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THRESHOLD AND HAUSDORFF SPECTRUM OF DISCONTINUOUS MEASURES

Abstract

Let m be a finite Borel measure on $[0,1]^d$. Consider the L^q-spectrum of m: $\tau_m(q) = \liminf_{n\to\infty} -n^{-1}\log_b \sum_{Q\in\mathcal{G}_n,\ m(Q)\neq 0} m(Q)^q$, where $\mathcal{G}_n$ is the set of b-adic cubes of generation n. Let $q_\tau = \inf\{q : \ \tau_m(q) = 0\}$ and $H_\tau = \tau_m'(q_\tau^-)$. When m is a mono-dimensional continuous measure of information dimension D, $(q_\tau, H_\tau) = (1, D)$. When m is purely discontinuous, its information dimension is $D = 0$, but the non-trivial pair (q_τ, H_τ) may contain relevant information on the distribution of m. The connection between (q_τ, H_τ) and the large deviation spectrum of m is studied in a companion paper. This paper shows that when a discontinuous measure m possesses self-similarity properties, the pair (q_τ, H_τ) may store the main multifractal properties of m, in particular the Hausdorff spectrum. This is observed thanks to a threshold performed on m.

1 Introduction and Statements of Results.

In a companion paper [5], we introduced new information parameters associated with any positive Borel measure m on $[0,1]^d$. Let us recall their definitions. Let $b \geq 2$ be an integer and let $\mathcal{G}_n$ be the partition of $[0,1]^d$ into b-adic

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boxes $\prod_{i=1}^{d}[b^{-n}k_i, b^{-n}(k_i+1))$ with $(k_1,\dots,k_d) \in \{0,1,\dots,b^n-1\}^d$. The L^q-spectrum of m is the mapping defined for any $q \in \mathbb{R}$ by

$$\tau_m(q) = \liminf_{n\to\infty} -\frac{1}{n}\log_b s_n(q) \text{ where } s_n(q) = \sum_{\substack{Q\in\mathcal{G}_n \\ m(Q)\neq 0}} m(Q)^q.$$

It is easy to see that the restriction to $\mathbb{R}_+$ of τ_m does not depend on b. Two parameters are naturally associated with the measure m:

$$q_\tau(m) = \inf\{q \in \mathbb{R} : \tau_m(q) = 0\} \text{ and } H_\tau(m) = \tau_m'(q_\tau(m)^-).$$

The motivation of the introduction of these parameters was the following. For purely discontinuous measures, the classical measure dimensions vanish [25, 11, 16, 18, 7]. Nevertheless, these measures may have very interesting multifractal spectra [15, 1, 9, 14, 6, 24, 2, 4], and there is a need for other relevant parameters. The study of the pair $(q_\tau(m), H_\tau(m))$ and their relationships with the so-called large deviation spectrum is achieved in [5] and recalled below in Section 2. As we wished, these parameters are very pertinent for purely discontinuous measures m, i.e. measures constituted only by positive Dirac masses of the form

$$m = \sum_{k\geq 1} M_k \cdot \delta_{X_k}, \tag{1}$$

with $\widetilde{M} = (M_k)_{k\geq 1} \in (\mathbb{R}^+)^{\mathbb{N}^*}$, $\sum_k M_k < \infty$ and $\widetilde{X} = (X_k)_{k\geq 1} \in ([0,1]^d)^{\mathbb{N}^*}$ such that the X_k's are pairwise distinct. This paper aims at showing that for certain classes of purely discontinuous measures denoted ν in the following, these parameters not only store information about the large deviation spectrum of ν, but also store essential information about the multifractal Hausdorff spectrum of ν. To achieve this, we apply a threshold procedure to such measures ν by keeping only the Dirac masses naturally associated with the information parameters introduced in [5]. We prove that the obtained measure, denoted by $\nu^{\widetilde{\varepsilon}}$, has the same multifractal behavior as ν itself. Since the threshold procedure puts to zero the largest part of the Dirac masses of ν, it is thus very interesting to understand why the multifractal properties of ν are essentially the same as those of $\nu^{\widetilde{\varepsilon}}$.

From now on we shall work in the one-dimensional context. Extensions to higher dimensions are immediate, though more technical. Let us recall the definition of the Hausdorff spectrum of any measure m. First, for $x \in \mathrm{Supp}(m)$ (the support of m), the pointwise Hölder exponent of m at x is defined by

$$h_m(x) = \liminf_{r\to 0^+} \frac{\log m(B(x,r))}{\log r}.$$

Then, for every $h \geq 0$, one defines the level sets of the pointwise Hölder exponent of m and the multifractal Hausdorff spectrum of m as

$$E_h^m = \{x \in \mathrm{Supp}(m) : h_m(x) = h\} \text{ and } d_m : h \geq 0 \mapsto \dim E_h^m$$

where dim stands for the Hausdorff dimension. This spectrum is used to describe the geometrical properties of measures at small scales. Recall that if g is a function from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$, its Legendre transform is the mapping $g^* : h \mapsto \inf_{q \in \mathbb{R}}(hq - g(q)) \in \mathbb{R} \cup \{-\infty\}$. For every $h \geq 0$, we always have $d_m(h) \leq \tau_m^*(h)$ (see [8]), and the *multifractal formalism* is said to hold at h when the equality holds; i.e., when $d_m(h) = \tau_m^*(h)$.

The measures ν we consider are introduced in [2]. The scheme of their construction is the following. Let μ be a Borel probability measure on $[0,1]$ and let

$$\nu = \sum_{j \geq 1} \sum_{\substack{0 \leq k \leq b^j - 1 \\ k \not\equiv 0 \bmod b}} \nu_{j,k} \delta_{kb^{-j}} \text{ , with } \nu_{j,k} = \frac{1}{j^2} \mu([kb^{-j}, (k+1)b^{-j})). \tag{2}$$

The jump points are located at the b-adic points, and an heterogeneity in the Dirac masses distribution is created by the measure μ. It turns out that when μ is a Gibbs measure, this class of measures (2) (which is included in the class of purely discontinuous measures of the form (1)) has a fruitful multifractal structure, studied in details in [4, 2].

Theorem 1.1. *Let μ be a Gibbs measure as defined in Section 3.2. The measure ν defined by (2) obeys the multifractal formalism at every $h > 0$ such that $\tau_\nu^*(h) > 0$, as well as at 0. More precisely, $H_\tau(\nu) = H_\tau(\mu)$ and*

$$\tau_\nu(q) = \begin{cases} \tau_\mu(q) & \text{if } \tau_\nu(q) < 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad d_\nu(h) = \begin{cases} h & \text{if } 0 \leq h \leq H_\tau(\nu), \\ d_\mu(h) & \text{otherwise.} \end{cases}$$

Let us describe the thresholding procedure applied to ν. Let $\widetilde{\varepsilon} = (\varepsilon_j)_{j \geq 0}$ be a non-increasing positive sequence converging to 0. Consider the atomic measure ν of (2) and let

$$\nu^{\widetilde{\varepsilon}} = \sum_{j \geq 1} \sum_{0 \leq k \leq b^j - 1 :\, k \not\equiv 0 \bmod b} t_{j,k}\, \nu_{j,k}\; \delta_{kb^{-j}} \tag{3}$$

$$\text{with } \forall j \geq 1, \quad \forall k, \quad t_{j,k} = \mathbf{1}_{[H_\tau(\nu) - \varepsilon_j, H_\tau(\nu) + \varepsilon_j]}\Big(\frac{\log \nu_{j,k}}{\log b^{-j}} \Big). \tag{4}$$

Heuristically, the measure $\nu^{\widetilde{\varepsilon}}$ contains only the Dirac masses $\nu_{j,k}\delta_{kb^{-j}}$ such that $\nu_{j,k}$ is approximately equal to $b^{-jH_\tau(\nu)}$. A more complete explanation of such a formula comes from the companion paper [5], and is detailed in Section 2.

We obtain the following remarkable result which illustrates the amount of information potentially stored in the pair $(q_\tau(\nu), H_\tau(\nu))$.

Theorem 1.2. *Let μ be a Gibbs measure as in Section 3.2. Consider the thresholded measure $\nu^{\widetilde{\varepsilon}}$ (3). There exists a non-increasing positive sequence $\widetilde{\varepsilon}$ converging to 0 such that $d_{\nu^{\widetilde{\varepsilon}}}(h) = d_\nu(h)$ for every $h > 0$ such that $\tau_\nu^*(h) > 0$. Moreover $\nu^{\widetilde{\varepsilon}}$ obeys the multifractal formalism at 0 and at every $h > 0$ such that $\tau_\nu^*(h) > 0$. Finally, the L^q-spectra of ν and $\nu^{\widetilde{\varepsilon}}$ coincide ($\tau_\nu = \tau_{\nu^{\widetilde{\varepsilon}}}$).*

Actually, a slightly more general result will be proved (Theorem 2.2).

Theorem 1.2 shows the role played by the information parameters $q_\tau(\nu)$ and $H_\tau(\nu)$ for discontinuous measures having a nice structure close to statistical self-similarity. There is no doubt about the fact that Theorem 1.2 can be extended to other nice families of measures, such as the inverse of Gibbs measures on cookie-cutters [21] and the self-similar sums of Dirac masses introduced in [24]. These measures will be studied in a forthcoming paper. However it seems difficult to get similar results for measures without any structure.

It will be justified in the next section that at each generation j, approximately $b^{jH_\tau(\nu)}$ Dirac masses among b^j are kept after threshold. Since generally $H_\tau(\nu)$ equals $H_\tau(\mu)$ and is strictly lower than 1 when μ is non trivial, the threshold we realize is very severe. The situation $H_\tau(\nu) = 1$ corresponds for instance to the choice $\mu = \ell$ (the Lebesgue measure). It is a typical example of a homogeneous sum of Dirac masses ν_ℓ, for which there exists a positive sequence $\widetilde{\varepsilon}$ going to 0 at ∞ such that $\nu_\ell^{\widetilde{\varepsilon}} = \nu_\ell$.

2 Detailed Exposition of the Result.

2.1 More on the Information Parameters.

The connection between $(q_\tau(m), H_\tau(m))$ and the more usual Hausdorff, packing or entropy dimensions of m is the following. When $q_\tau(m) = 1$ and $H_\tau(m) = \tau_m'(1)$ exists, then $H_\tau(m)$ defines without ambiguity the dimension of the measure m [25, 16, 18, 11, 7].

The pair $(q_\tau(m), H_\tau(m))$ is also connected to the *large deviation spectrum* f_m of m. This spectrum describes the statistical distribution of m at small scales in the following sense. This spectrum f_m of m is defined as

$$h \geq 0 \mapsto f_m(h) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log_b \#\mathcal{S}_n^m(h, \varepsilon),$$

where for $\varepsilon > 0$, $h \geq 0$ and $n \in \mathbb{N}$,

$$\mathcal{S}_n^m(h,\varepsilon) = \left\{Q \in \mathcal{G}_n : b^{-n(h+\varepsilon)} \leq m(Q) \leq b^{-n(h-\varepsilon)}\right\}.$$

Very classical considerations [12, 8, 22, 17, 5] show that $\forall\, h \geq 0$, $d_m(h) \leq f_m(h) \leq \tau_m^*(h)$. Hence when the multifractal formalism holds at h, we also have $d_m(h) = f_m(h)$.

As a consequence of the fact that $f_m(\alpha) = \tau_m^*(\alpha)$ for all α of the form $\tau_m'(q^-)$ (see [23]), $H_\tau(m) = \max\{h \geq 0 : f_m(h) = q_\tau(m)h\}$ when $q_\tau(m) > 0$. For a discontinuous measure $m = \sum_{k\geq 1} M_k\, \delta_{X_k}$ on $[0,1]^d$, the relationships between the large deviation spectrum f_m restricted to $[0, H_\tau(m)]$ and the pair $(q_\tau(m), H_\tau(m))$ are investigated in [5]. Under a weak assumption on the distribution of the masses, it is shown that there exists a real number $H_g(m) \in (0, H_\tau(m)]$ depending on the sequences $(\widetilde{M}, \widetilde{X})$ (used in (1)) such that $f_m(h) = q_\tau(m)h$ over $[0, H_g(m)]$. In addition, $H_g(m)$ is equal to $H_\tau(m)$ if $q_\tau(m) \in (0,1)$, but it may differ from $H_\tau(m)$ if $q_\tau(m) = 1$.

We do not go into much details on $H_g(m)$. (This was the purpose of [5].) This linear increasing part in the large deviation spectrum confirms the observations made on special classes of homogeneous and heterogeneous sums of Dirac masses studied in the last fifteen years [1, 15, 9, 23, 2, 4]. Moreover, the elements of these classes of measures (which contain the measures (2) and (6)) verify that $H_g(m) = H_\tau(m)$ even when $q_\tau(m) = 1$. This is always assumed hereafter.

The starting point of the threshold performed in this article is provided by two important remarks made in [5] (Proposition 3.3, [5]):

- For every $n \geq 1$, most of the cubes in $\mathcal{S}_n^m(H_\tau(m), \varepsilon)$ contain a point X_k such that $b^{-n(H_\tau(m)+\varepsilon)} \leq M_k \leq b^{-n(H_\tau(m)-\varepsilon)}$. (Recall that $\mathcal{S}_n^m(H_\tau(m),\varepsilon)$ is the set of b-adic cubes Q of generation n such that $b^{-n(H_\tau(m)+\varepsilon)} \leq m(Q) \leq b^{-n(H_\tau(m)-\varepsilon)}$.) Hence, the m-mass of these cubes is approximately due to the presence of a single Dirac mass.

- The b-adic cubes which contain such a point X_k are responsible for the linear shape of f_m on $[0, H_\tau(m)]$.

Consequently, a certain amount of information is contained in the set of pairs (X_k, M_k) defined for any $\varepsilon > 0$ by

$$\mathcal{P}(H_\tau(m),\varepsilon) = \left\{(M_k, X_k) : \begin{cases} \exists\, n \geq 1, \ \exists\, Q \in \mathcal{S}_n^m(H_\tau(m),\varepsilon), \\ X_k \in Q, \ b^{-n(H_\tau(m)+\varepsilon)} \leq M_k \leq b^{-n(H_\tau(m)-\varepsilon)} \end{cases}\right\}.$$

A natural way to study this set of pairs (X_k, M_k) is to consider the measure

$$m^\varepsilon = \sum_{k\geq 1} \mathbf{1}_{\mathcal{P}(H_\tau(m),\varepsilon)}((M_k, X_k))\, M_k\, \delta_{X_k}. \tag{5}$$

This measure shall be viewed as a thresholded version of the initial measure m (1). It can be deduced from [5] that the measure has the same large deviation spectrum as m over $[0, H_\tau(m)]$.

This raises the following question. Do the measures m^ε still contain enough Dirac masses to have the same Hausdorff spectrum as m? This is the question investigated below.

2.2 The Measures $\nu_{\gamma,\sigma}$ and a More General Result.

Let μ be a Borel probability measure on $[0,1]$, $\gamma \geq 0$ and $\sigma \geq 1$, and

$$\nu_{\gamma,\sigma} = \sum_{j\geq 1} \sum_{\substack{0\leq k\leq b^j-1 \\ k\not\equiv 0 \mod b}} \nu_{j,k}\ \delta_{kb^{-j}}, \text{ with } \nu_{j,k} = \frac{b^{-j\gamma}}{j^2}\mu([kb^{-j},(k+1)b^{-j}))^\sigma. \tag{6}$$

The condition $k \not\equiv 0 \mod b$ in the definition (2) of $\nu_{\gamma,\sigma}$ is not required in [2]. This is unessential, since the two measures (with or without the condition) are equivalent, and thus have the same multifractal nature.

Theorem 2.1. *[2] Let μ be a Gibbs measure as in Section 3.2, $\gamma \geq 0$ and $\sigma \geq 1$. The measure $\nu_{\gamma,\sigma}$ given by formula (6) obeys the multifractal formalism at every $h > 0$ such that $\tau^*_{\nu_{\gamma,\sigma}}(h) > 0$, as well as at 0. Moreover, we have*

$$\tau_{\nu_{\gamma,\sigma}}(q) = \begin{cases} \gamma q + \tau_\mu(\sigma q) & \textit{if } \tau_{\nu_{\gamma,\sigma}}(q) < 0, \\ 0 & \textit{otherwise.} \end{cases}$$

These measures $\nu_{\gamma,\sigma}$ are generalized versions of the measures ν considered by Theorem 1.2. (Indeed, $\nu_{0,1}$ is the measure ν of the introduction.) Main Theorem 2.2 deals with $\nu_{\gamma,\sigma}$, and is thus more general than Theorem 1.2.

For $j \geq 1$ and $k \in [0,\ldots,b^j-1]$, we set $I_{j,k} = [kb^{-j},(k+1)b^{-j})$. The measure $\nu_{\gamma,\sigma}$ is of the form (1) if we take for the points X_k the b-adic numbers lb^{-j} with $l \not\equiv 0 \mod b$ and for the corresponding M_k the mass $M_{j,l} = \nu_{j,l}$. It is then easily seen that there exists a universal constant K such that $M_{j,l} \leq \nu_{\gamma,\sigma}(I_{j,l}) \leq KM_{j,l}$.

Consequently, in this case, requiring that $I_{j,l} \in \mathcal{S}_j^{\nu_{\gamma,\sigma}}(H_\tau,\varepsilon)$ is equivalent to requiring that $b^{-j(H_\tau+\varepsilon)} \leq M_{j,l} \leq b^{-j(H_\tau-\varepsilon)}$.

We apply the threshold procedure (3-4) to the class of measures $\nu_{\gamma,\sigma}$ defined by (6). This procedure is finer than (5). Recall that, if $\widetilde{\varepsilon} = (\varepsilon_j)_{j\geq 0}$ is a positive sequence converging to 0, then we set

$$\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}} = \sum_{j\geq 1} \sum_{0\leq k\leq b^j-1:\, k\not\equiv 0 \mod b} t_{j,k}\, \nu_{j,k}\ \delta_{kb^{-j}} \tag{7}$$

with

$$t_{j,k} = \mathbf{1}_{[H_\tau(\nu_{\gamma,\sigma})-\varepsilon_j, H_\tau(\nu_{\gamma,\sigma})+\varepsilon_j]}\left(\frac{\log \nu_{j,k}}{\log b^{-j}}\right)$$

defined as in (4.) ($\nu_{\gamma,\sigma}$ is used in the definition of $t_{j,k}$ instead of simply ν.)

Theorem 2.2. *Let μ be a Gibbs measure as in Section 3.2, $\gamma \geq 0$ and $\sigma \geq 1$. Consider $\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}$ defined by (7). There exists a non-increasing positive sequence $\widetilde{\varepsilon}$ converging to 0 such that:*

1. $(q_\tau(\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}), H_\tau(\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}})) = (q_\tau(\nu_{\gamma,\sigma}), H_\tau(\nu_{\gamma,\sigma}))$.
2. *For every $0 \leq h \leq H_\tau(\nu_{\gamma,\sigma})$, $d_{\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}}(h) = d_{\nu_{\gamma,\sigma}}(h)$ and $\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}$ obeys the multifractal formalism at h.*
3. *If $\gamma = 0$ and $\sigma = 1$, then the claims above reduce to Theorem 1.2.*

When $q_\tau(\nu) < 1$, the Hausdorff spectrum of $\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}$ may differ from $d_{\nu_{\gamma,\sigma}}$ on $(H_\tau(\nu_{\gamma,\sigma}), \infty)$. To see this heuristically, notice that the total mass conserved at each scale in $\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}$ is negligible with respect to the total mass of $\nu_{\gamma,\sigma}$, since approximatively there are at most $2^{jq_\tau H_\tau(\nu)}$ terms weighted by $2^{-jH_\tau(\nu)}$. Hence the amount of lost "information" is large. Nevertheless, it is remarkable that the Dirac masses we keep are enough to recover the spectrum on $[0, H_\tau(\nu_{\gamma,\sigma})]$.

Sections 3 gives some background necessary to establish Theorem 2.2, while Sections 4 is devoted to the proof of Theorem 2.2.

3 Scaling Properties of Gibbs Measures.

For $x \in (0,1)$, $I_j(x)$ is the unique b-adic interval of scale $j \geq 1$, semi-open to the right, containing x, and for every $\epsilon \in \{-1,0,1\}$, $I_j^{(\epsilon)}(x) = I_j(x) + \epsilon b^{-j}$. In the following, $|B|$ always denotes the diameter of the set B. Eventually, for the rest of the paper, the convention $\log(0) = -\infty$ is adopted.

3.1 Some Dimension and Large Deviation Bounds.

Definition 3.1. Let μ be a positive Borel measure on $[0,1]$. For $x \in (0,1)$, recall the definition (1) of the Hölder exponent of μ at x and of the level sets

E^μ_α defined for every $\alpha \geq 0$ by $E^\mu_\alpha = \{x : h_\mu(x) = \alpha\}$.
For $\tilde{\xi} = (\xi_j)_{j\geq 1}$ a positive non-increasing sequence converging to zero, we set

$$\widetilde{E}^\mu_{\alpha,\widetilde{\xi}} = \left\{ x : \begin{cases} \text{there is a scale } J_x \text{ such that for every } j \geq J_x, \\ \forall \epsilon \in \{-1,0,1\},\ b^{-j(\alpha+\xi_j)} \leq \mu(I_j^{(\epsilon)}(x)) \leq b^{-j(\alpha-\xi_j)} \end{cases} \right\}.$$

For any $\tilde{\xi}$, it is obvious that $\widetilde{E}^\mu_{\alpha,\widetilde{\xi}} \subset E^\mu_\alpha$. The level sets $\widetilde{E}^\mu_{\alpha,\widetilde{\xi}}$ contain points around which the local μ-behavior can be very precisely controlled.

As a simple consequence of [8, 17], we get the following.

Proposition 3.2. *Let μ be a positive Borel measure on $[0,1]$, and let $(h_{\min}, h_{\max})$ be the maximal open interval on which $\tau^*_\mu > 0$.*

1. *For every $\alpha \geq 0$ such that $\tau^*_\mu(\alpha) \geq 0$ and for any non-increasing sequence $\widetilde{\xi}$ converging to zero, $\dim \widetilde{E}^\mu_{\alpha,\widetilde{\xi}} \leq d_\mu(\alpha) \leq f_\mu(\alpha) \leq \tau^*_\mu(\alpha)$.*
2. *If μ obeys the multifractal formalism at every $\alpha \in (h_{\min}, \tau'_\mu(0^+)]$, then for every $\alpha \in (h_{\min}, \tau'_\mu(0^+)]$, $\dim \left(\bigcup_{\alpha' \leq \alpha} E^\mu_{\alpha'}\right) = \dim E^\mu_\alpha$.*
3. *If μ obeys the multifractal formalism at every $\alpha \in [\tau'_\mu(0^+), h_{\max})$, then for every $\alpha \in [\tau'_\mu(0^+), h_{\max})$, $\dim \left(\bigcup_{\alpha' \geq \alpha} E^\mu_{\alpha'}\right) = \dim E^\mu_\alpha$.*

Definition 3.3. Let λ be a positive Borel measure on $\mathbb{R}$. Let us define, $\forall \alpha \geq 0$, $J \geq 0$ and $K \in \{0, \dots, b^J - 1\}$, $\eta > 0$, $j \geq J+1$,

$$N_{J,K}(\lambda, j, \eta, \alpha) = \# \left\{ k \not\equiv 0 \bmod b : \begin{cases} I_{j,k} \subset I_{J,K}, \\ b^{-(j-J)(\alpha+\eta)} \leq \frac{\lambda(I_{j,k})}{\lambda(I_{J,K})} \leq b^{-(j-J)(\alpha-\eta)} \end{cases} \right\}.$$

Heuristically, $N_{J,K}(\lambda, j, \eta, \alpha)$ is the number of intervals $I_{j,k} \subset I_{J,K}$ such that, when forgetting what happens before j, the rescaled λ-measure of $I_{j,k}$, $\frac{\lambda(I_{j,k})}{\lambda(I_{J,K})}$, is approximately equal to $b^{-(j-J)\alpha} = \left(\frac{|I_{j,k}|}{|I_{J,K}|}\right)^\alpha$.

3.2 Gibbs Measures and Their Multifractal Properties.

Here are defined the Gibbs measures used in Theorems 2.1 and 2.2. We summarize some of their scaling and multifractal properties.

3.2.1 Definition.

Let c be an integer greater than 2 and let ℓ stand for the Lebesgue measure on $[0,1]$. Let ϕ be a 1-periodic Hölder continuous function on $\mathbb{R}$ and $\omega = (\omega_n)_{n\geq 0}$ be a sequence of independent random phases uniformly distributed in $[0,1]$.

Let T be the shift transformation on $[0,1)$: $T(t)=ct \mod 1$. For $n\geq 1$ and $t\in[0,1)$ let us consider the Birkhoff sums

$$S_n(\phi)(t)=\sum_{k=0}^{n-1}\phi(T^k t) \text{ and } S_n(\phi,\omega)(t)=\sum_{k=0}^{n-1}\phi(T^k t+\omega_k).$$

Also let

$$Q_n(t)=\frac{\exp\big(S_n(\phi)(t)\big)}{\int_{[0,1]}\exp\big(S_n(\phi)(u)\big)\,du} \text{ and } Q_n(t,\omega)=\frac{\exp\big(S_n(\phi,\omega)(t)\big)}{\int_{[0,1]}\exp\big(S_n(\phi,\omega)(u)\big)\,du}.$$

It follows from the thermodynamic formalism [19, 13] that $\mu_n=Q_n(\cdot)\cdot\ell$ (resp. $\mu_n^\omega=Q_n(\cdot,\omega)\cdot\ell$) converges (resp. almost surely), as $n\to\infty$, to a deterministic Gibbs (resp. random Gibbs) measure denoted μ (resp. μ^ω).

The multifractal analysis of μ and μ^ω is performed for instance in [8, 20, 10, 13]. With ϕ and ω are associated the analytic functions

$$\begin{aligned} P: q\mapsto \log(c)+\lim_{n\to\infty} n^{-1}\log\int_{[0,1)}\exp(qS_n(\phi(t)))\,dt \\ \text{and } \widetilde{P}: q\mapsto \log(c)+\lim_{n\to\infty} n^{-1}\mathbb{E}\log\int_{[0,1)}\exp(qS_n(\phi(t,\omega)))\,dt, \end{aligned}$$

which respectively are the topological pressures of ϕ relative to T and $\widetilde{T}:(t,\omega)\mapsto\big(T(t),\theta(\omega)\big)$, where $\theta(\omega)=(\omega_{n+1})_{n\geq 0}$. We have $\tau_\mu(q)=\frac{qP(1)-P(q)}{\log(c)}$, and a.s. $\tau_{\mu^\omega}(q)=\frac{q\widetilde{P}(1)-\widetilde{P}(q)}{\log(c)}$.

Gibbs measures considered here obey the multifractal formalism. In particular, for every $h\geq 0$, $d_\mu(h)=\tau_\mu^*(h)$ as soon as $\tau_\mu^*(h)>0$. Actually, Theorems 1.1, 1.2, 2.1 and 2.2 hold for the elements of a larger class of measures described in [3], which also contains the multinomial measures and their random counterpart.

3.2.2 Properties of Gibbs Measures.

In this section, we fix a Gibbs measure μ as defined above. In the random case, μ is a realization of μ^ω and the following results hold almost surely. We fix another integer $b\geq 2$ in order to consider the b-adic grid defined in Section 1.

Fine properties on the measure μ are required to prove Theorem 2.2. Let $(h_{\min},h_{\max})$ be defined as in Proposition 3.2.

• **Property P1 (lower and upper bound for the scaling properties):** We have $h_{\min}>0$ and $h_{\max}<+\infty$. The measure μ obeys the multifractal formalism at any $h \in (h_{\min}, h_{\max})$. For j large enough, for every $0 \le k \le b^j - 1$, $b^{-2h_{\max} j} \le \mu(I_{j,k}) \le b^{-h_{\min} j/2}$.

• **Property P2 (Gibbs states as analyzing measures):**
Let $\mathcal{L}$ be a compact subset of $(h_{\min}, h_{\max})$. There is a sequence $\widetilde{\xi} = (\xi_j)_j$ such that for every $\alpha \in \mathcal{L}$, one can find a Borel measure m_α on $[0,1]$ such that $m_\alpha(\widetilde{E}^{\mu}_{\alpha,\widetilde{\xi}}) > 0$ and $m_\alpha(E) = 0$ for every Borel set $E \subset [0,1]$ such that $\dim E < \tau_\mu^*(\alpha)$. (This yields $\dim \widetilde{E}_{\alpha,\widetilde{\xi}} = \dim E^\mu_\alpha = \tau^*_\mu(\alpha)$.) Let q_α be the unique $q \in \mathbb{R}$ such that $\alpha = \tau'_\mu(q)$. A possible choice for m_α is the Gibbs measure μ_{q_α} constructed as μ with the potential $q_\alpha \phi$. We have $\tau^*_\mu(\alpha) = \tau'_{\mu_{q_\alpha}}(1)$.

• **Property P3 (Heterogeneous ubiquity):** It follows from [2].
For $\rho \ge 1$, $\alpha > 0$ and for a positive sequence $\widetilde{\xi} = (\widetilde{\xi}_j)_{j \ge 1}$ define the limsup set

$$S_\mu(\rho, \alpha, \widetilde{\xi}) = \bigcap_{J \ge 0} \bigcup_{j \ge J} \bigcup_{\substack{k \in \{0,\dots,b^j-1\}:\, k \not\equiv 0 \mod b \\ b^{-j(\alpha+\xi_j)} \le \mu(I_{j,k}) \le b^{-j(\alpha-\xi_j)}}} [kb^{-j}, kb^{-j} + b^{-j\rho}]. \tag{8}$$

Let $\mathcal{L}$ be a compact subset of $(h_{\min}, h_{\max})$. There exists a positive sequence $\widetilde{\xi}$ converging to 0 such that for every $\rho \ge 1$ and $\alpha \in \mathcal{L}$, one can find a positive Borel measure $m_{\alpha,\rho}$ such that:
- $m_{\alpha,\rho}(E) = 0$ for every Borel set E such that $\dim E < \tau^*_\mu(\alpha)/\rho$,
- $m_{\alpha,\rho}\big(S_\mu(\rho, \alpha, \widetilde{\xi})\big) > 0$.

In particular, dim $S_\mu(\rho, \alpha, \widetilde{\xi}) \ge \tau^*_\mu(\alpha)/\rho$.

• **Property P4 (Uniform renewal speed of large deviations spectrum):** This property is proved in [3].

Let $\mathcal{L}$ be a compact subinterval of $(h_{\min}, h_{\max})$. Let $\eta > 0$, and let us consider the sequence defined for $j \ge 1$ by

$$\gamma_j := \sqrt{\frac{\log(j)^{1+\eta}}{j^{1/4}}}. \tag{9}$$

There exists a constant $M > 0$ and a scale $J_0 \ge 1$ such that for every $J \ge J_0$ and $K \in \{0, \dots, b^J - 1\}$, for every integer $j \ge J + [\exp(\sqrt{(1+\eta)\log(J)})]$ and $\alpha \in \mathcal{L}$, we have

$$b^{(j-J)(\tau^*_\mu(\alpha) - M\gamma_{j-J})} \le N_{J,K}(\mu, j, \gamma_{j-J}, \alpha) \le b^{(j-J)(\tau^*_\mu(\alpha) + M\gamma_{j-J})}. \tag{10}$$

Remark 3.4. Properties **P1** and **P2** are well known for Gibbs measures associated with a smooth enough potential (among many references, see [8, 10, 20]). Properties **P3** and **P4** rely on finer properties without the restriction

$k \not\equiv 0 \mod b$, but simple verifications show that the results also hold with this restriction.

It is important for the sequel to make it precise that in Properties **P2** and **P3**, $\widetilde{\xi}$ can be taken equal to the sequence $(\gamma_j)_{j\geq 1}$ of **P4**.

4 Proof of Theorem 2.2.

4.1 Proof of item *3.* of Theorem 2.2.

We begin by the last assertion. In this section, $\gamma = 0$ and $\sigma = 1$; thus $\nu_{0,1}$ is simply denoted ν. A b-adic number kb^{-j} is said to be *irreducible* if the fraction k/b^j is irreducible. Let $\widetilde{\gamma} = (\gamma_j)_{j\geq 1}$ be the sequence defined by (9). For $j \geq 1$, define

$$\varepsilon_j = 2\gamma_{[\frac{j}{\log j}]} + 6\frac{h_{\max}}{\log j}. \tag{11}$$

Due to the last remark of Section 3.2.2, Properties **P2** and **P3** hold true with $\widetilde{\xi} := \widetilde{\varepsilon}/2$.

For simplicity of notation, we consider the measure $\nu^t := \nu^{\widetilde{\varepsilon}} = \nu_{0,1}^{\widetilde{\varepsilon}}$ (3) associated with the sequence $\widetilde{\varepsilon} = (\varepsilon_j)_j$. We also denote $t_{j,k}\nu_{j,k}$ by $\nu_{j,k}^t$. We deduce from Theorem 2.1 that $H_\tau := H_\tau(\nu) = \tau_\mu'(1)$, and thus by construction $\tau_\mu^*(H_\tau) = H_\tau$. We are going to show that $d_{\nu^t}(h) = d_\nu(h)(= \tau_\nu^*(h))$ for all $h \in [0, h_{\max})$. Since $\tau_\nu \leq \tau_{\nu^t}$, we have $\tau_\nu^* = \tau_{\nu^t}^*$ on $\mathbb{R}_+$ and thus $\tau_\nu = \tau_{\nu^t}$ (remember that τ_ν and τ_{ν^t} are non-decreasing).

4.1.1 First Results on the Local Regularity of ν^t.

It is easy to verify that for every $x \in [0, 1]$,

$$h_\nu(x) \leq h_{\nu^t}(x) \quad \text{and} \quad h_\nu(x) \leq h_\mu(x). \tag{12}$$

The first inequality is due to the fact that by construction, for any Borel set $B \subset [0, 1]$, $\nu^t(B) \leq \nu(B)$. The second one follows from the fact that for any b-adic interval $I_{j,k}$, $\nu(I_{j,k}) \geq j^{-2}\mu(I_{j,k})$.

Proposition 4.1. *For every $\varepsilon > 0$, there is an integer J_ε such that for any $\beta \in [h_{\min}/2, 2h_{\max}]$, $\forall J \geq J_\varepsilon$, for every integer K such that Kb^{-J} is irreducible,*

$$\mu(I_{J,K}) = b^{-J\beta} \Rightarrow b^{-J(\beta+\varepsilon)} \leq \nu^t(I_{J,K}) \leq b^{-J(\beta-\varepsilon)}.$$

PROOF. Let $\varepsilon > 0$. Let J_1 be large enough so that $j \geq J_1$ implies $0 < \max(\gamma_j, \varepsilon_j) \leq \varepsilon/2$ and $b^{-2jh_{\max}} \leq \mu(I_{j,k}) \leq b^{-jh_{\min}/2}$ for all $0 \leq k \leq b^j - 1$. Let Kb^{-J} be an irreducible b-adic number such that $J \geq J_1$, and let β be defined by $\mu(I_{J,K}) = b^{-J\beta}$.

• Let us first notice that (recall the definition (2) of the measure ν)

$$\begin{aligned}\nu(I_{J,K}) &= \frac{1}{J^2}\mu(I_{J,K}) + \sum_{j\geq J+1}\frac{1}{j^2}\sum_{\substack{k=0,\dots,b^j-1:\\ k\not\equiv 0 \mod b,\ kb^{-j}\in I_{J,K}}}\mu([kb^{-j},(k+1)b^{-j})) \\ &\leq \frac{1}{J^2}\mu(I_{J,K}) + \sum_{j\geq J+1}\frac{1}{j^2}\mu(I_{J,K}).\end{aligned}$$

If J is greater than some fixed integer J_2 large enough, then $\nu(I_{J,K}) \leq \mu(I_{J,K})b^{J\varepsilon/2} \leq b^{-J(\beta-\varepsilon/2)}$. Now it is obvious that by construction, for any subset B of $[0,1]$, $\nu^t(B) \leq \nu(B)$. Hence we get the first inequality $\mu(I_{J,K}) = b^{-J\beta} \Rightarrow \nu^t(I_{J,K}) \leq b^{-J(\beta-\varepsilon)}$ for any $J \geq \max(J_1, J_2)$.

• The converse inequality is more difficult to obtain. Let us show that $\nu^t(I_{J,K}) \geq b^{-J(\beta+\varepsilon)}$. By definition, we have

$$\nu^t(I_{J,K}) = \nu^t_{J,K} + \sum_{j\geq J+1}\sum_{\substack{k=0,\dots,b^j-1:\\ k\not\equiv 0 \mod b,\ kb^{-j}\in I_{J,K}}}\nu^t_{j,k}\delta_{kb^{-j}}. \tag{13}$$

1. If $\beta = H_\tau$: By construction, for J large enough, we have $\nu^t_{J,K} = J^{-2}\mu(I_{J,K})$, and $\nu^t(I_{J,K}) \geq \nu^t_{J,K} \geq b^{-J(\beta+\varepsilon)}$.

2. If $\beta > H_\tau$: Let us recall (13). To find a lower bound for $\nu^t(I_{J,K})$, we must look for non-zero Dirac masses (after threshold) in the sum (13).

Let us use Property **P4** applied with $\alpha = H_\tau$. Let $\eta > 0$. There exists a constant $M > 0$ and a scale $J_0 \geq 1$ such that for every $J \geq J_0$ and $K \in \{0,\dots,b^J-1\}$, for every $j \geq J+\exp(\sqrt{(1+\eta)\log(J)})$, (10) holds with $\alpha = H_\tau$. In particular, for every $j \geq J + \exp(\sqrt{(1+\eta)\log(J)})$, we get

$$N_{J,K}(\mu, j, \gamma_{j-J}, H_\tau) \geq b^{(j-J)(H_\tau - M\gamma_{j-J})}. \tag{14}$$

Let $I_{j,k}$ be any of the intervals such that $I_{j,k} \subset I_{J,K}$, $k \not\equiv 0 \mod b$, and

$$\mu(I_{J,K})b^{-(j-J)(H_\tau+\gamma_{j-J})} \leq \mu(I_{j,k}) \leq \mu(I_{J,K})b^{-(j-J)(H_\tau-\gamma_{j-J})}.$$

We have $b^{-j\alpha^1_{j,J}} \leq \mu(I_{j,k}) \leq b^{-j\alpha^2_{j,J}}$ with

$$\begin{aligned}\alpha^1_{j,J} &= H_\tau + \gamma_{j-J} - \frac{J}{j}(H_\tau - \beta + \gamma_{j-J}), \\ \text{and}\quad \alpha^2_{j,J} &= H_\tau - \gamma_{j-J} - \frac{J}{j}(H_\tau - \beta - \gamma_{j-J}).\end{aligned}$$

In order to ensure that $\nu^t_{j,k} \neq 0$, it is sufficient to have

$$[\alpha^2_{j,J}, \alpha^1_{j,J}] \subset [H_\tau - \varepsilon_j + \log_b(j^2)/j, H_\tau + \varepsilon_j + \log_b(j^2)/j].$$

This is achieved as follows.

Let $\theta > 0$. There exists a scale J_3 such that for every $J \geq J_3$, for every $j \geq J + J^{1+\theta}$,

$$\frac{j}{\log j} \leq j - J \quad \text{and} \quad \frac{6h_{\max}}{\log j} \geq \frac{J}{j}(2h_{\max} + H_\tau + \gamma_{j-J}) + \frac{\log_b(j^2)}{j}.$$

Let $J_4 = \max(J_1, J_2, J_3)$ (J_4 is independent of β). Then by (11), for every $J \geq J_4$, as soon as $j \geq J + J^{1+\theta}$, we obtain

$$H_\tau - \varepsilon_j + \log_b(j^2)/j \leq \alpha^2_{j,J} \leq \alpha^1_{j,J} \leq H_\tau + \varepsilon_j + \log_b(j^2)/j.$$

Hence those intervals $I_{j,k} \subset I_{J,K}$ (with $j \geq J + J^{1+\theta}$) such that $k \not\equiv 0 \mod b$ and $b^{-j\alpha^1_{j,J}} \leq \mu(I_{j,k}) \leq b^{-j\alpha^2_{j,J}}$ give rise to non-zero masses in the sum (13).

Using (13) and (14), we obtain that for every $J \geq J_4$, for every K such that Kb^{-J} is irreducible, for every $j_0 = J + J^{1+\theta}$,

$$\begin{aligned}\nu^t(I_{J,K}) &\geq \sum_{\substack{k=0,\dots,b^{j_0}-1: \\ k\not\equiv 0 \mod b,\ kb^{-j_0}\in I_{J,K}}} \nu^t_{j_0,k} \geq \frac{1}{j_0^2} N_{J,K}(\mu, j_0, \gamma_{j_0-J}, H_\tau) b^{-j_0\alpha^1_{j_0,J}} \\ &\geq \frac{1}{j_0^2} b^{(j_0-J)(H_\tau - M\gamma_{j_0-J})} b^{-j_0(H_\tau + \gamma_{j_0-J} - \frac{J}{j_0}(H_\tau - \beta + \gamma_{j_0-J}))}.\end{aligned}$$

Hence $\nu^t(I_{J,K}) \geq b^{-J\beta}\frac{b^{(j_0-J)(M+1)\gamma_{j_0-J}}}{j_0^2}$. Since $\gamma_j = \sqrt{\frac{\log(j)^{1+\eta}}{j^{1/4}}}$, we deduce that

$$\begin{aligned}(j_0 - J)(M+1)\gamma_{j_0-J} &= J^{1+\theta}(M+1)\gamma_{J^{1+\theta}} = (M+1)J^{1+\theta}\sqrt{\frac{\log(J^{1+\theta})^{1+\eta}}{J^{(1+\theta)/4}}} \\ &= (M+1)(1+\theta)^{(1+\eta)/2} J^{7(1+\theta)/8} \log(J)^{(1+\eta)/2}.\end{aligned}$$

Choosing $\theta \in (0, 1/7)$, there is a scale J_5 (independent of β) such that for every $J \geq J_5$, we have $(j_0 - J)(M+1)\gamma_{j_0-J} \leq \varepsilon J$. It is also obvious that $\frac{1}{j_0^2} \geq b^{-\varepsilon J}$ for J large enough. Finally, for J large enough, we obtain

$$\nu^t(I_{J,K}) \geq b^{-J(\beta+2\varepsilon)}.$$

3. If $\beta < H_\tau$: The same arguments as above yield the same result. $\square$

We emphasize that the order of magnitude of the sequence γ_j plays a crucial role in the previous computation.

Proposition 4.2. *For every $\varepsilon > 0$, there is an integer J_ε such that for any $\beta \in [h_{\min}/2, 2h_{\max}]$, for every $J \geq J_\varepsilon$, for every integer K such that Kb^{-J} is irreducible,*

$$\mu(I_{J,K}) = b^{-J\beta} \Rightarrow b^{-J(\beta+\varepsilon)} \leq \nu^t(I_{J,K}) \leq \nu(I_{J,K}).$$

Proof. The right inequality is immediate, and the left one is a consequence of Proposition 4.1. □

Using Propositions 4.1 and 4.2, we can specify (12) and assert that,

$$\text{for every } x \in [0,1],\ h_\nu(x) \leq h_{\nu^t}(x) \leq h_\mu(x). \tag{15}$$

Proposition 4.3. *Let $\rho \geq 1$. Consider the limsup set $S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2)$ defined in (8). For every $x \in S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2)$, $h_{\nu^t}(x) \leq H_\tau/\rho$.*

Proof. By definition of $S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2)$, for every $x \in S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2)$, there is an infinite number of scales j_n such that for some $k_n \in \{0, \dots, b^{j_n} - 1\}$ with $k_n b^{-j_n}$ irreducible, $|x - k_n b^{-j_n}| \leq b^{-j_n \rho}$ and simultaneously $b^{-j_n(H_\tau + \varepsilon_{j_n}/2)} \leq \mu(I_{j_n,k_n})) \leq b^{-j_n(H_\tau - \varepsilon_{j_n/2})}$. This implies that

$$\nu^t(B(x, b^{-j_n\rho})) \geq \frac{1}{j_n^2}\mu(I_{j_n,k_n})) \geq b^{-j_n(H_\tau+\varepsilon_{j_n})} = b^{-j_n\rho\left(\frac{H_\tau+\varepsilon_{j_n}}{\rho}\right)}.$$

Hence $\frac{\log \nu^t(B(x,b^{-j_n\rho}))}{\log b^{-j_n\rho}} \leq \frac{H_\tau+\varepsilon_{j_n}}{\rho}$. Since $\varepsilon_j \to 0$, by letting j_n go to $+\infty$, we obtain that $h_{\nu^t}(x) \leq H_\tau/\rho$. □

4.1.2 Upper Bound for the Multifractal Spectrum of d_{ν^t}.

Proposition 4.4. *For every $h \in [0, \tau_\mu'(0)]$, $d_{\nu^t}(h) \leq d_\nu(h) = \tau_\mu^*(h)$.*

Proof. We use (15). For every $x \in [0,1]$, $h_{\nu^t}(x) \geq h_\nu(x)$. This implies that for every $h \in [0, \tau_\mu'(0)]$, $E_h^{\nu^t} \subset \bigcup_{h' \leq h} E_{h'}^\nu$. By Proposition 3.2 and Theorem 2.1, for every $h \in [0, \tau_\mu'(0)]$, we obtain that $\dim \bigcup_{h' \leq h} E_{h'}^\nu \leq \dim E_h^\nu = d_\nu(h)$; hence the result. □

Proposition 4.5. *For every $h \in (\tau_\mu'(0), h_{\max}]$, $d_{\nu^t}(h) \leq d_\nu(h) = \tau_\mu^*(h)$.*

Proof. Let $h \in (\tau_\mu'(0), h_{\max}]$. By (15), $E_h^{\nu^t} \subset \bigcup_{h' \geq h} E_h^\mu$. By Proposition 3.2, $d_{\nu^t}(h) = \dim E_h^{\nu^t} \leq \dim \bigcup_{h' \geq h} E_h^\mu \leq \tau_\mu^*(h)$. □

4.1.3 Lower Bound for the Multifractal Spectrum of d_{ν^t}.

Proposition 4.6. *For every $h \in [0, H_\tau]$, $d_{\nu^t}(h) \geq d_\nu(h) = h$.*

PROOF. We first apply property **P3**. Let $h \in (0, H_\tau]$, and consider $\rho = H_\tau/h$ and $\alpha = H_\tau$. Property **P3** provides us with a measure $m_{\alpha,\rho}$ and the set $S = S_\mu(\rho, \alpha, \widetilde{\varepsilon}/2)$. By Proposition 4.3, every $x \in S$ satisfies $h_{\nu^t}(x) \leq H_\tau/\rho = h$. Hence $S \subset \bigcup_{h' \leq h} E_{h'}^{\nu^t}$. By Proposition 3.2, for all $i \geq 1$, $\dim \bigcup_{h' \leq h-1/i} E_{h'}^{\nu^t} \leq \tau_{\nu^t}^*(h - 1/i)$. Moreover, $\tau_{\nu^t} \geq \tau_\nu$ so $\tau_{\nu^t}^*(h-1/i) \leq \tau_\nu^*(h-1/i) < \tau_\nu^*(h) = H_\tau/\rho$. Hence $m_{\alpha,\rho}(\bigcup_{i\geq 1} \bigcup_{h' \leq h-1/i} E_{h'}^{\nu^t}) = 0$. We deduce that $S \setminus \bigcup_{i \geq 1} \bigcup_{h' \leq h-1/i} E_{h'}^{\nu^t} \subset E_h^{\nu^t}$. Since $\varepsilon_j \geq \xi_j$, by construction $m_{\alpha,\rho}(S) > 0$, and $m_{\alpha,\rho}(S \setminus \bigcup_{i\geq 1} \bigcup_{h' \leq h-1/i} E_{h'}^{\nu^t}) > 0$. As a conclusion, $\dim E_h^{\nu^t} \geq h$. □

Proposition 4.7. *For every $h \in (H_\tau, h_{\max})$, $d_{\nu^t}(h) \geq d_\nu(h)$.*

PROOF. We need a lemma extracted from the proof of Proposition 8 in [2].

Lemma 4.8. *Let $h \in [H_\tau, h_{\max})$. Let m_h be a measure as in* **P2**. *Then there exists a subset S of $\widetilde{E}^\mu_{h,\widetilde{\xi}}$ such that $m_h(S) > 0$ and $S \subset E_h^\nu$.*

Let $h \in (H_\tau, h_{\max})$. Consider a set S and a measure m_h as in Lemma 4.8. Let $x \in S \subset \widetilde{E}^\mu_{h,\widetilde{\xi}} \cap E_h^\nu$. Note that at every scale j, at least one of $I_j^{(-1)}(x)$, $I_j^0(x)$, $I_j^{(+1)}(x)$, is irreducible. Hence, for this irreducible b-adic interval I, by Proposition 4.1 we have $\nu^t(I) \geq \mu(I) b^{-j\varepsilon} \geq b^{-j(h+\xi_j+\varepsilon)}$. This holds for every j large enough, and then for every ε small enough. Hence $h_{\nu^t}(x) \leq h$. But $h_{\nu^t}(x)$ is always larger than $h_\nu(x)$, which equals h since $S \subset E_h^\nu$. Hence $h_{\nu^t}(x) = h$, and $S \subset E_h^{\nu^t}$. As a consequence, $m_h(E_h^{\nu^t}) \geq m_h(S) > 0$, and $\dim E_h^{\nu^t} \geq d_\nu(h)$. □

4.2 Proof of Item *2.* of Theorem 2.2.

We come back to the general measures $\nu_{\gamma,\sigma}$ and to the general form of their thresholded versions $\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}$. Let $\widetilde{\xi}$ be a sequence such that **P3** holds with $\alpha = \tau_\mu'(\sigma q_\tau(\nu))$, and let $\widetilde{\varepsilon} = (\varepsilon_j)_{j\geq 1}$ be defined by $\varepsilon_j = \sigma \widetilde{\xi}_j + 2\log_b(j)/j$.

Let $\mu_{\sigma q_\tau}$ be the Gibbs measure constructed as μ, but with the potential $\sigma q_\tau(\nu)\phi$. We deduce from Theorem 2.1 that $q_\tau(\nu_{\gamma,\sigma}) H_\tau(\nu_{\gamma,\sigma}) = \tau_\mu^*(\sigma q_\tau(\nu_{\gamma,\sigma}))$. Then, the same arguments as those used in the proof of Proposition 4.3 show that for $\rho \geq 1$, if $x \in S_\mu(\rho, \tau_\mu'(\sigma q_\tau(\nu_{\gamma,\sigma})), \widetilde{\xi})$, then $h_{\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}}(x) \leq H_\tau(\nu_{\gamma,\sigma})/\rho$.

Since by construction $\tau_{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}} \geq \tau_{\nu_{\gamma,\sigma}}$, for every $h \in [0, H_\tau(\nu_{\gamma,\sigma})]$ we have $\dim \bigcup_{h'\leq h} E_{h'}^{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}} \leq \tau^*_{\nu_{\gamma,\sigma}}(h) = q_\tau(\nu_{\gamma,\sigma})h$. This is enough to conclude as in the proof of Proposition 4.6.

4.3 Proof of Item *1.* of Theorem 2.2.

It follows from the item *2* . of Theorem 2.2 that

$$d_{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}}(H_\tau(\nu_{\gamma,\sigma})) = q_\tau(\nu_{\gamma,\sigma})H_\tau(\nu_{\gamma,\sigma}).$$

Moreover, due to the threshold operation, we have $\tau_{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}} \geq \tau_{\nu_{\gamma,\sigma}}$ on $\mathbb{R}_+$, so that $q_\tau(\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}) \leq q_\tau(\nu_{\gamma,\sigma})$.

On the other hand, $d_{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}}(H_\tau(\nu_{\gamma,\sigma})) \leq \tau^*_{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}}(H_\tau(\nu_{\gamma,\sigma})) \leq q_\tau(\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma})H_\tau(\nu_{\gamma,\sigma})$. This yields $q_\tau(\nu^{H_\tau(\nu),\widetilde{\varepsilon}}_{\gamma,\sigma}) = q_\tau(\nu_{\gamma,\sigma})$ and then $H_\tau(\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}) \leq H_\tau(\nu_{\gamma,\sigma})$ again because $\tau_{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}} \geq \tau_{\nu_{\gamma,\sigma}}$ and these functions are concave.

Coming back to $d_{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}}(H_\tau(\nu_{\gamma,\sigma})) = q_\tau(\nu_{\gamma,\sigma})H_\tau(\nu_{\gamma,\sigma})$ and the fact that for any positive Borel measure m on $[0,1]$, we have $d_m(h) \leq \tau^*_m(h) < q_\tau(m)h$ if $h > H_\tau(m)$, we finally obtain $H_\tau(\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}) = H_\tau(\nu_{\gamma,\sigma})$.

References

[1] V. Aversa, C. Bandt, *The multifractal spectrum of discrete measures*. 18th Winter School on Abstract Analysis (1990), Acta Univ. Carolin. Math. Phys. **31(2)** (1990), 5–8.

[2] J. Barral, S. Seuret, *Combining multifractal additive and multiplicative chaos*, Commun. Math. Phys., **257(2)** (2005), 473–497.

[3] J. Barral, S. Seuret, *Inside singularity sets of random Gibbs measures*, J. Stat. Phys., **120** (2005), 1101–1124.

[4] J. Barral, S. Seuret, *The multifractal nature of heterogeneous sums of Dirac masses*, submitted (2006).

[5] J. Barral, S. Seuret, *Information parameters and large deviations spectrum of discontinuous measures*, Real Analysis Exchange, this issue.

[6] J. Barral, S. Seuret, The singularity spectrum of Lévy processes in multifractal time, Adv. Math., to appear.

[7] A. Batakis, Y. Heurteaux, *On relations between entropy and Hausdorff dimension of measures*, Asian J. Math., **6** (2002), 399–408.

[8] G. Brown, G. Michon, J. Peyrière, *On the multifractal analysis of measures*, J. Stat. Phys., **66** (1992), 775–790.

[9] K. J. Falconer, *Representation of families of sets by measures, dimension spectra and Diophantine approximation*, Math. Proc. Cambridge Philos. Soc., **128** (2000), 111–121.

[10] A.-H. Fan, *Multifractal analysis of infinite products*. J. Stat. Phys., **86** (1997), 1313–1336.

[11] A.-H. Fan, K.-S. Lau, H. Rao, *Relationships between different dimensions of a measure*, Monatsh. Math., **135** (2002), 191–201.

[12] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, *Fractal measures and their singularities: the characterization of strange sets*, Phys. Rev. A, **33** (1986), 1141.

[13] Y. Kifer, *Fractals via random iterated function systems and random geometric constructions*, Fractal geometry and stochastics (Finsterbergen, 1994) Progr. Probab., **37**, 145–164, Birkhäuser, Basel (1995)

[14] S. Jaffard, *Old friends revisited: the multifractal nature of some classical functions*, J. Fourier Anal. Appl., **3(1)** (1997), 1–21.

[15] S. Jaffard, *The multifractal nature of Lévy processes*, Probab. Theory Relat. Fields, **114** (1999), 207–227.

[16] S.-M. Ngai, *A dimension result arising from the L^p-spectrum of a measure*, Proc. Amer. Math. Soc., (1997) **125**, 2943–2951.

[17] L. Olsen, *A multifractal formalism*, Adv. in Math., (1995) **116**, 92–195.

[18] L. Olsen, *Dimension inequalities of multifractal Hausdorff measures and multifractal packing measures*, Math. Scand., **86, no. 1** (2000), 109–129.

[19] W. Parry and M. Policott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics*, Société Mathématique de France, Astérisque, **187–188** (1990), 268.

[20] Y. Pesin, H. Weiss, *The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples*, Chaos, **7(1)** (1997), 89–106.

[21] D. A. Rand, *The singularity spectrum $f(\alpha)$ for cookie-cutters*, Ergodic Th. Dyn. Sys., **9** (1989), 527–541.

[22] R. Riedi, *An improved multifractal formalism and self-similar measures*, J. Math. Anal. Appl., **189** (1995), 462–490.

[23] R. Riedi, *Multifractal processes*, Theory and applications of long-range dependence, 625–716 (2003), Birkhuser Boston, Boston, MA.

[24] R. Riedi, B. B. Mandelbrot, *Exceptions to the multifractal formalism for discontinuous measures*, Math. Proc. Cambridge Philos. Soc., **123** (1998), 133–157.

[25] L. S. Young, *Dimension, entropy and Lyapounov exponents*, Ergodic Th. Dyn. Sys., **2** (1982), 109–124.