RESEARCH

Real Analysis Exchange
Vol. 32(2), 2006/2007, pp. 313–318

Aleksander Maliszewski, Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93–005 Łódź, Poland. email: `AMal@p.lodz.pl`

THE MAXIMAL CLASS WITH RESPECT TO MAXIMUMS FOR THE FAMILY OF ALMOST CONTINUOUS FUNCTIONS

Abstract

It is shown that a function $f\colon \mathbb{R} \to \mathbb{R}$ is Darboux and upper semicontinuous if and only if its maximum with each almost continuous function is almost continuous. This result generalizes an old theorem due to J. Farková.

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. Let $f\colon \mathbb{R} \to \mathbb{R}$. We say that f is *Darboux* if it maps intervals onto connected sets. We say that f is *almost continuous* in the sense of Stallings [7], if for every open set $U \subset \mathbb{R}^2$ containing f there is a continuous function $h\colon \mathbb{R} \to \mathbb{R}$ with $h \subset U$. (We make no distinction between a function and its graph.) Recall that almost continuous functions possess the Darboux property, and that the converse is not true [7]. However, in Baire class one these two notions coincide [1].

Denote by $\mathcal{M}$ the maximal class with respect to maximums for the family of almost continuous functions; i.e., let $\mathcal{M}$ consist of all functions f such that $\max\{f, g\}$ is almost continuous whenever g is so. It is well-known that $\mathcal{M}$ contains all continuous functions [5, Proposition 2] and that if $f \in \mathcal{M}$ is almost continuous, then f is upper semicontinuous [3]. (Recall that the class of Darboux upper semicontinuous functions is the maximal class with respect to maximums for the family of Darboux functions [2, Theorems 1 and 2].) T. Natkaniec asked for a characterization of $\mathcal{M}$ [5, Problem], [6, Problem 6.5]. Corollary 4 is the solution to this problem.

We start with a simple lemma.

Key Words: Darboux property, almost continuity, maximum of functions.
Mathematical Reviews subject classification: Primary: 26A21, 54C30; Secondary: 26A15, 54C08
Received by the editors February 12, 2006
Communicated by: Udayan B. Darji

Lemma 1. *Let $M \in \mathbb{R}$. Assume that a function $g\colon [a,b] \to (-\infty, M)$ is upper semicontinuous both at a and at b. Then there is a continuous function $\psi\colon [a,b] \to [\min\{g(a), g(b)\}, M]$ such that $\psi = g$ on $\{a,b\}$ and $\psi > g$ on (a,b).*

Proof. Let $\delta_0 = (b-a)/2$. For each $n \in \mathbb{N}$, find a $\delta_n \in (0, \delta_{n-1}/2)$ such that $g < g(a) + n^{-1}$ on $[a, a+\delta_n]$ and $g < g(b) + n^{-1}$ on $[b - \delta_n, b]$. Let

$$\psi(x) = \begin{cases} M & \text{if } x \in [a+\delta_1, b-\delta_1], \\ \min\bigl\{g(a) + n^{-1}, M\bigr\} & \text{if } x = a + \delta_{n+1},\ n \in \mathbb{N}, \\ \min\bigl\{g(b) + n^{-1}, M\bigr\} & \text{if } x = b - \delta_{n+1},\ n \in \mathbb{N}, \\ g(x) & \text{if } x \in \{a, b\}, \end{cases}$$

and let ψ be linear on the intervals $[a + \delta_{n+1}, a + \delta_n]$ and $[b - \delta_n, b - \delta_{n+1}]$ ($n \in \mathbb{N}$). One can easily see that ψ has all required properties. □

Theorem 2. *Suppose f is almost continuous and g is Darboux and upper semicontinuous. Then $\varphi = \max\{f, g\}$ is almost continuous.*

Proof. By [4] or [6, Corollary 2.2], it suffices to show that $\varphi\restriction[\alpha,\beta]$ is almost continuous whenever $\alpha < \beta$. Fix $\alpha < \beta$ and let $U \subset \mathbb{R}^2$ be an open set such that $\varphi\restriction[\alpha,\beta] \subset U$. Denote by $\mathfrak{J}$ the family of all compact (possibly degenerate) intervals $[a,b]$ for which there exists a continuous function $h\colon [a,b] \to \mathbb{R}$ such that $h \subset U$, $h = \varphi$ on $\{a,b\}$, and $h > g$ on (a,b).

The rest of the proof is divided into claims. The end of the proof of each claim is marked with a triangle $\triangleleft$.

Claim 1. For each $x \in [\alpha,\beta]$ and each $\varepsilon > 0$, there are $a, b \in (x - \varepsilon, x + \varepsilon)$ such that $a < x < b$ and $[a,b] \in \mathfrak{J}$.

Let $\delta \in (0, \varepsilon)$ be such that $\bigl((x-\delta, x+\delta) \times (\varphi(x) - \delta, \varphi(x) + 2\delta)\bigr) \subset U$. We consider two cases. If $f(x) > g(x)$, then choose $y \in (g(x), f(x))$, and let $I \subset (x - \delta, x + \delta)$ be an open interval such that $I \ni x$ and $g < y$ on I. Since f is almost continuous, there are $a, b \in I$ such that $a < x < b$ and $f(t) \in (\max\{y, f(x) - \delta\}, f(x) + \delta)$ for $t \in \{a,b\}$. Let $h = f$ on $\{a,b\}$ and let h be linear on $[a,b]$. Clearly h proves $[a,b] \in \mathfrak{J}$.

Now assume $f(x) \le g(x)$. There is an open interval $I \subset (x - \delta, x + \delta)$ such that $I \ni x$ and $g < g(x) + \delta$ on I. Since g is Darboux, there are $a, b \in I$ such that $a < x < b$ and $g > g(x) - \delta$ on $\{a,b\}$. Use Lemma 1 to construct a continuous function $\psi\colon [a,b] \to (g(x) - \delta, g(x) + \delta]$ such that $\psi = g$ on $\{a,b\}$ and $\psi > g$ on (a,b). Define the open set $\widetilde{U}$ by

$$\widetilde{U} = U \cup \bigl\{\langle t, y\rangle \in \mathbb{R}^2 \colon t \notin [a,b] \text{ or } y < \psi(t)\bigr\}.$$

If $t \in [a,b]$ and $f(t) \geq \psi(t)$, then

$$\langle t, f(t)\rangle = \langle t, \varphi(t)\rangle \in U \subset \widetilde{U}.$$

Thus, $f{\restriction}[a,b] \subset \widetilde{U}$. By [6, Lemma 6.2], we can find a continuous function $\widetilde{h}\colon [a,b] \to \mathbb{R}$ such that $\widetilde{h} \subset \widetilde{U}$ and $\widetilde{h} = f$ on $\{a,b\}$. Put $h = \max\bigl\{\psi, \widetilde{h}\bigr\}$. We will show that h proves $[a,b] \in \mathcal{J}$.

Indeed, let $t \in [a,b]$. If $h(t) \geq g(x) + \delta$, then $h(t) \geq \psi(t)$, whence

$$\langle t, h(t)\rangle = \langle t, \widetilde{h}(t)\rangle \in U.$$

(Recall that $\widetilde{h} \subset \widetilde{U}$.) On the other hand, if $h(t) < g(x) + \delta$, then

$$\langle t, h(t)\rangle \in \bigl(\{t\} \times (g(x) - \delta, g(x) + \delta)\bigr) \subset U.$$

(We used the fact that $h(t) \geq \psi(t) > g(x) - \delta$.) The conditions '$h = \varphi$ on $\{a,b\}$' and '$h > g$ on (a,b)' are evident. $\lhd$

Let $S = \bigl\{x \in (-\infty, \beta]\colon [x,\beta] \in \mathcal{J}\bigr\}$, and note that $\beta \in S$.

Claim 2. $\inf S < \alpha$.

Suppose $\bar{\alpha} = \inf S \geq \alpha$. By Claim 1, there are $a, b \in (-\infty, \beta)$ with $a < \bar{\alpha} < b$ such that $[a,b] \in \mathcal{J}$. We will show that $a \in S$, which is impossible.

Let $h_1\colon [a,b] \to \mathbb{R}$ correspond to $[a,b] \in \mathcal{J}$. Choose an $x \in S \cap [\bar{\alpha}, b)$, and let $h_2\colon [x,\beta] \to \mathbb{R}$ correspond to $[x,\beta] \in \mathcal{J}$. First assume that $h_1(b) > h_2(b)$. Define the open set $\widetilde{U}$ by

$$\widetilde{U} = U \cup \bigl\{\langle t, y\rangle \in \mathbb{R}^2\colon t \notin [b,\beta] \text{ or } y < h_2(t)\bigr\},$$

and observe that $f{\restriction}[b,\beta] \subset \widetilde{U}$. Construct a continuous function $\widetilde{h}\colon [b,\beta] \to \mathbb{R}$ such that $\widetilde{h} \subset \widetilde{U}$ and $\widetilde{h} = f$ on $\{x,b\}$. Let $h = h_1 \cup \max\bigl\{h_2{\restriction}[b,\beta], \widetilde{h}\bigr\}$. One can easily show that h proves $[a,\beta] \in \mathcal{J}$. (Cf. the argument used in Claim 1.)

The case $h_1(x) < h_2(x)$ is analogous. If neither of the above two cases holds, then $(h_1(x) - h_2(x))(h_1(b) - h_2(b)) \leq 0$, so $h_1(t) = h_2(t)$ for some $t \in [x,b]$. Put $h = h_1{\restriction}[a,t] \cup h_2{\restriction}[t,\beta]$. Clearly h proves that $[a,\beta] \in \mathcal{J}$. Consequently, $a \in S$. But $a < \bar{\alpha} = \inf S$, an impossibility. $\lhd$

By Claim 2, there exists a continuous function $h\colon [\alpha,\beta] \to \mathbb{R}$ such that $h \subset U$. Since U was arbitrary open neighborhood of $\varphi{\restriction}[\alpha,\beta]$, we conclude that $\varphi{\restriction}[\alpha,\beta]$ is almost continuous. Since $\alpha < \beta$ were arbitrary, φ is almost continuous as well. This completes the proof. $\square$

The following theorem, which is interesting by itself, is due to D. Preiss.

Theorem 3. *Let $g\colon \mathbb{R} \to \mathbb{R}$ and assume that the function $\varphi_c = \max\{c, g\}$ is almost continuous for each $c \in \mathbb{R}$. Then the function g is almost continuous.*

PROOF. Clearly g is Darboux. To show that g is almost continuous fix $\alpha < \beta$ and let $U \subset \mathbb{R}^2$ be an open set such that $g{\restriction}[\alpha, \beta] \subset U$. Denote by S the set of all $x \in [\alpha, \beta]$ for which there is a continuous function $h\colon [x, \beta] \to \mathbb{R}$ such that $h \subset U$ and $h = g$ on $\{x, \beta\}$. Observe that $\beta \in S$. Lry $\bar\alpha = \inf S$.

The rest of the proof is divided into claims. The end of the proof of each claim is marked with a triangle $\lhd$.

Claim 1. $\bar\alpha \in S$.

Let $c < \min\bigl\{g(\bar\alpha), g(\beta)\bigr\}$. Since the function φ_{c-1} is almost continuous, there is a continuous function $h_0\colon [\bar\alpha, \beta] \to \mathbb{R}$ such that $h_0 \subset U \cup \bigl(\mathbb{R} \times (-\infty, c)\bigr)$ and $h_0 = \varphi_{c-1} = g$ on $\{\bar\alpha, \beta\}$. Let $\delta \in (0, g(\bar\alpha) - c)$ be such that

$$\bigl((\bar\alpha - \delta, \bar\alpha + \delta) \times (g(\bar\alpha) - \delta, g(\bar\alpha) + \delta)\bigr) \subset U.$$

There is a $\tau \in \bigl(0, \min\{\delta, \beta - \bar\alpha\}\bigr)$ such that

$$|h_0(x) - g(\bar\alpha)| < \delta \text{ whenever } x \in [\bar\alpha, \bar\alpha + \tau].$$

Since g is Darboux, there is an $x_0 \in (\bar\alpha, \bar\alpha + \tau)$ with $|g(\bar\alpha) - g(x_0)| < \delta$. Take an arbitrary $x_1 \in S \cap (\bar\alpha, x_0)$ and let $h_1 \subset U$ correspond to $x_1 \in S$. We consider three cases.

If $h_1(x_1) \ge h_0(x_1)$, then find a continuous function $h_2\colon [\bar\alpha, x_1] \to \mathbb{R}$ with $h_2 \subset U \cup \bigl(\mathbb{R} \times (-\infty, c)\bigr)$ such that $h_2 = g$ on $\{\bar\alpha, x_1\}$. (Again we use the fact that the function φ_{c-1} is almost continuous.) Let

$$h = \max\{h_0{\restriction}[\bar\alpha, x_1], h_2\} \cup h_1.$$

Clearly the function h proves $\bar\alpha \in S$.

If $h_1(x_0) \le h_0(x_0)$, then choose a $c_1 < \min\bigl(h_1[[x_1, \beta]] \cup \{c\}\bigr)$. Use the fact that the function φ_{c_1-1} is almost continuous to find a continuous function $h_2\colon [x_0, \beta] \to \mathbb{R}$ such that $h_2 \subset U \cup \bigl(\mathbb{R} \times (-\infty, c_1)\bigr)$ and $h_2 = g$ on $\{x_0, \beta\}$. Let

$$h(x) = \begin{cases} \max\{h_1(x), h_2(x)\} & \text{if } x \in [x_0, \beta], \\ g(\bar\alpha) & \text{if } x = \bar\alpha, \\ \text{linear} & \text{on } [\bar\alpha, x_0]. \end{cases}$$

Note that

$$g(\bar\alpha) - \delta < g(x_0) = h_2(x_0) \le h(x_0) = \max\{h_1(x_0), g(x_0)\} < g(\bar\alpha) + \delta.$$

So, $h \subset U$ and the function h proves $\bar{\alpha} \in S$.

Finally if $h_1(x_1) < h_0(x_1)$ and $h_1(x_0) > h_0(x_0)$, then $h_1(x_2) = h_0(x_2)$ for some $x_2 \in (x_1, x_0)$. Let

$$h = h_0 \restriction [\bar{\alpha}, x_2] \cup h_1 \restriction [x_2, \beta].$$

Clearly the function h proves $\bar{\alpha} \in S$. $\triangleleft$

Claim 2. $\bar{\alpha} = \alpha$.

Indeed, suppose that $\bar{\alpha} > \alpha$. Let $h_1 \subset U$ correspond to $\bar{\alpha} \in S$. (Cf. Claim 1.) Let $\delta \in (0, \bar{\alpha} - \alpha)$ be such that

$$\big((\bar{\alpha} - \delta, \bar{\alpha} + \delta) \times (g(\bar{\alpha}) - \delta, g(\bar{\alpha}) + \delta)\big) \subset U.$$

Since g is Darboux, there is an $x_0 \in (\bar{\alpha} - \delta, \bar{\alpha})$ such that $|g(x_0) - g(\bar{\alpha})| < \delta$. Let

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in [\bar{\alpha}, \beta], \\ g(x_0) & \text{if } x = x_0, \\ \text{linear} & \text{on } [x_0, \bar{\alpha}]. \end{cases}$$

Then the function h proves $x_0 \in S$. But $x_0 < \bar{\alpha} = \inf S$, an impossibility. $\triangleleft$

By Claim 2, the restriction $g \restriction [\alpha, \beta]$ is almost continuous. Since $\alpha < \beta$ were arbitrary, the function g is almost continuous as well. $\square$

Corollary 4. *The family* $\mathcal{M}$ *coincides with the family of all Darboux upper semicontinuous functions.*

Proof. The inclusion '$\supset$' follows by Theorem 2.

To prove the opposite inclusion let $g \in \mathcal{M}$. Then by Theorem 3, the function g is almost continuous. So, by the results of [3], the function g is both Darboux and upper semicontinuous. $\square$

Acknowledgement. I would like to thank Prof. D. Preiss for Theorem 3, which contains a solution of a natural problem and which helped me to shorten the proof of Corollary 4.

References

[1] J. B. Brown, *Almost Continuous Darboux Functions and Reed's Pointwise Convergence Criteria*, Fund. Math., **86** (1974), 1–7.

[2] J. Farková, *About the Maximum and the Minimum of Darboux Functions*, Mat. Časopis Sloven. Akad. Vied, **21(2)** (1971), 110–116.

[3] J. M. Jastrzębski, J. M. Jędrzejewski and T. Natkaniec, *On Some Subclasses of Darboux Functions*, Fund. Math., **138** (1991), 165–173.

[4] K. R. Kellum, *Sums and Limits of Almost Continuous Functions*, Colloq. Math., **31** (1974), 125–128.

[5] T. Natkaniec, *On Lattices Generated by Darboux Functions*, Bull. Polish Acad. Sci. Math., **35(9–10)** (1987), 549–552.

[6] T. Natkaniec, *Almost Continuity*, Real Anal. Exchange, **17(2)** (1991–92), 462–520.

[7] J. Stallings, *Fixed Point Theorems for Connectivity Maps*, Fund. Math., **47** (1959), 249–263.