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DETERMINATION OF MEASURES

Abstract

This paper contains two results related to the question of when a measure on a metric space is determined by its value on certain subsets. The first is that two finite positive measures on a countable metric abelian group G which agree on all balls of some fixed non-zero radius agree on G. The second relates to measures on a compact metric space that agree on all intersections of pairs of balls.

1 Introduction

Let M denote a metric space and μ_1, μ_2 two (positive Borel) measures on M. A natural question is: for a given subset T of the set $\mathcal{B}(M)$ of all Borel sets of M are the following statements (1) or (2) true?

$$\mu_1(S) = \mu_2(S)\ \forall S \in T \Rightarrow \mu_1 = \mu_2 \tag{1}$$

$$\mu_1(S) = \mu_2(S)\ \forall S \in T \Rightarrow \mu_1(M) = \mu_2(M) \tag{2}$$

There are many results on this when T is the set O of all open balls, or the set of all balls of particular radii; see for example the survey by J.P.R. Christensen in [1]. Of course for any T, if the σ-class $\mathcal{D}(T)$ generated by T (i.e., the smallest subset of $\mathcal{B}(M)$ containing T and closed under complements and disjoint unions) is $\mathcal{B}(M)$, then (1) holds. In [3] Steve Jackson and R. Daniel Mauldin showed that if $M = \mathbb{R}^n$ with a metric induced from a norm, then $\mathcal{D}(O) = \mathcal{B}(M)$, and hence (1) holds. (This result was also shown for $M = \mathbb{R}^n$ with the Euclidean metric at about the same time by M. Zelený in [7].) However, T. Keleti and D. Preiss showed in [4] that if M is an arbitrary separable infinite-dimensional Hilbert space, then $\mathcal{D}(O) \neq \mathcal{B}(M)$, although D. Preiss and J. Tišer had previously shown in [6] that (1) nevertheless holds in this case (or in fact for M any separable Banach space).

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In Section 2 it is shown that if M is a countable metric abelian group, the measures are finite, and T is the set of all balls of a given fixed non-zero radius, then (2) holds; i.e., the value of the measure on the whole space is determined by its value on closed balls of a fixed non-zero radius.

In the negative direction, a well-known example of R.O. Davies [2] shows that (1) does not hold in general if M is a compact metric space and T is the set of all balls. One can ask what happens if T is extended to the set of all intersections of pairs of closed balls of fixed radius. In Section 3 it is shown that if M is a finite metric space, then in this case (2) holds. This does not answer the question above, but shows that the method of construction used in [2] does not extend directly, since this is based on finding finite metric spaces where (2) fails.

2 Measures on a Countable Abelian Group

Definition 2.1. A metric d on an abelian group G is invariant if for all $x, y, g \in G$, $d(x+g, y+g) = d(x, y)$. A measure μ on a topological abelian group G is invariant if $\mu(S) = \mu(S+g)$ for any Borel set S and any $g \in G$.

If G is countable, then the unique invariant measure is (a multiple of) counting measure.

Definition 2.2. An abelian group G has a slowly growing measure if there exists an invariant metric d on G and an invariant measure ν such that

$$\forall k > 1, \lim_{n\to\infty} \nu(B_n)/k^n = 0 \tag{3}$$

where $B_n = \{x \in G : d(0, x) \leq n\}$ is the closed ball of radius n.

Proposition 2.3. *Any countable abelian group has a slowly growing measure.*

Proof. The measure can be taken to be counting measure; so the claim is that there is an invariant metric on G such that the number of elements in balls of radius n grows fairly slowly, as specified by (3).

To define an invariant metric on G is equivalent to defining a function $f : G \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in G$ (a) $f(x) \geq 0$, (b) $f(x) = f(-x)$, (c) $f(x+y) \leq f(x) + f(y)$.

First prove the existence of a map f with the required properties for $G = \mathbb{Z}^\infty$, the direct sum of countably many copies of $\mathbb{Z}$. If $x = (x_i) \in \mathbb{Z}^\infty$, let $r(x) = \max\{i : x_i \neq 0\}$ and $|x| = \max\{|x_i|\}$. Now let

$$f(x) = \begin{cases} \max\{2^{r(x)-1}, |x|\} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Properties (a) and (b) are immediate and (c) clearly holds for $|x|$. Since $r(x+y) \leq \max\{r(x), r(y)\}$, (c) holds for any increasing function of $r(x)$, in particular for $2^{r(x)-1}$; hence (c) holds for f. Now using the metric defined by $f, |B_n| = (2n+1)^{\log_2 n+1}$, which satisfies (3) since $(\log n)^2 << n$.

Now if G is any countable abelian group, $G \cong \mathbb{Z}^\infty/I$ for some subgroup I of G. Let f be as above, and define $g : G \to \mathbb{R}_{\geq 0}$ by $g(y) = min\{f(x) : x \in G, I + x = y\}$. Since the metric on $\mathbb{Z}^\infty$ is discrete and hence I is a closed subset of $\mathbb{Z}^\infty$, this yields a well-defined invariant metric on G, and $|B_n(G)| \leq |B_n(\mathbb{Z}^\infty)|$ so counting measure has the required property with respect to this metric. □

Theorem 2.4. *Let G be a countable abelian group, μ_1, μ_2 be two (finite, positive) measures on G. Suppose that for some non-empty subset S of G, $\mu_1(g+S) = \mu_2(g+S)$ for all $g \in G$. Then $\mu_1(G) = \mu_2(G)$.*

PROOF. Let $\mu = \mu_1 - \mu_2$; so μ is a finite signed measure on G which is zero on all translates of S. Let d be the invariant metric on G so that the counting measure ν satisfies (3); all balls below are with respect to d, and B_n denotes the ball of radius n centre 0. Since μ_1 and μ_2 are finite, so is $|\mu|$, say $|\mu|(G) = K$. Given any $\epsilon > 0$, pick n such that $|\mu|(G - B_n) < \epsilon$.

Let $c_m = \sup_g\{\nu(S \cap (B_m + g))\}$. This exists since $\nu(S \cap (B_m + g)) \leq \nu(B_m)$ and this latter quantity is finite by (3). Let $\delta = \min\{\epsilon/K, 1\}, \alpha = \frac{1}{2n}$. If there do not exist arbitrarily large m such that $c_{m+n} \leq (1+\delta)c_{m-n}$, then there exists $J > 0$ such that for large m, $\nu(B_m) \geq c_m \geq J(1+\delta)^{\alpha m}$, contradicting (3). Hence there exist arbitrarily large m such that $c_{m+n} \leq (1+\delta)c_{m-n}$; let m be such an integer $> n$.

Now let $p \in G$ be such that $\nu(S \cap (B_{m-n} + p)) = c_{m-n}$. Replacing S by $S - p$ we have that $c_{m-n} = \nu(S \cap B_{m-n})$; i.e., 0 is the centre of one of the densest balls of radius $m - n$.

Now define $\omega : G \times G \to \mathbb{R}$ by $\omega(x,y) = \begin{cases} \mu(y) & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$. Let $D_r = \{(x,y) \in G \times G : y - x \in B_r\}$. While ω does not make sense as a signed measure on $G \times G$, it does on any D_r $(r > 0)$, since $\sum_{(x,y) \in D_r} |\omega(x,y)| = \sum_{g \in B_r} \sum_{x \in G} |\omega|(x, x+g) = \sum_{g \in B_r} \sum_{x \in S} |\mu|(x+g) \leq \sum_{g \in B_r} |\mu|(G) < \infty$. Now consider the following subsets of D_m : $R = B_{m-n} \times B_n$, $E = \{(x,y) \in D_m : y \notin B_n\}$, $F = \{(B_{m+n} - B_{m-n}) \times B_n\} \cap D_m$. Since D_m is equal to the disjoint union of R, E and F, $\omega(D_m) = \omega(R) + \omega(E) + \omega(F)$ and hence $|\omega(R)| \leq |\omega(E)| + |\omega(F)| + |\omega(D_m)|$. Now

$$\omega(D_m) = \sum_{g \in B_m} \sum_{x \in G} \omega(x, x+g) = \sum_{g \in B_m} \sum_{x \in S} \mu(x+g) = \sum_{g \in B_m} \mu(S+g) = 0$$

$$|\omega(E)| \le |\omega|(E) = \sum_{y \notin B_n} |\mu|(y)\nu\left(S \cap (B_m + y)\right) \le |\mu|(G - B_n)c_m \le \epsilon c_m$$
$$\le \epsilon c_{m+n} \le \epsilon(1+\delta)c_{m-n} \le 2\epsilon c_{m-n},$$

and

$$\begin{aligned} |\omega(F)| &\le |\omega|(F) \le |\omega|\left((B_{m+n} - B_{m-n}) \times B_n\right) \\ &= |\mu|(B_n)\nu\left(B_{m+n} \cap S - B_{m-n} \cap S\right) \le K\left(c_{m+n} - c_{m-n}\right) \\ &\le K\left[(1+\delta)\, c_{m-n} - c_{m-n}\right] = K\delta c_{m-n} \le K\frac{\epsilon}{K}c_{m-n} \le \epsilon c_{m-n} \end{aligned}$$

and

$$|\omega(R)| = \nu\left(S \cap B_{m-n}\right)|\mu(B_n)| = c_{m-n}|\mu(B_n)|.$$

Thus $|\mu(B_n)| \le 3\epsilon$ and hence $\mu(G) \le |\mu(B_n)| + |\mu(G - B_n)| \le 4\epsilon$. Since this is true for arbitrary ϵ, we have $\mu(G) = 0$; i.e., $\mu_1(G) = \mu_2(G)$. □

The next results follows immediately by taking S to be the ball of radius r at the origin.

Corollary 2.5. *Let G be a countable abelian group with invariant metric, and μ_1, μ_2 be two (finite, positive) measures on G which agree on all balls of some fixed non-zero radius r. Then $\mu_1(G) = \mu_2(G)$.*

3 Measures Agreeing on Intersections of Pairs of Balls

Here $B_r(x)$ denotes the closed ball of radius r, centre x.

Proposition 3.1. *Let M be a finite metric space, $r > 0$, and μ a signed measure on M such that $\mu(B_r(x) \cap B_r(y)) = 0$ for all $x, y \in M$. Then $\mu(M) = 0$.*

Proof. Write $M = \{x_1, \ldots, x_n\}$, $d_i = \mu(x_i)$ and define the symmetric matrix A and diagonal matrix D by $A_{ij} = \begin{cases} 1 & \text{if } d(x_i, x_j) \le r \\ 0 & \text{otherwise} \end{cases}$, $D_{ij} = \delta_{ij} d_i$. Then $(ADA)_{ij} = \sum_k a_{ik} d_k a_{kj} = \sum_k a_{ik} a_{jk} d_k = \mu\left(B_r(x_i) \cap B_r(x_j)\right) = 0$; so $(AD)^2 = 0$. Hence $tr(AD) = 0$; i.e., $\sum_i d_i = 0$ and so $\mu(M) = 0$. □

It would be interesting to know if Proposition 3.1 holds for a countable metric space. The problem is that the statement

$$B^2 = 0 \Rightarrow tr(B) = 0, \tag{4}$$

trivial for a finite matrix, does not in general hold for infinite matrices: a proof by A. M. Davie (see [5]) gives a random construction of a row-finite matrix B with square zero and non-zero trace. It is also noted in [5] that if the matrix B is row finite with $\sum_i (\max_j |b_{ij}|)^{2/3} < \infty$, then (4) does hold.

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