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A GENERALIZATION OF THE DENSITY TOPOLOGY

Abstract

Wilczynski's definition of the Lebesgue density point given in [W1] opened the possibility of more subtle properties of the notion of density point and the density topology, their various modifications and most of all category analogues. In this paper we introduce a notion of an $\mathcal{A}_d-$density point of a measurable set on the real line. The notion is a generalization of Lebesgue density and is based on the definition given by Wilczyński. We prove that the $\mathcal{A}_d-$density topology generated by this notion is strictly finer than the Lebesgue density topology and we examine several of its properties.

Let S be the σ-algebra of Lebesgue measurable subsets of the real line $\mathbb{R}$ and I the σ-ideal of null sets. We shall say that the sets A, $B \in S$ are equivalent ($A \sim B$), if and only if $\lambda(A \triangle B) = 0$, where λ stands for Lebesgue measure on the real line. Recall that the point $x \in \mathbb{R}$ is a density point of a set $A \in S$, if and only if

$$\lim_{h \to 0} \frac{\lambda\left(A \cap [x-h, x+h]\right)}{2h} = 1$$

W. Wilczyński in [W1] introduced an equivalent definition of a density point of a set $A \in S$, in terms of convergence of characteristic functions of dilations of the set A.

A point $x \in \mathbb{R}$ is a density point of a set $A \in S$ if and only if every subsequence $\left\{\chi_{(n_m \cdot (A-x)) \cap [-1,1]}\right\}_{m \in \mathbb{N}}$ of $\left\{\chi_{(n \cdot (A-x)) \cap [-1,1]}\right\}_{n \in \mathbb{N}}$ contains a subsequence $\left\{\chi_{\left(n_{m_p} \cdot (A-x)\right) \cap [-1,1]}\right\}_{p \in \mathbb{N}}$ which converges to $\chi_{[-1,1]}$ I-almost everywhere on $[-1,1]$ (which means except on a set belonging to I).

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The above definition, as proved in [PWW2] (Corollary 1 p. 556), is equivalent to the following one (for detailed discussion see [W2] pp. 680–681).

A point $x \in \mathbb{R}$ is a density point of a set $A \in S$ if and only if, for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$, such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on $[-1,1]$ to $\chi_{[-1,1]}$.

Wilczynski's definition presented the opportunity for the study of more subtle properties of the notion of density point and the density topology, their various modifications and most of all category analogues (see [PWW1], [PWW2], [CLO]).

Recently the notions of simple density point and complete density point were introduced with associated $\mathcal{T}_s$ and $\mathcal{T}_c$ density topologies respectively, essentially different from the density topology $\mathcal{T}$ (see [AW] and [WW]). Actually, we have the following sequence of inclusions:

$$\mathcal{T}_n \subsetneqq \mathcal{T}_c \subsetneqq \mathcal{T}_s \subsetneqq \mathcal{T},$$

where $\mathcal{T}_n$ is the natural topology on the real line.

We shall follow this approach to consider some new generalization of density point, leading to a new density topology $\mathcal{T}_{\mathcal{A}_d}$ that extends the sequence of inclusions to

$$\mathcal{T}_n \subsetneqq \mathcal{T}_c \subsetneqq \mathcal{T}_s \subsetneqq \mathcal{T} \subsetneqq \mathcal{T}_{\mathcal{A}_d}.$$

We shall consider the following families of sets**:**

a) $\mathcal{A}_{[-1,1]}$— the family of subsets of the interval $[-1,1]$ of Lebesgue measure two,
b) $\mathcal{A}_{[-\alpha,\alpha]}$— the family of measurable subsets of interval $[-1,1]$ of full measure on interval $[-\alpha,\alpha]$, where $0 < \alpha \leq 1$,
c) $\mathcal{A}_d$— the family of measurable subsets of $[-1,1]$ that have Lebesgue density one at 0.

We have $\mathcal{A}_{[-1,1]} \subset \mathcal{A}_{[-\alpha,\alpha]} \subset \mathcal{A}_d$.

Definition 1. We shall say that x is an $\mathcal{A}_d$-density point of $A \in S$ if, for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there exists subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_d$, such that $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ converges I-almost everywhere on $[-1,1]$ to χ_B. (In other words, for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_d$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges on $[-1,1]$ in measure to χ_B).

By analogy, we define a notion of $\mathcal{A}_{[-\alpha,\alpha]}$-density point and $\mathcal{A}_{[-1,1]}$-density point of $A \in S$. The family $\mathcal{A}_{[-1,1]}$ corresponds precisely to the definition of a Lebesgue density point. The set of all $\mathcal{A}_d$-density points, $\mathcal{A}_{[-\alpha,\alpha]}$-density points and Lebesgue density points of $A \in S$ will be denoted by $\Phi_{\mathcal{A}_d}(A)$, $\Phi_{\mathcal{A}_{[-\alpha,\alpha]}}(A)$ and $\Phi(A)$, respectively.

Proposition 1. *For each $A \in S$, $\Phi(A) \subset \Phi_{\mathcal{A}_{[-\alpha,\alpha]}}(A) \subset \Phi_{\mathcal{A}_d}(A)$.*

PROOF. Obvious. □

Lemma 1. *Let $A \subset [0,1]$ be a measurable set, $\{a_n\}_{n\in\mathbb{N}}$ a sequence of positive numbers converging to 1, $a_n < \frac{3}{2}$. Then the sequence of characteristic functions $\{\chi_{a_n \cdot A}\}_{n\in\mathbb{N}}$ converges in measure to χ_A.*

PROOF. Let $\epsilon > 0$, let F be a closed set and let G be an open set such that $F \subset A \subset G$ and $\lambda(G \setminus F) < \frac{\epsilon}{6}$. The set G is a union of a family of open intervals that is an open cover of the compact set F. Then, there is a finite subcover of F with union H, such that $\lambda(F \triangle H) < \frac{\epsilon}{6}$. Since H is a finite union of open intervals, we can find an $n_0 \in \mathbb{N}$, such that for $n > n_0$,

$$\lambda((a_n \cdot H) \triangle H) < \frac{\epsilon}{6},$$

$$\begin{aligned}&\lambda((a_n \cdot A) \triangle (a_n \cdot H)) = \lambda(a_n \cdot (A \triangle H))) = a_n \lambda(A \triangle H)) \\ &< \frac{3}{2}\lambda(A \triangle H) \le \frac{3}{2}(\lambda(A \triangle F) + \lambda(F \triangle H)) \le \frac{3}{2}\left(\frac{\epsilon}{6} + \frac{\epsilon}{6}\right) = \frac{\epsilon}{2},\end{aligned}$$

and

$$\begin{aligned}&\lambda((a_n \cdot A) \triangle A) \\ &\le \lambda((a_n \cdot A) \triangle (a_n \cdot H)) + \lambda((a_n \cdot H) \triangle H) + \lambda(H \triangle F) + \lambda(F \triangle A) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon, \text{ for } n > n_0.\end{aligned}$$

Thus, the sequence $\{\chi_{a_n \cdot A}\}_{n\in\mathbb{N}}$ of characteristic functions converges in measure to χ_A. □

Lemma 2. *Let $A \subset [0,1]$ be a measurable set, $\{a_n\}_{n\in\mathbb{N}}$ a sequence of positive numbers converging to 1, $a_n < \frac{3}{2}$, and $\{A_n\}_{n\in\mathbb{N}}$ a sequence of measurable sets convergent in measure to A. Then the sequence $\{\chi_{a_n \cdot A_n}\}_{n\in\mathbb{N}}$ of characteristic functions converges in measure to χ_A.*

PROOF. The proof is a simple consequence of the above Lemma 1. Let $\epsilon > 0$. Since $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of measurable sets convergent in measure to A, we can find $n_1 \in \mathbb{N}$ such that $\lambda(A_n \triangle A) < \frac{\epsilon}{3}$, for $n > n_1$. By the Lemma 1, we can find $n_2 \in \mathbb{N}$ such that $\lambda((a_n \cdot A) \triangle A) \leq \frac{\epsilon}{2}$ for $n > n_2$. Take $n_0 = \max(n_1, n_2)$.

For $n > n_0$, we have

$$\begin{aligned}\lambda((a_n \cdot A_n) \triangle A) &\leq \lambda((a_n \cdot A_n) \triangle (a_n \cdot A)) + \lambda((a_n \cdot A) \triangle A) \\ &= \lambda(a_n \cdot (A_n \triangle A)) + \lambda((a_n \cdot A) \triangle A) \\ &= a_n \lambda((A_n \triangle A)) + \lambda((a_n \cdot A) \triangle A) \\ &\leq \frac{3}{2}\lambda((A_n \triangle A)) + \lambda((a_n \cdot A) \triangle A) < \frac{3}{2}\frac{\epsilon}{3} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus, the sequence $\{\chi_{a_n \cdot A_n}\}_{n\in\mathbb{N}}$ of characteristic functions converges in measure to χ_A. □

Proposition 2. *There exists a set A such that $\Phi(A) \subsetneqq \Phi_{\mathcal{A}_d}(A)$.*

PROOF. We start with the notion of the density from the right. We shall define a set A such that:
1) 0 is not a density point of A from the right,
2) 0 is not a density point of $\mathbb{R} \setminus A$ from the right,
3) 0 is a $\mathcal{A}_d$-density point of A from the right.
Let $D \in \mathcal{A}_d$ be such that $\lambda(D \cap (0,1)) < 1$ (for example the interval $\left(0, \frac{1}{2}\right)$); and $\{c_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, $c_1 < 1$, such that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. We define a set $A \in S$ as

$$A = \bigcup_{n=1}^{\infty} \left[(c_n \cdot D) \cap (c_{n+1}, c_n)\right].$$

Now let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. We can find subsequences $\{t_{n_r}\}_{r\in\mathbb{N}}$ and $\{c_{m_r}\}_{r\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$, respectively, such that $c_{m_r} \leq t_{n_r}$, and there are no elements of $\{t_n\}_{n\in\mathbb{N}}$ nor of $\{c_n\}_{n\in\mathbb{N}}$ between c_{m_r} and t_{n_r}.

Consider the sequence $\left\{c_{m_r} \cdot \frac{1}{t_{n_r}}\right\}_{r\in\mathbb{N}} \subset (0,1]$. We can find a convergent subsequence $\left\{c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\right\}_{k\in\mathbb{N}}$.

There are two possibilities.

a) $\lim_{k\to\infty}\left(c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\right) = a \neq 0$; i.e., $\lim_{k\to\infty}\left(c_{m_{r_k}} \cdot \frac{1}{a t_{n_{r_k}}}\right) = 1$.

In this case, by Lemma 2 $\left\{\chi_{\frac{c_{m_{r_k}}}{a t_{n_{r_k}}}\cdot\left(\frac{c_{m_{r_k}+1}}{c_{m_{r_k}}},1\right)}\right\}_{k\in\mathbb{N}}$ converges in measure to $\chi_{[0,1]}$, and $\left\{\chi_{\frac{c_{m_{r_k}}}{a t_{n_{r_k}}}\cdot\left[\left(\frac{c_{m_{r_k}+1}}{c_{m_{r_k}}},1\right)\cap D\right]}\right\}_{k\in\mathbb{N}}$ converges in measure to χ_D on $[0,1]$. Equivalently, $\left\{\chi_{\frac{1}{a t_{n_{r_k}}}\cdot\left[\left(c_{m_{r_k}+1},c_{m_{r_k}}\right)\cap\left(c_{m_{r_k}}\cdot D\right)\right]}\right\}_{k\in\mathbb{N}}$ converges in measure to χ_D on $[0,1]$. Thus, since

$$\left(c_{m_{r_k}+1}, c_{m_{r_k}}\right)\cap\left(c_{m_{r_k}}\cdot D\right)=\left(c_{m_{r_k}+1}, c_{m_{r_k}}\right)\cap A,$$

the sequence $\chi_{\left(\frac{1}{a\cdot t_{n_{r_k}}}\cdot A\right)\cap[0,1]}$ converges in measure to χ_D on $[0,1]$, and, consequently, $\chi_{\left(\frac{1}{t_{n_{r_k}}}\cdot A\right)\cap[0,a]}$ converges in measure to $\chi_{(a\cdot D)\cap[0,a]}$ on $[0,a]$.

We can find one more subsequence $\left\{c_{m_{r_{k_p}}}\cdot\frac{1}{t_{n_{r_{k_p}}}}\right\}_{p\in\mathbb{N}}$, such that $\chi_{\left(\frac{1}{t_{n_{r_{k_p}}}}\cdot A\right)\cap[0,a]}$ converges to $\chi_{(a\cdot D)\cap[0,a]}$ I-almost everywhere on $[0,a]$, and we obtain B on $[0,a]$ as

$$B\cap[0,a]=(a\cdot D)\cap[0,a]\,.$$

If $a=1$, the proof is complete; $B=D\cap[0,1]$ has density 1 at 0 from the right. If $a<1$, we have to determine B on $(a,1]$ as well. By definition of $\{c_n\}_{n\in\mathbb{N}}$, we have $\lim_{p\to\infty}\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_k}p}}=\infty$. on the other hand

$$\begin{aligned}\lim_{p\to\infty}\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}&=\lim_{p\to\infty}\left(\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}\cdot\frac{c_{m_{r_{k_p}}}}{c_{m_{r_{k_p}}}}\right)\\&=\lim_{p\to\infty}\left(\frac{c_{m_{r_{k_p}}-1}}{c_{m_{r_{k_p}}}}\cdot\frac{c_{m_{r_{k_p}}}}{t_{n_{r_{k_p}}}}\right)=a\cdot\lim_{p\to\infty}\frac{c_{m_{r_{k_p}}-1}}{c_{m_{r_{k_p}}}}=\infty.\end{aligned}$$

Hence, as D has density 1 at 0 from the right, we have

$$\lim_{p\to\infty}\lambda\left[\left(\frac{1}{t_{n_{r_{k_p}}}}\cdot A\right)\cap(a,1]\right]=1-a,$$

and we can find a subsequence $\left\{t_{n_{r_{k_{p_l}}}}\right\}_{l\in\mathbb{N}}$, such that $\chi_{\left(\frac{1}{t_{n_{r_{k_{p_l}}}}}\cdot A\right)\cap(a,1]}$ converges to $\chi_{(a,1]}$, I-almost everywhere on $(a,1]$, and we determine B on $[0,1]$ as

$$B\cap[0,a]=(a\cdot D)\cap[0,a]\ \text{ and } B\cap(a,1]=(a,1].$$

b) $\lim_{k\to\infty}\left(c_{m_{r_k}}\cdot\frac{1}{t_{n_{r_k}}}\right)=0$.

In this case, we have two possible situations again.

b1) The sequence $\left\{\frac{c_{m_{r_k}}-1}{t_{n_{r_k}}}\right\}_{k\in\mathbb{N}}$ is bounded from above.

We take a subsequence $\left\{\frac{c_{m_{r_{k_p}}}-1}{t_{n_{r_{k_p}}}}\right\}_{p\in\mathbb{N}}$, such that $\lim_{p\to\infty}\frac{c_{m_{r_{k_p}}}-1}{t_{n_{r_{k_p}}}}=b<\infty$, and proceed similarly to the argument in a). We have $b\geq 1$, and $\chi_{\left(\frac{1}{t_{n_{r_{k_p}}}}\cdot A\right)\cap[0,1]}$ converges in measure to $\chi_{(b\cdot D)\cap[0,1]}$, on $[0,1]$, and we can find a subsequence $\chi_{\left(\frac{1}{t_{n_{r_{k_{p_l}}}}}\cdot A\right)\cap[0,1]}$ convergent to $\chi_{(b\cdot D)\cap[0,1]}$ I-almost everywhere on $[0,b]$, and obtain B on $[0,1]$ as

$$B\cap[0,1]=(b\cdot D)\cap[0,1]\,.$$

b2) The sequence $\left\{\frac{c_{m_{r_k}}-1}{t_{n_{r_k}}}\right\}_{k\in\mathbb{N}}$ is not bounded from above.

We take a subsequence $\left\{\frac{c_{m_{r_{k_p}}}-1}{t_{n_{r_{k_p}}}}\right\}_{p\in\mathbb{N}}$ such that $\lim_{p\to\infty}\frac{c_{m_{r_{k_p}}}-1}{t_{n_{r_{k_p}}}}=\infty$. As D has density 1 at 0 from the right and $\lim_{k\to\infty}\left(c_{m_{r_k}}\cdot\frac{1}{t_{n_{r_k}}}\right)=0$, we have

$$\lim_{k\to\infty}\lambda\left[\left(\frac{1}{t_{n_{r_{k_p}}}}\cdot A\right)\cap(0,1]\right]=1,$$

and we can find a subsequence $\left\{t_{n_{r_{k_{p_l}}}}\right\}_{l\in N}$, such that $\chi_{\left(\frac{1}{t_{n_{r_{k_{p_l}}}}}\cdot A\right)\cap[0,1]}$ converges to $\chi_{[0,1]}$ I-almost everywhere on $[0,1]$, and we determine B on $[0,1]$ as

$$B\cap[0,1]=[0,1]$$

and B has density 1 at 0 from the right.

Finally, 0 is a $\Phi_{\mathcal{A}_d}$-density point of $-A\cup A$ but not a Φ-density point of $-A\cup A$ nor of $\mathbb{R}\setminus(-A\cup A)$. $\square$

Theorem 1. *The mapping $\Phi_{\mathcal{A}_d}:S\to 2^{\mathbb{R}}$ has the following properties:*
(1) For each $A\in S$, $\Phi_{\mathcal{A}_d}(A)\in S$.
(2) For each $A\in S$, $A\sim\Phi_{\mathcal{A}_d}(A)$.
(3) For each A,B $\in S$, if $A\sim B$, then $\Phi_{\mathcal{A}_d}(A)=\Phi_{\mathcal{A}_d}(B)$.
(4) $\Phi_{\mathcal{A}_d}(\emptyset)=\emptyset$, $\Phi_{\mathcal{A}_d}(\mathbb{R})=\mathbb{R}$.
(5) For each A,B $\in S$, $\Phi_{\mathcal{A}_d}(A\cap B)=\Phi_{\mathcal{A}_d}(A)\cap\Phi_{\mathcal{A}_d}(B)$.

Proof. (1) From Proposition 1, $\Phi_{\mathcal{A}_d}(A) = \Phi(A) \cup (\Phi_{\mathcal{A}_d}(A) \setminus \Phi(A))$. The set $(\Phi_{\mathcal{A}_d}(A) \setminus \Phi(A))$ is a subset of $\mathbb{R} \setminus ((\Phi(A) \cup \Phi(\mathbb{R} \setminus A)))$, a set of measure zero. Then $\Phi_{\mathcal{A}_d}(A)$ is an union of a measurable set $\Phi(A)$ and a null set; hence, a measurable set.

(2) is clear in view of $A \sim \Phi(A)$ and of the fact that $\Phi_{\mathcal{A}_d}(A)$ and $\Phi(A)$ differ by a null set.

(3) is a simple consequence of the fact that I-almost everywhere convergence is involved in the definition of $\Phi_{\mathcal{A}_d}(A)$.

(4) Obvious.

(5) Observe first that if $A \subset B$, $A, B \in S$, then $\Phi_{\mathcal{A}_d}(A) \subset \Phi_{\mathcal{A}_d}(B)$, so $\Phi_{\mathcal{A}_d}(A \cap B) \subset \Phi_{\mathcal{A}_d}(A) \cap \Phi_{\mathcal{A}_d}(B)$. To prove the opposite inclusion, assume $x \in \Phi_{\mathcal{A}_d}(A) \cap \Phi_{\mathcal{A}_d}(B)$. Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. From $x \in \Phi_{\mathcal{A}_d}(A)$, by definition, there exist a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $A_1 \in \mathcal{A}_d$, such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on $[-1,1]$ to $\chi_{A_1\cap[-1,1]}$. Similarly, for $\{t_{n_m}\}_{m\in\mathbb{N}}$ from $x \in \Phi_{\mathcal{A}_d}(B)$, by definition, there exists a subsequence $\left\{t_{n_{m_k}}\right\}_{k\in\mathbb{N}}$ and a set $B_1 \in \mathcal{A}_d$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot(A-x)\cap[-1,1]}\right\}_{k\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on $[-1,1]$ to $\chi_{B_1\cap[-1,1]}$. It is clear that the sequence $\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot((A\cap B)-x)\cap[-1,1]}\right\}_{k\in\mathbb{N}}$ converges I-almost everywhere on $[-1,1]$ to $\chi_{(A_1\cap B_1)\cap[-1,1]}$; i.e., x is a $\Phi_{\mathcal{A}_d}-$density point of $A \cap B$. □

Proposition 3. *If* 0 *is an* $\mathcal{A}_d$*-density point of a set* A*, then*

$$\liminf_{h\to 0^+} \frac{\lambda(A\cap[-h,0])}{h} > 0 \text{ and } \liminf_{h\to 0^+} \frac{\lambda(A\cap[0,h])}{h} > 0.$$

Proof. The assertion is a simple consequence of Definition 1. □

Remark 1. It is an immediate consequence of (1), (2) and (3) of Theorem 1 that $\Phi_{\mathcal{A}_d}$ is idempotent; i.e., $\Phi_{\mathcal{A}_d}(A) = \Phi_{\mathcal{A}_d}(\Phi_{\mathcal{A}_d}(A))$. We have also $\Phi_{\mathcal{A}_d}(A) \cap \Phi_{\mathcal{A}_d}(\mathbb{R} \setminus A) = \emptyset$.

Theorem 2. *The family* $\mathcal{T}_{\mathcal{A}_d} = \{A \in S : A \subset \Phi_{\mathcal{A}_d}(A)\}$ *is a stronger topology than the density topology* $\mathcal{T}$.

Proof. From Theorem 1, (4) $\emptyset$ and $\mathbb{R} \in \mathcal{T}_{\mathcal{A}_d}$, and the family is closed under finite intersections by (5). To prove that $\mathcal{T}_{\mathcal{A}_d}$ is closed under arbitrary unions, observe that from Theorem 1, $\Phi_{\mathcal{A}_d}(A) \setminus A$ is a null set for each $A \in S$, and then apply the proof in [W2]. Take a family $\{A_t\}_{t\in T} \subset \mathcal{T}_{\mathcal{A}_d}$. We have $A_t \subset \Phi_{\mathcal{A}_d}(A_t)$ for each t. Choose a sequence $\{t_n\}_{n\in\mathbb{N}}$, such that for each $t \in T$, we have $\lambda\left(A_t \setminus \bigcup_{n=1}^{\infty} A_{t_n}\right) = 0$. It is possible by the CCC property of (S, I). Then $\Phi_{\mathcal{A}_d}(A_t) = \Phi_{\mathcal{A}_d}\left(\left(A_t \cap \bigcup_{n=1}^{\infty} A_{t_n}\right) \cup \left(A_t - \bigcup_{n=1}^{\infty} A_{t_n}\right)\right) \subset \Phi_{\mathcal{A}_d}\left(\bigcup_{n=1}^{\infty} A_{t_n}\right)$, for each $t \in T$. Hence,

$$\bigcup_{n=1}^{\infty} A_{t_n} \subset \bigcup_{t\in T} A_t \subset \bigcup_{t\in T} \Phi_{\mathcal{A}_d}(A_t) \subset \Phi_{\mathcal{A}_d}\left(\bigcup_{n=1}^{\infty} A_{t_n}\right).$$

The first and the last set in the above sequence of inclusions differ by a null set and both are measurable, so $\bigcup_{t\in T} A_t \in S$. Also, $\bigcup_{t\in T} A_t \subset \Phi_{\mathcal{A}_d}\left(\bigcup_{t\in T} A_t\right)$, by central inclusion and the monotonicity of $\Phi_{\mathcal{A}_d}$, so finally $\bigcup_{t\in T}^{\infty} A_t \in \mathcal{T}_{\mathcal{A}_d}$.

The set $(-A \cup A) \cup \{0\}$, where A is defined in Proposition 2, with D additionally open, belongs to $\mathcal{T}_{\mathcal{A}_d}$ but not to $\mathcal{T}$ topology. $\square$

Remark 2. Like the density topology, the $\mathcal{A}_d$-density topology can be described in the form $\mathcal{T}_{\mathcal{A}_d} = \{\Phi_{\mathcal{A}_d}(A) \setminus P : A \in S \text{ and } P \in \mathcal{I}\}$, for if $A \subset \mathcal{T}_{\mathcal{A}_d}$, then $A \subset \Phi_{\mathcal{A}_d}(A)$. Consequently, $A = \Phi_{\mathcal{A}_d}(A) \setminus (\Phi_{\mathcal{A}_d}(A) \setminus A)$ and we take $P = \Phi_{\mathcal{A}_d}(A) \setminus A \in \mathcal{I}$. Now, if $B = \Phi_{\mathcal{A}_d}(A) \setminus P$, for some $A \in S$ and $P \in \mathcal{I}$, then

$$\begin{aligned}\Phi_{\mathcal{A}_d}(B) &= \Phi_{\mathcal{A}_d}(\Phi_{\mathcal{A}_d}(A) \setminus P) = \Phi_{\mathcal{A}_d}(\Phi_{\mathcal{A}_d}(A)) \\ &= \Phi_{\mathcal{A}_d}(A) \supset \Phi_{\mathcal{A}_d}(A) \setminus P = B\end{aligned}$$

from Theorem 1 (2), (3) and the above observation.

The $\mathcal{A}_d-$density topology has properties similar to the density topology.

Theorem 3. *For an arbitrary set $A \subset \mathbb{R}$*

$$\mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(A) = A \cap \Phi_{\mathcal{A}_d}(B),$$

where B is a measurable kernel of A.

PROOF. We can follow the proof of Theorem 2.5 from [W2], with Φ replaced by $\Phi_{\mathcal{A}_d}$. $\square$

Theorem 4. *A set $A \in \mathcal{T}_{\mathcal{A}_d}$ is $\mathcal{T}_{\mathcal{A}_d}$-regular open if and only if $A = \Phi_{\mathcal{A}_d}(A)$.*

PROOF. We can adopt here the proof of Theorem 2.6 from [W2]. The inclusion $\Phi_{\mathcal{A}_d}(A) \subset \mathrm{Cl}_{\mathcal{A}_d}(A)$, in the first part of the proof, can now be verified as follows.

$$\begin{aligned}Cl_{\mathcal{A}_d}(A) &= \mathbb{R} \setminus \mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(\mathbb{R} \setminus A) \\ &= \mathbb{R} \setminus ((\mathbb{R} \setminus A) \cap \Phi_{\mathcal{A}_d}(\mathbb{R} \setminus A)) \\ &= A \cup (\mathbb{R} \setminus \Phi_{\mathcal{A}_d}(\mathbb{R} \setminus A)) \supset A \cup \Phi_{\mathcal{A}_d}(A),\end{aligned}$$

since $\Phi_{\mathcal{A}_d}(A) \subset \mathbb{R} \setminus \Phi_{\mathcal{A}_d}(\mathbb{R} \setminus A)$. □

Theorem 5.

$$\begin{aligned}\mathcal{I} &= \{A \subset \mathbb{R} : A \text{ is } \mathcal{T}_{\mathcal{A}_d}\text{-nowhere dense set}\} \\ &= \{A \subset \mathbb{R} : A \text{ is } \mathcal{T}_{\mathcal{A}_d}\text{-first category set}\} \\ &= \{A \subset \mathbb{R} : A \text{ is } \mathcal{T}_{\mathcal{A}_d}\text{-closed } \mathcal{T}_{\mathcal{A}_d}\text{-discrete set}\}.\end{aligned}$$

PROOF. We can follow here the proof of Theorem 2.8 from [W2]. □

Theorem 6. *The σ-algebra of $\mathcal{T}_{\mathcal{A}_d}$-Borel sets coincides with S.*
If $E \subset \mathbb{R}$ is a $\mathcal{T}_{\mathcal{A}_d}$-compact set, then E is finite.
The space $(\mathbb{R}, \mathcal{T}_{\mathcal{A}_d})$ is neither first countable, nor second countable, nor Lindelöf, nor separable.
$(\mathbb{R}, \mathcal{T}_{\mathcal{A}_d})$ is a Baire space.

PROOF. We can follow here the proofs of Theorems 2.9-2.12 from [W2]. □

Remark 3. In the proofs of the above theorems, we used a classical argument referring only to the results for Lebesgue density topology from [W1]. However, since $\mathcal{T}_{\mathcal{A}_d} \subset S$ and $\Phi_{\mathcal{A}_d}$ is a closed lower density operator (i.e., $\Phi_{\mathcal{A}_d}(A) \in S$), we could rely on more recent results from [RJH] given in more general settings. Namely, the idempotency of the operator $\Phi_{\mathcal{A}_d}$ was shown to hold even for non-closed lower density operators on page 312 of [RJH]. The proof that $\mathcal{T}_{\mathcal{A}_d}$ is a topology was shown in a different way in Theorem 9 of [RJH]. Theorem 4 is Corollary 6.3 of [RJH] and Theorem 5 is Corollary 8.1 of [RJH]. Theorem 6 is a combination of results from Theorem 7 and Corollary 8.2 of [RJH].

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References

[AW] V. Aversa and W. Wilczyński, *Simple Density Topology*, Rend. Circ. Mat. Palermo, Serie II-Tomo, **LIII** (2004), 344–352.

[CLO] K. Ciesielski, L. Larson and K. Ostaszewski, *I-Density Continuous Functions,* Memoirs of Amer. Math. Soc., **107**, No 515 (1994).

[GNN] C. Goffman, C. J. Neugebauer, and T. Nishura, *The Density Topology and Approximate Continuity*, Duke Math. J., **28** (1961), 497–506.

[PWW1] W. Poreda , E. Wagner-Bojakowska and W. Wilczyński, *A Category Analogue of the Density Topology*, Fund. Math., **CXXV** (1985), 167–173.

[PWW2] W. Poreda, E. Wagner-Bojakowska and W. Wilczyński, *Remarks on $\mathcal{I}$-density and $\mathcal{I}$-approximately Continuous Functions*, Comm. Math. Univ. Carolinae, **26(3)** (1985), 553–563.

[RJH] D. A. Rose , D. Jankowić and T. R. Hamlett, *Lower Density Topologies*, Ann. New York Acad. Sci., Papers on General Topology and Applications, **704** (1993), 309–321.

[W1] W. Wilczyński, *A Generalization of the Density Topology*, Real. Anal. Exchange, **8(1)** (1982/83), 16–20.

[W2] W. Wilczyński, *Density Topologies*, Handbook of Measure Theory, E. Pap, ed., Elsevier (2002), 675–702.

[WW] W. Wilczyński and W. Wojdowski, *Complete Density Topology*, in preparation.