Real Analysis Exchange
Vol. 28(2), 2002/2003, pp. 471–475

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DARBOUX SYMMETRICALLY CONTINUOUS FUNCTIONS

Abstract

For a symmetrically continuous function $f : \mathbb{R} \to [0, 1]$, a reduction formula is obtained which gives a Darboux symmetrically continuous function $g_f : \mathbb{R} \to [0, 1]$ such that the set $C(f)$ of continuity points of f is a subset of $C(g_f)$. Under additional conditions, g_f and the oscillation function ω_f of f are Croft-like functions. One consequence of g_f being Darboux is that the absolutely convergent values $s(x)$ of a real trigonometric series $\sum_{n=1}^{\infty} \rho_n \sin(nx + x_n)$, with $\sum_{n=1}^{\infty} |\rho_n| = \infty$ and with an uncountable set E of points of absolute convergence, almost has the intermediate value property except for countably many values $s(x)$ and countably many points of E.

A function $f : \mathbb{R} \to \mathbb{R}$ is *symmetrically continuous* if for each $x \in \mathbb{R}$, $\lim_{h\to 0}(f(x+h) - f(x-h)) = 0$. The Stein-Zygmund and Pesin-Preiss Theorems [5] state that a symmetrically continuous function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and its set $D(f)$ of discontinuities is an F_σ set of measure zero. Also, the Denjoy-Luzin Theorem [6] states that the set E of points of absolute convergence of a real trigonometric series $\sum_{n=1}^{\infty} \rho_n \sin(nx + x_n)$, with $\sum_{n=1}^{\infty} |\rho_n| = \infty$, is an F_σ set of measure 0. Furthermore, according to Preiss [4], the function $f : \mathbb{R} \to [0, 1]$ defined by

$$f(x) = \lim_{m\to\infty} (1 + \sum_{n=1}^{m} |\rho_n \sin(nx + x_n)|)^{-1}$$

is upper semicontinuous, symmetrically continuous, $C(f) = \mathbb{R} \setminus E = f^{-1}(0)$, and E can be uncountable. We redefine this f at countably many points to get a Darboux function g_f with these same properties of f. So g_f is just like the Croft function on [0,1] described in [1] in that they both are Darboux, Baire class 1, and equal 0 a.e. but not everywhere. However, Croft's function is not symmetrically continuous.

Key Words: symmetric continuity, Darboux function, trigonometric series
Mathematical Reviews subject classification: 26A15
Received by the editors October 8, 2002

A *Darboux* function maps connected sets to connected sets. Suppose $B \subset E$ and $C \subset R$. A function $f : E \to \mathbb{R}$ has the *intermediate value property relative to* $(E \setminus B) \times (\mathbb{R} \setminus C)$ if whenever $a, b \in E \setminus B, a < b$, and the number $y \in \mathbb{R} \setminus C$ lies between $f(a)$ and $f(b)$, then there exists $x \in (E \setminus B) \cap (a, b)$ such that $f(x) = y$. A point $(x, y) \in \mathbb{R}^2$ is a *bilateral limit ($\mathfrak{c}$-limit) point* of the graph of $f : \mathbb{R} \to \mathbb{R}$ if for every open neighborhood U of (x, y), both $((-\infty, x) \times \mathbb{R}) \cap U \cap f$ and $((x, \infty) \times \mathbb{R}) \cap U \cap f$ are infinite (have cardinality $\mathfrak{c}$). The graph of f is *bilaterally dense ($\mathfrak{c}$-dense) in itself* if every point $(x, f(x))$ is a bilateral limit ($\mathfrak{c}$-limit) point of f.

Given $f : \mathbb{R} \to \mathbb{R}$, define the set $B_f = \{x \in \mathbb{R} : (x, f(x))$ is not a bilateral $\mathfrak{c}$-limit point of $f\}$. If f is bounded, define $g_f : \mathbb{R} \to \mathbb{R}$ by $g_f(x) = \sup\{z : (x, z)$ is a bilateral $\mathfrak{c}$-limit point of $f\}$ and denote the oscillation of f at x by $\omega_f(x) = \limsup_{h \to 0+}\{|f(y) - f(z)| : y, z \in (x - h, x + h)\}$.

Lemma 1. (Maliszewski [3]) *If* $f : \mathbb{R} \to \mathbb{R}$, *then* $\mathrm{card}(B_f) < \mathfrak{c}$. *If* f *is also Lebesgue measurable (Borel measurable), then* B_f *has measure 0 (is countable).*

Theorem 1. *If* $f : \mathbb{R} \to [0, 1]$ *is symmetrically continuous, then* g_f *is upper semicontinuous, Darboux, symmetrically continuous, and* $C(f) \subset C(g_f)$. *Moreover, if* $D(f) = \mathbb{R} \setminus f^{-1}(0)$, *then* $C(g_f) = g_f^{-1}(0)$ *and* $\mathrm{card}(D(g_f)) = \mathfrak{c}$ *whenever* $\mathrm{card}(D(f)) = \mathfrak{c}$.

PROOF. According to Lemma 1, B_f had cardinality $< \mathfrak{c}$, and therefore $g_f(x)$ exists for each $x \in \mathbb{R}$ because f is bounded and symmetrically continuous. Notice that $g_f \geq f$ on $\mathbb{R} \setminus B_f$ and $C(f) \subset C(g_f)$. So if $D(f) = \mathbb{R} \setminus f^{-1}(0)$, then $C(g_f) = g_f^{-1}(0)$ and $\mathrm{card}(D(g_f)) = \mathfrak{c}$ whenever $\mathrm{card}(D(f)) = \mathfrak{c}$. Because g_f is a sup and f is symmetrically continuous, g_f is upper semicontinuous. The graph of g_f is bilaterally $\mathfrak{c}$-dense in itself because B_f has cardinality $< \mathfrak{c}$. Therefore by [2], every Baire class 1, bilaterally dense in itself function, which g_f is, must have a connected graph and so must be Darboux.

To see that g_f is symmetrically continuous, let $\varepsilon > 0$ and $x \in \mathbb{R}$. There exists $\delta > 0$ such that if $0 < h < \delta$, then $|f(x + h) - f(x - h)| < \frac{\varepsilon}{3}$. We may as well suppose $g_f(x - h) < g_f(x + h)$. Since $(x + h, g_f(x + h))$ is a bilateral $\mathfrak{c}$-limit point of f, there exists a sequence $f(x + h_n) \to g_f(x + h)$ as $h_n \to h$ such that $x \pm h_n \in \mathbb{R} \setminus B_f$. Since f is symmetrically continuous at x, $\{f(x - h_n)\}$ has a subsequence $\{f(x - h_{i_n})\}$ converging to some $z \leq g_f(x - h)$ such that $(x - h, z)$ is a bilateral $\mathfrak{c}$-limit point of f. Therefore for h_{i_n} close enough to h with $h_{i_n} < \delta$, $|f(x - h_{i_n}) - z| < \frac{\varepsilon}{3}$ and $|g_f(x + h) - f(x + h_{i_n})| < \frac{\varepsilon}{3}$. Since $z \leq g_f(x - h) < g_f(x + h)$,

$$\begin{aligned} |g_f(x + h) - g_f(x - h)| \leq &|g_f(x + h) - z| \leq |g_f(x + h) - f(x + h_{i_n})| \\ &+ |f(x + h_{i_n}) - f(x - h_{i_n})| + |f(x - h_{i_n}) - z| < \varepsilon. \end{aligned}$$

$\square$

Apply Lemma 1 and Theorem 1 to Preiss' usc function f and observe $f = g_f$ on $\mathbb{R} \setminus B_f$ because f is usc order to obtain immediately the first corollary. The second corollary follows from $\sum_{n=1}^{\infty} |\rho_n \sin(nx+x_n)| = \frac{1}{f(x)} - 1$ for $f(x) \in (0,1]$.

Corollary 1. *If $f(x) = \lim_{m\to\infty}(1+\sum_{n=1}^{m} |\rho_n \sin(nx+x_n)|)^{-1}$, where $\sum\limits_{n=1}^{\infty} |\rho_n|$ $= \infty$ and $D(f)$ is uncountable, then g_f is an upper semicontinuous, symmetrically continuous, and Darboux 2π-periodic function with $D(g_f)$ uncountable and $C(g_f) = g_f^{-1}(0)$. Moreover, f has the intermediate value property relative to$(\mathbb{R} \setminus B_f) \times (\mathbb{R} \setminus g_f(B_f))$, where B_f is countable.*

Corollary 2. *Let B be the set of all $x \in \mathbb{R}$ such that*

$$s(x) = \sum_{n=1}^{\infty} |\rho_n \sin(nx + x_n)| < \infty$$

and such that the graph of s does not have a bilateral $\mathfrak{c}$-limit point at $(x, s(x))$. If $\sum_{n=1}^{\infty} |\rho_n| = \infty$ and the set E of points of absolute convergence of the real trigonometric series $\sum_{n=1}^{\infty} \rho_n \sin(nx + x_n)$ is uncountable, then s has the intermediate value property relative to $(E \setminus B) \times \big(\mathbb{R} \setminus (\frac{1}{g_f} - 1)(B)\big)$, where B is countable and f is as in Corollary 1.

A result for convergence instead of absolute convergence can be found in [6], Thm 2.20, p.323. If $\sum_{k=1}^{n} k\rho_k = o(n)$, then the set E_0 of points of convergence of $\sum_{n=1}^{\infty} \rho_n \sin(nx+x_n)$ has cardinality $\mathfrak{c}$ in every interval and its sum is Darboux with respect to $E_0 \times \mathbb{R}$. Observe that $s(x) = \sum_{n=1}^{\infty} \frac{1}{n} |\sin 2^n x|$ is bilaterally dense in itself at each dyadic rational $\frac{p}{2^q}$ times π. In particular, as $x_n = \frac{\pi}{2^n} \to 0$, then for $n > 1$,

$$s(x_n) = \sum_{k=1}^{\infty} \frac{1}{k} |\sin 2^k x_n| = \sum_{k=1}^{n-1} \frac{1}{n-k} \sin\frac{\pi}{2^k} < \sum_{k=1}^{n-1} \frac{\pi}{(n-k)2^k} < \frac{\pi \sum\limits_{k=1}^{2^n} k\rho_k}{2^n} \to 0.$$

According to a reduction theorem in [5], Cor. 2.6, if $f : \mathbb{R} \to [0,1]$ is symmetrically continuous, then the oscillation $\omega_f : \mathbb{R} \to [0,1]$ is upper semicontinuous, symmetrically continuous, and f is continuous exactly at the points $\omega_f^{-1}(0)$ where ω_f is continuous. Alternatively, in lieu of this reduction theorem, Theorem 1 can be used at the end of the proof of the Pesin-Preiss Theorem in [5] to show that a symmetrically continuous function $f : \mathbb{R} \to [0,1]$

continuous on a dense set is measurable because f is continuous a.e. on account of $C(f) \subset C(g_f)$.

For a symmetrically continuous function $f : \mathbb{R} \to [0,1]$, ω_f is symmetrically continuous by the above reduction theorem. Thus by Theorem 1, g_{w_f} is upper semicontinuous, symmetrically continuous, and Darboux , but f might be discontinuous at only countably many of the points $g_{w_f}^{-1}(0)$ where g_{w_f} is continuous. Also, f can be symmetrically continuous and bilaterally $\mathfrak{c}$-dense in itself, yet ω_f not be Darboux. The example

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

illustrates both situations. However, if $f : \mathbb{R} \to [0,1]$ is symmetrically continuous, bilaterally $\mathfrak{c}$-dense in itself and $C(f) \subset f^{-1}(0)$, then $\omega_f = g_f$, which is Darboux by Theorem 1.

Theorem 2. *If $f : \mathbb{R} \to [0,1]$ is symmetrically continuous, just bilaterally dense in itself, and $C(f) \subset f^{-1}(0)$, then ω_f is Darboux. Consequently, if $f^{-1}(0) \neq \mathbb{R}$, then ω_f acts like Croft's function.*

PROOF. By the reduction theorem in [5], ω_f is upper semicontinuous, symmetrically continuous and $C(\omega_f) = \omega_f^{-1}(0) = C(f) \subset f^{-1}(0)$. Since f is symmetrically continuous, $C(f)$ is dense in $\mathbb{R}$ [5]. To see that the graph of ω_f is bilaterally dense in itself, let $x \in \mathbb{R}$ and $\varepsilon > 0$. Because $C(f) \subset f^{-1}(0)$ and $C(f)$ is dense in $\mathbb{R}$,

$$\begin{aligned} \omega_f(x) &= \lim_{h\to 0+} \sup\{|f(y) - f(z)| : y, z \in (x-h, x+h)\} \\ &= \lim_{h\to 0+} \sup\{f(y) : y \in (x-h, x+h)\} \geq f(x). \end{aligned}$$

Since f is symmetrically continuous at x and ω_f is upper semicontinuous, there exists $\delta > 0$ such that: if $0 < h < \delta$, then $|f(x+h) - f(x-h)| < \varepsilon$ and if $|x-y| < \delta$, then $\omega_f(y) < \omega_f(x) + \varepsilon$. Then, since f is bilaterally dense in itself and $C(f) \subset f^{-1}(0)$, for every $0 < h < \delta$ there exist $t,\ t' \in (x-h, x+h) \setminus \{x\}$ symmetric with respect to x (i.e., $t + t' = 2x$) such that $|f(t) - \omega_f(x)| < \varepsilon$ and $|f(t') - f(t)| < \varepsilon$. Therefore,

$$|f(t') - \omega_f(x)| \leq |f(t') - f(t)| + |f(t) - \omega_f(x)| < 2\varepsilon.$$

Then $|\omega_f(t) - \omega_f(x)| < \varepsilon$ because $\omega_f(t) < \omega_f(x) + \varepsilon$, $|f(t) - \omega_f(x)| < \varepsilon$ and $\omega_f(t) \geq f(t)$. Also $|\omega_f(t') - \omega_f(x)| < 2\varepsilon$ because $\omega_f(t') < \omega_f(x) + \varepsilon$, $|f(t') - \omega_f(x)| < 2\varepsilon$ and $\omega_f(t') \geq f(t')$. So ω_f is Darboux due to it being Baire 1 and its graph having each $(x, \omega_f(x))$ as a bilateral limit point [2]. $\square$

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