For brevity, denote: $X_\mu = \{x_\mu^n, y_\mu^n : n \in \omega\}$ for any $\mu < \kappa$. We will show that there is no four-points-block $\mathcal{B} \subseteq X$ with center from $\mathbb{R} \setminus A$. So suppose that $\mathcal{B} \subseteq X$ is a four-points-block. We will show that there exists $\mu < \kappa$ such that $\mathcal{B} \subseteq X_\mu$. By way of contradiction, consider the following cases:
CASE 1: There exists $\zeta < \kappa$ such that $|\mathcal{B} \cap X_\zeta| = 1$ and $|\mathcal{B} \cap X_\zeta^*| = 3$.

So, let $n \in \omega$ be a natural number such that $\mathcal{B} \cap X_\zeta = \{a_\zeta \pm \frac{d_\zeta}{2^n}\}$. Let $x, y, z \in \mathcal{B} \cap X_\zeta^*$ be distinct elements. Then we have (after possibly exchanging x and y): $x - y = z - (a_\zeta \pm \frac{d_\zeta}{2^n})$, and hence $d_\zeta = \pm 2^n \cdot (-x + y + z - a_\zeta)$. By Assumption 1, this is impossible.
CASE 2: There exists $\zeta < \kappa$ such that $|\mathcal{B} \cap X_\zeta| = 2$ and $|\mathcal{B} \cap X_\zeta^*| = 2$.

So let $n, m \in \omega$ be natural numbers such that $\mathcal{B} \cap X_\zeta = \{a_\zeta \pm \frac{d_\zeta}{2^n}, a_\zeta \pm \frac{d_\zeta}{2^m}\}$. Let $x, y \in \mathcal{B} \cap X_\zeta^*$ be distinct elements. Then we have two cases:

Case (a) $x - (a_\zeta \pm \frac{d_\zeta}{2^n}) = y - (a_\zeta \pm \frac{d_\zeta}{2^m})$. If $m \neq n$, then $d_\zeta = \frac{x-y}{\pm\frac{1}{2^n} \mp \frac{1}{2^m}}$, which is impossible by Assumption 2, and if $n = m$, then (after possibly exchanging x and y) $x - (a_\zeta - \frac{d_\zeta}{2^n}) = y - (a_\zeta + \frac{d_\zeta}{2^n})$ so $d_\zeta = \frac{x-y}{-\frac{2}{2^n}}$ which is impossible by Assumption 3.

Case (b) $x-(a_\zeta \pm \frac{d_\zeta}{2^n}) = (a_\zeta \pm \frac{d_\zeta}{2^m})-y$. If $n \neq m$, then $d_\zeta = \frac{x+y-2a_\zeta}{\pm\frac{1}{2^m}\pm\frac{1}{2^n}}$ which is impossible by Assumption 4. If $m = n$, then $x-(a_\zeta \pm \frac{d_\zeta}{2^n}) = (a_\zeta \mp \frac{d_\zeta}{2^n})-y$ which is impossible because four-points-block $\mathcal{B}$ would have a center at $a_\zeta \in A$.

CASE 3: There exists $\zeta < \kappa$ such that $|\mathcal{B} \cap X_\zeta| = 3$ and $|\mathcal{B} \cap X_\zeta^*| = 1$.

So, let $\mathcal{B} \cap X_\zeta = \{a_\zeta \pm \frac{d_\zeta}{2^n}, a_\zeta \pm \frac{d_\zeta}{2^m}, a_\zeta \pm \frac{d_\zeta}{2^l}\}$, and let $\mathcal{B} \cap X_\zeta^* = \{x\}$. Then we have $(a_\zeta \pm \frac{d_\zeta}{2^n}) - (a_\zeta \pm \frac{d_\zeta}{2^m}) = (a_\zeta \pm \frac{d_\zeta}{2^l}) - x$. Hence, $d_\zeta = \frac{a_\zeta - x}{\pm\frac{1}{2^n}\pm\frac{1}{2^m}\mp\frac{1}{2^l}}$, which is a contradiction by Assumption 5. In this way, we conclude that there exists $\mu < \kappa$ such that $\mathcal{B} \subseteq X_\mu$ for some $\mu < \kappa$. This is, however, possible only in the case where $\mathcal{B} = \{a_\mu - \frac{d_\mu}{2^n}, a_\mu - \frac{d_\mu}{2^m}, a_\mu + \frac{d_\mu}{2^m}, a_\mu + \frac{d_\mu}{2^n}\}$. Hence, $\mathcal{B}$ has the center at the point a_μ.

This shows that condition 3 from Lemma 3.3 is satisfied. Condition 1 from the Lemma is satisfied by Assumption 6, and Condition 4 from the Lemma is satisfied by Assumption 7. □

PROOF OF THEOREM 3.1. It will be simpler to construct a function f with $T(f) = \mathbb{R} \setminus A$ (simply switch A with $\mathbb{R} \setminus A$ to get f as in the Theorem). Let $h_1 \colon \mathbb{R} \to \omega$ be a function such that $T(h_1) = \mathbb{R}$ which exists by the Corollary 1.2 from [1][1]. Define $h \colon \mathbb{R} \to \omega$ by $h(x) = 5 \cdot h_1(x)$. We also have $T(h) = \mathbb{R}$.

Let $\{a_\mu \colon \mu < \kappa = |A|\}$ be an enumeration of A, and the set

$$X = \{x_\mu^n, y_\mu^n \colon \mu < \kappa, n \in \omega\}$$

is as in Lemma 3.3. We will also use the auxiliary sets X_ζ and X_ζ^* from the proof of Lemma 3.3. Let us define a function $g \colon X \to \omega$ by induction:

Suppose that we have already constructed functions $g_\mu \colon X_\mu \to \omega$ for $\mu < \zeta$, where $\zeta < \kappa$. We want to define $g_\zeta \colon X_\zeta \to \omega$. Consider the following cases:
CASE 1: $a_\zeta \notin X_\zeta^*$

Then define $g_\zeta(x_n^\zeta) = h(a_\zeta) - 1$ and $g_\zeta(y_n^\zeta) = h(a_\zeta) + 1$.
CASE 2: $a_\zeta \in X_\zeta^*$

Then $a_\zeta \in X_\xi$ for some $\xi < \zeta$. Define $g_\zeta(x_n^\zeta) = g_\zeta(y_n^\zeta) = g_\xi(a_\zeta)$. In such a way, we have defined the function g. Now define the final function $f \colon \mathbb{R} \to \omega$:

$$f(x) = \begin{cases} h(x) & \text{if } x \in \mathbb{R} \setminus X \\ g(x) & \text{if } x \in X. \end{cases}$$

[1]To obtain such a function just put $h_1(x) = 3^{f(x)}$, where f is a function from [1].

We will check that such defined function f has the property $T(f) = \mathbb{R} \setminus A$. It is easy to see that if $a \in A$, then $a \in \mathbb{R} \setminus T(f)$. Indeed, there is $\zeta < \kappa$ such that $a_\zeta = a$. Hence, by the construction of the function g, $f(x_n^\zeta) = g(x_n^\zeta) = h(a_\zeta) - 1$ and $f(y_n^\zeta) = g(y_n^\zeta) = h(a_\zeta) + 1$, or $f(x_n^\zeta) = g(x_n^\zeta) = g(a_\zeta)$ and $f(y_n^\zeta) = g(y_n^\zeta) = g(a_\zeta)$. In both cases, $f(x_n^\zeta) + f(y_n^\zeta) - 2f(a_\zeta) = 0$, and hence $a \in \mathbb{R} \setminus T(f)$.

On the other hand, suppose that $b \in \mathbb{R} \setminus A$ and let $h_n \searrow 0$ be any sequence. By way of contradiction, suppose that

$$\lim_{n\to\infty} |f(b+h_n) + f(b-h_n) - 2f(b)| \to 0.$$

At first, observe that since $f(b+h_n) + f(b-h_n) - 2f(b) \in \mathbb{Z}$, we can assume without loss of generality that $\forall_{n\in\omega} f(b+h_n) + f(b-h_n) - 2f(b) = 0$.

We have $f(b+h_n) = 5k_n + s_n$; $f(b-h_n) = 5k'_n + s'_n$ for some $k_n, k'_n \in \omega \setminus \{0\}$ and $s_n, s'_n \in \{-1, 0, 1\}$. Also, we have $f(b) = 5k + s$ for some $k \in \omega \setminus \{0\}$ and $s \in \{-1, 0, 1\}$. Then, we have $5k_n + 5k'_n - 2 \cdot 5k + s_n + s'_n - 2 \cdot s = 0$, and therefore $s_n + s'_n = 2 \cdot s$.

We will verify that this last equation cannot be satisfied.

If $s \in \{-1, 1\}$, then $s_n = s'_n = s$, but this shows that $\forall_{n\in\omega}\ b - h_n, b + h_n \in X$, which is impossible by Condition 3 of Lemma 3.3.

If $s = 0$, then $s_n = s'_n = 0$ or $s_n \cdot s'_n = -1$.

By the same argument as above, we have that $\exists_{M\in\omega}\forall_{n>M}\ s_n = s'_n = 0$, which is impossible, since $b \in T(h) = \mathbb{R}$. $\square$

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