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THE QUASICONTINUITY OF DELTA-FINE FUNCTIONS

1 Introduction

We wish to take advantage of the exploratory nature of the Inroads section to report on progress toward answering a question we posed with Richard O'Malley in [3]. There we noted the difficulty we were having trying to find an effective characterization of the class UPA of *universally polygonally approximable* functions. While several related subclasses of Baire one functions have aesthetically pleasing characterizations, UPA strikes us as more elusive.

One difficulty is that it is not closed in the sup metric [3]. Thus, if one is looking for a geometric characterization, one should perhaps look, instead, for a characterization of its closure, $\overline{\text{UPA}}$. In [3] we defined the class DF of delta-fine functions, which is closed, showed that UPA $\subseteq$ DF, but were unable to determine if $\overline{\text{UPA}} = \text{DF}$.

Although the theorem presented in this paper doesn't decide this question, it does provide additional insight into the similarity of these two function classes. In [2] we showed that the set of points at which a UPA function fails to be quasicontinuous is very small in the sense of porosity; indeed, it was shown to be σ-$(1-\epsilon)$-symmetrically porous for every $\epsilon > 0$. (In [1] we examined how tantalizingly close this result is to being sharp.) Here we show that the same exceptional behavior is true for the class DF; that is, for every $\epsilon > 0$ the set of nonquasicontinuity points for a delta-fine function is σ-$(1-\epsilon)$-symmetrically porous.

Although there are obvious similarities between the proof presented here and the UPA case proved in [2], the proofs differ at critical points. The main

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point of this note then is to remark that one cannot distinguish these classes by the nature of their exceptional sets of quasicontinuity.

2 Definitions and Notation

Definition 1. Let $f : [0,1] \to \mathbb{R}$.

a) We say that a function $h : [0,1] \to \mathbb{R}$ is a *polygonal function for* f if there is a partition $\tau = \{0 = a_0 < a_1 < a_2 < \cdots < a_m = 1\}$ such that h agrees with f at each partition point and is linear on the intervening closed intervals. We call $a_0, a_1, \ldots, a_m$ the *nodes of* h and $(a_0, h(a_0)), (a_1, h(a_1)), \ldots, (a_m, h(a_m))$ the *vertices of* h. The maximum distance between adjacent nodes is called the *mesh of* h.

b) We say that a sequence $\{h_n\}$ of polygonal functions for f *polygonally approximates* f provided both $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in [0,1]$ and $\lim_{n\to\infty} mesh(h_n) = 0$. In this case we say that f is polygonally approximable. Further, if all the nodes of the polygonal functions, other than 0 and 1, belong to the set of points of continuity, $C(f)$, we say that $\{h_n\}$ $C(f)$*-polygonally approximates* f and that f is *universally polygonally approximable.* The collection of all universally polygonally approximable functions is *UPA*.

As noted in [3], if $f \in$ UPA in the above sense, then given *any* dense subset D in $[0,1]$, it is possible to construct a sequence $\{h_n\}$ of polygonal functions for f which polygonally approximates f and has the property that all nodes of each h_n lie in $D \cup \{0,1\}$. This is the reason for using the word *universal.*

If $f : [0,1] \to \mathbb{R}$ and $\tau = \{a = a_0 < a_1 < a_2 < \cdots < a_n = b\}$ is a partition of $[a,b]$, we let $P_{f,\tau,[a,b]}$ denote that function which agrees with f at each point of τ and is linear on the intervening intervals. We omit the subscripts f and $[a,b]$ if the function f and interval $[a,b]$ are understood.

Definition 2. A function $f : [0,1] \to \mathbb{R}$ is said to have the *delta-fine property* if for each closed set W and each $\epsilon > 0$, there are two points $a < b$ in $C(f)$ such that $[a,b] \cap W \neq \emptyset$ and such that for every $\delta > 0$ there is a δ-fine partition $\tau = \{a = a_0 < a_1 < a_2 < \cdots < a_n = b\}$ of $[a,b]$ consisting of points in $C(f)$ such that $|P_{f,\tau,[a,b]}(x) - f(x)| < \epsilon$ for every $x \in W \cap [a,b]$. We let *DF* denote the class of all functions having the delta-fine property.

Definition 3. A function $f : [0,1] \to \mathbb{R}$ is *quasi-continuous at* x if every neighborhood of $(x, f(x))$ contains a point of $f|C(f)$. We let $Q(f)$ denote the set of points of quasi-continuity of f and $NQ(f) = [0,1] \backslash Q(f)$. If $Q(f) = [0,1]$, we say that f is a *quasi-continuous function* and we let QC denote the class of all such functions.

3 The Result

Theorem 1. *Let $f \in DF$. Then, for every $\epsilon > 0$, $NQ(f) = \bigcup_{i=1}^{\infty} E_i$ where each E_i is closed and $(1-\epsilon)$-symmetrically porous.*

PROOF. Let $\epsilon > 0$ be given and without loss of generality assume that $\epsilon < \frac{1}{2}$. Throughout this proof the numbers s, t, r_1, r_2 are assumed to be rational. Suppose $s < t$, set $\delta^* = (t-s)/2$, and define

$$E_{st} = \{x_o \in NQ(f) : \forall x \in C(f) \cap (x_o - \delta^*, x_o + \delta^*),\ f(x) \geq t \text{ or } f(x) \leq s\}.$$

Then $E_{s,t}$ is closed for each $s < t$, and $NQ(f)$ is clearly the union of the E_{st}'s. Now, fix $s < t$. Then for r_1 and r_2 satisfying

$$s < r_1 < r_2 < t, \text{ and } r_2 - r_1 < \frac{\epsilon \cdot \min(r_1 - s, t - r_2)}{6}$$

define $D_{r_1,r_2} = f^{-1}((r_1, r_2)) \cap E_{s,t}$. Then E_{st} can clearly be expressed as a countable union of the D_{r_1,r_2}'s. Fix such a pair r_1, r_2. As f belongs to Baire Class 1 and $E_{s,t}$ is closed, $D_{r_1,r_2} = \bigcup_{i=1}^{\infty} F_i$ where each F_i is closed. Let F denote one of the F_i's. As $f \in DF$, there is a portion, $(a,b) \cap F$, of F so that $\forall \delta > 0$, there is a partition, $\tau \equiv \tau_\delta$ of $[a,b]$ such that

$$|P_\tau(x) - f(x)| < \frac{r_2 - r_1}{2}, \tag{1}$$

whenever $x \in (a,b) \cap F$. Now, fix $0 < \delta < \delta^*$, suppose $x_0 \in F$, and let $y \equiv y_\delta < x_o < z_\delta \equiv z$ be the nodes of P_τ which span x_o. Then, $y, z \in C(f)$ so that $f(y), f(z) \notin (s,t)$. For definiteness, we suppose that $f(y) \leq s$ and $f(z) \geq t$. Since P_τ approximates f at x_o and $r_1 < f(x_o) < r_2$, the only other case is that where $f(z) \leq s$ and $f(y) \geq t$ which is similar to the case considered here. We also assume $t - r_2 \leq r_1 - s$. Let $t^* = t - 2(r_2 - r_1)$ and $s^* = 2P_\tau(x_o) - t^*$. Then,

$$\begin{aligned} s^* &> 2(r_1 - \frac{r_2 - r_1}{2}) - t^* = 3r_1 - r_2 - t^* \\ &= r_1 - (t - r_2) \geq r_1 - (r_1 - s) = s. \end{aligned}$$

Also,

$$r_2 - r_1 < \frac{\epsilon(t - r_2)}{6} = \frac{\epsilon}{6}[(t^* - r_2) + 2(r_2 - r_1)]$$

and thus,

$$r_2 - r_1 < \frac{\frac{\epsilon}{6}}{1 - \frac{2\epsilon}{6}}(t^* - r_2) = \frac{\epsilon}{6 - 2\epsilon}(t^* - r_2) < \frac{\epsilon}{5}(t^* - r_2) \text{ since } \epsilon < \frac{1}{2}.$$

Moreover,

$$\begin{aligned} r_1 - s^* &> r_1 - (2(r_2 + \frac{r_2 - r_1}{2}) - t^*) = (t^* - r_2) - 2(r_2 - r_1) \\ &> \frac{5}{\epsilon}(r_2 - r_1) - 2(r_2 - r_1) = (\frac{5 - 2\epsilon}{\epsilon})(r_2 - r_1) > \frac{4}{\epsilon}(r_2 - r_1), \end{aligned}$$

again using the fact that $\epsilon < \frac{1}{2}$. Hence,

$$r_2 - r_1 < \frac{\epsilon}{4}\min(t^* - r_2, r_1 - s^*). \tag{2}$$

Now, set $y' = P_\tau^{-1}(2r_1 - r_2)$ and suppose $x \in (y, y')$. Then, $P_\tau(x) = L_{yz}(x) < 2r_1 - r_2$. But if $x \in F$, $|P_\tau(x) - f(x)| < \frac{r_2 - r_1}{2}$ (using (1)) and as $P_\tau(x) < 2r_1 - r_2$,

$$f(x) \le P_\tau(x) + |f(x) - P_\tau(x)| < 2r_2 - r_1 + \frac{r_2 - r_1}{2} < r_1.$$

This, however, contradicts the fact that $F \subseteq D_{r_1 r_2}$. Hence, $x \notin F$ and it follows that $F \cap (y, y') = \emptyset$. In a completely analogous manner, one shows that $F \cap (z', z) = \emptyset$ where $z' = P_\tau^{-1}(2r_2 - r_1)$. The remainder of the proof is devoted to using the intervals (y, y') and (z', z) to compute a symmetric porosity ratio at x_o.

To this end, set $y_o = P_\tau^{-1}(s^*)$and $z_o = P_\tau^{-1}(t^*)$. Then $(y_o, y') \subseteq (y, y')$, $(z', z_o) \subseteq (z', z)$, and the intervals (y_o, y') and (z', z_o) are symmetric about x_o. Using similar triangles, we note that

$$\begin{aligned} \frac{y' - y_o}{x_o - y_o} &= \frac{P_\tau(y') - P_\tau(y_o)}{P_\tau(x_o) - P_\tau(y_o)} = \frac{2r_1 - r_2 - s^*}{P_\tau(x_o) - s^*} > \frac{2r_1 - r_2 - s^*}{r_2 + \frac{r_1 + r_2}{2} - s^*} \\ &= \frac{1 - \frac{r_2 - r_1}{r_1 - s^*}}{1 + \frac{3}{2}\frac{r_2 - r_1}{r_1 - s^*}} > \frac{1 - \frac{\epsilon}{4}}{1 + \frac{3\epsilon}{8}} \text{ using (2)} \\ &> 1 - \frac{5\epsilon}{8} > 1 - \epsilon. \end{aligned}$$

Since this holds for all sufficiently small δ, it follows that the symmetric porosity of F at x_o, $sp(F, x_o)$, is at least $1 - \epsilon$ and since x_o was an arbitrary point in $(a, b) \cap F$, we have that for all $x \in (a, b) \cap F$, $sp(F, x) \ge 1 - \epsilon$.

As a consequence, we can conclude that there is a dense open set $U \cap F$ of F having the property that for each $x \in U \cap F$, $sp(F,x) \geq 1-\epsilon$. Set $G^1 = U \cap F$ and $F^1 = F \backslash G^1$. As $f \in DF$, we can repeat the argument above with F replaced by F^1 to find a relatively open set $G^2 = U^2 \cap F^1$ and a closed set $F^2 = F^1 \backslash G^2$. Continuing in this manner we construct a descending (possibly transfinite) sequence of closed sets, $\{F^n\}$, each of which is $(1-\epsilon)$-symmetrically porous. As any such decreasing sequence is at most countable, the theorem obtains.

□

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