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ON THE CONTINUITY OF SYMMETRICALLY CLIQUISH OR SYMMETRICALLY QUASICONTINUOUS FUNCTIONS

Abstract

Let (X, T_X) and (Y, T_Y) be topological spaces and let (Z, ρ_Z) be a metric space. In this article we characterize the sets of all continuity points of symmetrically cliquish functions from $X \times Y$ to Z and the sets of continuity points of symmetrically quasicontinuous functions from $\mathbb{R}^2$ to $\mathbb{R}$.

If (X, T_X) and (Y, T_Y) are topological spaces and (Z, ρ) is a metric space, then a function $f : X \times Y \to Y$ is said to be:

1. quasicontinuous (resp. cliquish) at a point $(x, y) \in X \times Y$ if for every set $U \times V \in T_X \times T_Y$ containing (x, y) and for each positive real η, there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $f(U' \times V') \subset K(f(x, y), \eta) = \{t \in Z; \rho(t, f(x, y)) < \eta\}$ (resp. $\operatorname{diam}(f(U' \times V')) = \sup\{\rho(f(t, t'), f(u, u')); t, u \in U' \text{ and } t', u' \in V'\} < \eta$) ([3, 4]);

2. quasicontinuous at (x, y) with respect to x (alternatively y) if for every set $U \times V \in T_X \times T_Y$ containing (x, y) and for each positive real η there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x \in U'$ (alternatively $y \in V'$) and $f(U' \times V') \subset K(f(x, y), \eta)$ ([5]);

3. cliquish at (x, y) with respect to x (alternatively y) if for every set $U \times V \in T_X \times T_Y$ containing (x, y) and for each positive real η there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x \in U'$ (alternatively $y \in V'$) and $\operatorname{diam}(f(U' \times V')) < \eta$) ([1]);

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4. symmetrically quasicontinuous (resp. symmetrically cliquish) at (x, y) if it is quasicontinuous (alternatively cliquish) at (x, y) with respect to x and with respect to y ([5, 1]).

It is obvious that if the set $C(f)$ of all continuity points of a function $f : X \times Y \to Z$ is dense, then f is cliquish. Moreover if $X \times Y$ is a Baire space, then f is cliquish if and only if $C(f)$ is dense ([4]).

In the last observation the hypothesis that $X \times Y$ is a Baire space is important. For example, if $X = Y = Z = \mathbb{Q}$ ($\mathbb{Q}$ denotes the set of all rationals) and $T_X = T_Y$ is the topology generated by the natural metric $\rho(x, y) = |x - y|$ in $\mathbb{R}$, then for each enumeration (r_n) of all rationals such that $r_n \neq r_m$ for $n \neq m$, the function $f(r_n, r_m) = \frac{1}{nm}$ is symmetrically cliquish (and hence cliquish), but the set $C(f)$ is empty.

Remark 1. *Let $f : X \times Y \to Z$ be a function. If the vertical sections $(C(f))_x = \{v \in Y; (x, v) \in C(f)\}$, $x \in X$, (alternatively the horizontal sections $(C(f))^y = \{u \in X; (u, y) \in C(f)\}$, $y \in Y$), are dense in Y (resp. in X), then f is cliquish with respect to x (alternatively with respect to y).*

PROOF. Fix a point $(x_1, y_1) \in X \times Y$, sets $U \in T_X$ and $V \in T_Y$ with $(x_1, y_1) \in U \times V$ and a real $\eta > 0$. Since the section $(C(f))_{x_1}$ is dense, there is a point $y_2 \in Y$ with $(x_1, y_2) \in C(f)$. Consequently, there are sets $U_1 \in T_X$ and $V_1 \in T_Y$ such that $(x_1, y_2) \in U_1 \times V_1 \subset U \times V$ and

$$f(U_1 \times V_1) \subset K((f(x_1, y_2), \frac{\eta}{3}).$$

So $\operatorname{osc}_{U_1 \times V_1} f \leq \frac{2\eta}{3} < \eta$ and the proof of the cliquishness of f with respect to x is completed. The proof of its cliquishness with respect to y is analogous. □

Corollary 1. *Let $f : X \times Y \to Z$ be a function. If the vertical sections $(C(f))_x$, $x \in X$, and the horizontal sections $(C(f))^y$, $y \in Y$, are dense in Y and respectively in X, then f is symmetrically cliquish.*

Theorem 1. *Suppose that (Y, T_Y) (alternatively (X, T_X)) is a Baire space and a function $f : X \times Y \to Z$ is cliquish with respect to x (alternatively with respect to y). Then each section $(C(f))_x$, $x \in X$, (alternatively each section $(C(f))^y$, $y \in Y$), is dense in Y (alternatively in X).*

PROOF. For $n \geq 1$ let

$$U_n = \{(x, y) \in X \times Y; \operatorname{osc} f < \frac{1}{n} \text{ at } (x, y)\}.$$

The sets
$$U_n \in T_X \times T_Y \text{ and } C(f) = \bigcap_{n=1}^{\infty} U_n.$$
Fix a point $(x,y) \in X \times Y$, sets $U \in T_X$ and $V \in T_Y$ with $(x,y) \in U \times V$ and a positive integer n. Since f is cliquish with respect to x, there are sets $U_1 \in T_X$ and $V_1 \in T_Y$ such that $x \in U_1 \subset U$, $V_1 \subset V$ and $\operatorname{diam}(f(U_1 \times V_1)) < \frac{1}{n}$. So $V_1 \subset (U_n)_x \cap V$, and consequently the set $(U_n)_x$ is dense in Y. The section $(U_n)_x$ is open and dense in Y. Thus $Y \setminus (U_n)_x$ is closed and nowhere dense in Y. From this it follows that
$$Y \setminus (C(f))_x = Y \setminus \bigcap_{n=1}^{\infty} (U_n)_x = \bigcup_{n=1}^{\infty} (Y \setminus (U_n)_x)$$
is of the first category in Y. Since Y is a Baire space, the section $(C(f))_x$ is dense in Y. The proof of the second part is analogous. $\square$

The next assertion follows immediately from Theorem 1.

Corollary 2. *Suppose that (Y, T_Y) and (X, T_X) are Baire spaces and a function $f : X \times Y \to Z$ is symmetrically cliquish. Then the sections $(C(f))_x$, $x \in X$, and the sections $(C(f))^y$, $y \in Y$, are dense in Y and resp. in X.*

By a standard reasoning we can prove the following remark which we apply in the proof of next theorem.

Remark 2. *If a sequence of cliquish (quasicontinuous) with respect to x [alternatively y] functions $f_n : X \times Y \to Z$ uniformly converges to a function f, then f is also cliquish (quasicontinuous) with respect to x [alternatively y].*

Theorem 2. *Let $A \subset X \times Y$ be an F_σ-set such that the sections A_x, $x \in X$, (alternatively A^y, $y \in Y$,) are of the first category in Y (alternatively in X). Then there is a function $f : X \times Y \to \mathbb{R}$ which is cliquish with respect to x (alternatively to y) such that $C(f) = (X \times Y) \setminus A$.*

PROOF. There are closed sets A_n with $A = \bigcup_n A_n$ and $A_n \subset A_{n+1}$ for $n \geq 1$. Since for $n \geq 1$, the sections $((X \times Y) \setminus A_n)_x$, $x \in X$, (alternatively $((X \times Y) \setminus A_n)^y$, $y \in Y$,) are open, the sections $(A_n)_x$, $x \in X$, (alternatively $(A_n)^y$, $y \in Y$,) and $n = 1, 2, \ldots$, are closed and nowhere dense. Consequently, the characteristic functions $f_n = \chi_{A_n, X \times Y}$ are symmetrically cliquish with respect to x (alternatively y). Let $f = \sum_{n=1}^{\infty} \frac{f_n}{2^n}$ and for $n \geq 1$ let $s_n = \sum_{k=1}^{n} \frac{f_k}{2^k}$. Since for each $n \geq 1$ the sections $(A_n)_x$, $x \in X$, (alternatively $(A_n)^y$, $y \in Y$,) are nowhere dense, the function s_n is cliquish with respect to x

(alternatively y). But the convergence of the series is uniform, so the function f is cliquish with respect to x (alternatively y). Moreover from the equalities $C(s_n) = (X \times Y) \setminus A_n$, $n \geq 1$, we obtain $C(f) = (X \times Y) \setminus A$. $\square$

In the same manner we can prove the following theorem.

Theorem 3. *Let $A \subset X \times Y$ be an F_σ-set such that the sections A_x, $x \in X$, and A^y, $y \in Y$, are of the first category in Y and resp. in X. Then there is a symmetrically cliquish function $f : X \times Y \to \mathbb{R}$ such that $C(f) = (X \times Y) \setminus A$.*

It is obvious (compare [2]) that if a function $f : X \times Y \to Z$ is such that the graph $Gr(f \restriction C(f)$ of the restricted function $f \restriction C(f)$ is dense in the graph $Gr(f)$, then f is quasicontinuous. The converse is also true.

Remark 3. *If a function $f : X \times Y \to Z$ is quasicontinuous and the set $C(f)$ is dense in $X \times Y$, then the graph $Gr(f \restriction C(f)$ is dense in $Gr(f)$.*

PROOF. Fix a point $(x, y, f(x, y))$, where $(x, y) \in X \times Y$, sets $U \in T_X$, $V \in T_Y$ with $(x, y) \in U \times V$ and a real $\eta > 0$. From the quasicontinuity of f at (x, y) it follows that there are nonempty sets $U_1 \in T_X$ and $V_1 \in T_Y$ such that

$$U_1 \times V_1 \subset U \times V \text{ and } f(U_1 \times V_1) \subset K\Big(f(x, y), \frac{\eta}{2}\Big).$$

Since $C(f)$ is dense in $X \times Y$, there is a point $(x_1, y_1) \in (U_1 \times V_1) \cap C(f)$. From the continuity of f at (x_1, y_1) it follows that there are sets $U_2 \in T_X$ and $V_2 \in T_Y$ such that

$$(x_1, y_1) \in U_2 \times V_2 \subset U_1 \times V_1 \text{ and } f(U_2 \times V_2) \subset K\Big(f(x_1, y_1), \frac{\eta}{2}\Big).$$

$U_2 \times V_2$ is a nonempty open set contained in $U \times V$ and $f(U_2 \times V_2) \subset K(f(x, y), \eta)$. $\square$

There is, however, a symmetrically quasicontinuous function g with $C(g) = \emptyset$. In a suitable example we apply the following remark, which may be proved by a standard reasoning.

Remark 4. *Let a function $f : X \times Y \to \mathbb{R}$ be symmetrically quasicontinuous at a point (x, y) and let $g : X \times Y \to \mathbb{R}$ be continuous at (x, y). Then the sum $f + g$ is symmetrically quasicontinuous at (x, y).*

Example 1. In $X = Y = Z = \mathbb{R}$ we introduce the natural metric ρ and let

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } x, y > 0 \\ f(x, y) = 0 & \text{otherwise on } \mathbb{R}^2. \end{cases}$$

Then $f : \mathbb{R}^2 \to [0, \frac{1}{2}]$ is a symmetrically quasicontinuous function and $C(f) = \mathbb{R}^2 \setminus \{(0, 0)\}$. Let $((x_n, y_n))_n$ be an enumeration of all pairs of rationals such that $(x_n, y_n) \neq (x_m, y_m)$ for $n \neq m$. Observe that for each positive integer n the function

$$g_n(x, y) = \frac{f(x - x_n, y - y_n)}{2^n} \text{ for } (x, y) \in \mathbb{R}^2,$$

is symmetrically quasicontinuous on $\mathbb{R}^2$ and $C(g_n) = \mathbb{R}^2 \setminus \{(x_n, y_n)\}$. Let $g : Q \times Q \to \mathbb{R}$ be defined by

$$g(x, y) = \sum_{n=1}^{\infty} \frac{f(x - x_n, y - y_n)}{2^n}.$$

For each positive integer k we have

$$g(x, y) = \sum_{k \neq n=1}^{\infty} g_n(x, y) + g_k(x, y).$$

So g is the sum of a continuous function at the point (x_k, y_k) and the symmetrically quasicontinuous function g_k which is not discontinuous at (x_k, y_k). Consequently, by Remark 4, the function g is symmetrically quasicontinuous on $Q \times Q$ and $C(g) = \emptyset$.

Theorem 4. *Let $f : X \times Y \to Z$ be a function. If the graphs of the restrictions of the vertical sections $f_x \restriction C(f)_x$, $x \in X$, (alternatively the graphs of the restrictions of the horizontal sections $f^y \restriction C(f)^y$, $y \in Y$), are dense in the graphs of these sections f_x (alternatively f^y), then f is quasicontinuous with respect to x (alternatively with respect to y).*

PROOF. Fix a point $(x_1, y_1) \in X \times Y$, sets $U \in T_X$ and $V \in T_Y$ with $(x_1, y_1) \in U \times V$ and a real $\eta > 0$. Since the graph $Gr(f_{x_1} \restriction ((C(f))_{x_1})$ is dense in $Gr(f_{x_1})$, there is a point

$$y_2 \in Y \text{ with } (x_1, y_2) \in C(f) \text{ and } \rho(f(x_1, y_2), f(x_1, y_1)) < \frac{\eta}{2}.$$

By the continuity of f at (x_1, y_2), there are sets $U_1 \in T_X$ and $V_1 \in T_Y$ such that $(x_1, y_2) \in U_1 \times V_1 \subset U \times V$ and $f(U_1 \times V_1) \subset K((f(x_1, y_2), \frac{\eta}{3})$. Observe that $f(U_1 \times V_1) \subset K(f(x_1, y_1), \eta)$ and the proof of the quasicontinuity of f with respect to x is completed. The proof of its quasicontinuity with respect to y is analogous. $\square$

The next assertion follows immediately from Theorem 4.

Corollary 3. *Let $f : X \times Y \to Z$ be a function. If the graphs of the restrictions $f_x \restriction C(f)_x$, $x \in X$, are dense in $Gr(f_x)$ and the graphs of the restrictions $f^y \restriction C(f)^y$, $y \in Y$, are dense in the graphs $Gr(f^y)$, then f is symmetrically quasicontinuous.*

Theorem 5. *Suppose that (Y, T_Y) (alternatively (X, T_X)) is a Baire space and a function $f : X \times Y \to Z$ is quasicontinuous with respect to x (alternatively with respect to y). Then the graphs $Gr(f_x \restriction C(f)_x)$, $x \in X$, (alternatively the graphs $Gr(f^y \restriction C(f)^y)$, $y \in Y$), are dense in $Gr(f_x)$ (alternatively in $Gr(f^y)$).*

PROOF. Fix a point $(x, y) \in X \times Y$, sets $U \in T_X$ and $V \in T_Y$ with $(x, y) \in U \times V$ and a real $\eta > 0$. Since f is quasicontinuous with respect to x, there are sets $U_1 \in T_X$ and $V_1 \in T_Y$ such that

$$x \in U_1 \subset U,\ \ V_1 \subset V \text{ and } f(U_1 \times V_1)) \subset K(f(x, y), \eta).$$

By Theorem 1 the section $(C(f))_x$ is dense in Y, so there is a point $v \in V_1$ with $(x, v) \in C(f)$. Since $f_x(v) = f(x, v) \in K(f(x, y), \eta)$, the proof of the first part is completed. The proof of the second part is analogous. □

The next Corollary follows immediately from Theorem 5.

Corollary 4. *Suppose that (Y, T_Y) and (X, T_X) are Baire spaces and a function $f : X \times Y \to Z$ is symmetrically quasicontinuous. Then the graphs of the restrictions $f_x \restriction C(f)_x$, $x \in X$, are dense in the graphs of these sections f_x and the graphs of the restrictions $f^y \restriction C(f)^y$, $y \in Y$, are dense in the graphs of these sections f^y.*

Since every symmetrically quasicontinuous function $f : X \times Y \to Z$ is symmetrically cliquish, the set $D(f) = (X \times Y) \setminus C(f)$ is an F_σ-set with of the first category horizontal and vertical sections $(D(f))^y$ and $(D(f))_x$, $y \in Y$ and resp. $x \in X$.

Theorem 6. *Suppose that $X = Y = Z = \mathbb{R}$, $\rho(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ and that $T_X = T_Y$ is the natural topology generated by ρ. If $A \subset \mathbb{R}^2$ is an F_σ-set whose horizontal and vertical sections A^y and A_x, $x, y \in \mathbb{R}$, are of the first category, then there is a symmetrically quasicontinuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $C(f) = \mathbb{R}^2 \setminus A$.*

PROOF. Since A is an F_σ-set, there are nonempty compact sets A_n such that

$$A = \bigcup_n A_n \text{ and } A_n \subset A_{n+1} \text{ for } n \geq 1.$$

Without loss of the generality we can assume that $A_{n+1} \setminus A_n \neq \emptyset$ for $n \geq 1$. Since every set $A_n \subset A$, the sections $(A_n)_x$ and $(A_n)^y$, $x, y \in \mathbb{R}$, are nowhere dense in $\mathbb{R}$. Now we will construct by induction a sequence of functions (f_n).

For this for a point $c = (c_1, c_2) \in \mathbb{R}^2$ and a real $r > 0$ denote by Sqr (c, r) the closed square $[c_1 - r, c_1 + r] \times [c_2 - r, c_2 + r]$.

Step 1. Let $B_1 \subset A_1$ be a countable set dense in A_1. Without loss of the generality we can assume that B_1 is an infinite set. Enumerate all points of B_1 in a sequence $(b_{1,n})$. Since A_1 is a nowhere dense set, for each point $b_{1,n}$, $n \geq 1$, there are a sequence of different points $c_{1,n,k} \in \mathbb{R}^2 \setminus A$ and a sequence of pairwise disjoint closed squares $I_{1,n,k} = \text{Sqr}\,(c_{1,n,k}, r_{1,n,k})$, $k \geq 1$, such that

(1.1) for each $n \geq 1$ the limit $\lim_{k\to\infty} c_{1,n,k} = b_{1,n}$;

(1.2) if $(n_1, k_1) \neq (n_2, k_2)$, then $I_{1,n_1,k_1} \cap I_{1,n_2,k_2} = \emptyset$;

(1.3) $I_{1,n,k} \cap A_{n+k} = \emptyset$ for $k \geq 1$;

(1.4) for all $n, k \geq 1$ and $x \in I_{1,n.k}$ $\text{dist}(x, A_1) = \inf\{|x - y|; y \in A_1\} < \frac{1}{n}$.

Now for all positive integers $n, k \geq 1$ we find a real $t_{1,n,k} \in (0, r_{1,n,k})$ and denote by $J_{1,n,k}$ the closed square Sqr $(c_{1,n,k}, t_{1,n,k})$. For $n, k \geq 1$ let $f_{1,n,k} : I_{1,n,k} \to [0, 1]$ be a continuous function such that

$$f_{1,n,k}(c_{1,n,k}) = 1 \text{ and } f_{1,n,k}(x, y) = 0 \text{ for } (x, y) \in I_{1,n,k} \setminus J_{1,n,k},$$

and let

$$f_1(x, y) = \begin{cases} f_{1,n,k}(x, y) & \text{for } (x, y) \in I_{1,n,k},\ n, k \geq 1 \\ 0 & \text{otherwise on } \mathbb{R}^2. \end{cases}$$

Observe that $C(f_1) = \mathbb{R}^2 \setminus A_1$. We will prove that f_1 is symmetrically quasicontinuous. Obviously, it is symmetrically quasicontinuous at all points $(x, y) \in C(f_1) = \mathbb{R}^2 \setminus A_1$. Fix a point $t = (x, y) \in A_1$, a real $\eta > 0$ and open intervals I, J such that $(x, y) \in I \times J$. If there is a pair (n_1, k_1) such that $\{(x, v); v \in J\} \cap I_{1,n_1,k_1} \neq \emptyset$, then there is a point $y_2 \in J$ with $(x, y_2) \in I_{1,n_1,k_1} \setminus J_{1,n_1,k_1}$ and consequently there are open intervals $I_1 \subset I$ and $J_1 \subset J$ such that $x \in I_1$ and $f_1(I_1 \times J_1) = \{0\}$. If such a pair (n_1, k_1) does not exist, then for each point $v \in J \setminus (A_1)_x$ the point $(x, v) \in C(f_1)$ and consequently, there are open intervals $I_1 \subset I$ and $J_1 \subset J$ such that $x \in I_1$ and $f_1(I_1 \times J_1) = \{0\}$. Since $f_1(t) = f_1(x, y) = 0$, we obtain that f_1 is quasicontinuous at t with respect to x. In the same way we can prove that f_1 is quasicontinuous at t with respect to y. So f_1 is symmetrically quasicontinuous.

Step m $(m \geq 2)$. Let $B_m \subset A_m \setminus A_{m-1}$ be a countable set dense in $A_m \setminus A_{m-1}$. Without loss of the generality we can assume that B_m is an infinite set. Enumerate all points of B_m in a sequence $(b_{m,n})$ such that $b_{m,n_1} \neq b_{m,n_2}$ for $n_1 \neq n_2$. Since the sections $(A_m)_x$ and $(A_m)^y$, $x, y \in \mathbb{R}$, are nowhere dense sets, for each point $b_{m,n}$, $n \geq 1$, there are a sequence of different points $c_{m,n,k} \in \mathbb{R}^2 \setminus A$ and a sequence of pairwise disjoint closed squares

$$I_{m,n,k} = \text{Sqr}\,(c_{m,n,k}, r_{m,n,k}), \quad k \geq 1,$$

such that

(m.1) for each $n \geq 1$ the limit $\lim_{k\to\infty} c_{m,n,k} = b_{m,n}$;

(m.2) if $(n_1, k_1) \neq (n_2, k_2)$, then $I_{m,n_1,k_1} \cap I_{m,n_2,k_2} = \emptyset$;

(m.3) if $b_{m,n} \in \mathbb{R}^2 \setminus \bigcup_{i<m;\ j,k\geq 1} I_{i,j,k}$, then $I_{m,n,k} \subset \mathbb{R}^2 \setminus (A_m \cup \bigcup_{i<m;\ j,k\geq 1} I_{i,j,k})$;

(m.4) if $b_{m,n} \in I_{i,j,l}$ for some $i < m$ and $j, l \geq 1$, then $I_{m,n,k} \subset I_{i,j,l}$;

(m.5) $I_{m,n,k} \cap A_{m+n+k} = \emptyset$ for $n, k \geq 1$;

(m.6) for all $n, k \geq 1$ and $x \in I_{m,n.k}$ $\text{dist}(x, A_m) = \inf\{|x-y|; y \in A_1\} < \frac{1}{m+n}$.

Now for all positive integers $n, k \geq 1$ we find a real $s_{m,n,k} \in (0, r_{m,n,k})$ and denote by $J_{m,n,k}$ the closed square $\text{Sqr}\,(c_{m,n,k}, s_{m,n,k})$. For $n, k \geq 1$ let $f_{m,n,k} : I_{m,n,k} \to [0, 1]$ be a continuous function such that

$$f_{m,n,k}(c_{m,n,k}) = 1 \text{ and } f_{m,n,k}(x, y) = 0 \text{ for } (x, y) \in I_{m,n,k} \setminus J_{m,n,k}.$$

Moreover let

$$f_m(x, y) = \begin{cases} f_{m,n,k}(x, y) & \text{for } x \in I_{m,n,k},\ n, k \geq 1 \\ 0 & \text{otherwise on } \mathbb{R}^2. \end{cases}$$

In the same manner as in the case of f_1 we can prove that $C(f_j) = \mathbb{R}^2 \setminus cl(A_j \setminus A_{j-1})$ and that f_j are symmetrically quasicontinuous everywhere on $\mathbb{R}^2$. Let

$$s_0 = 0 \text{ and } s_j = \sum_{i\leq j} \frac{f_i}{2^i} \text{ for } j \in \{1, 2, \ldots, m\}.$$

Observe that if $(x, y) \notin A_m$, then the functions f_i, $i \leq m$, are continuous at (x, y), and consequently s_m is also continuous at (x, y). So, $\mathbb{R}^2 \setminus A_m \subset C(s_m)$. If $(x, y) \in A_m$, then either $(x, y) \in A_1$ or there is a positive integer $k < m$ such that $(x, y) \in A_{k+1} \setminus A_k$. If $(x, y) \in A_1$, then $s_m(x, y) = f_1(x, y) = 0$ and $\limsup_{(u,v)\to(x,y)} s_m(u, v) \geq \limsup_{(u,v)\to(x,y)} \frac{f_1(u,v)}{2} = \frac{1}{2}$, and s_m is not continuous at (x, y).

If there is a positive integer $k < m$ with $(x, y) \in A_{k+1} \setminus A_k$, then put $h = s_m - s_k$ and observe that s_k is continuous at (x, y). Similarly as above we can prove that $h(x, y) = 0$ and $\limsup_{(u,v)\to(x,y)} h(u, v) > 0$. So h is not continuous at (x, y). Since $s_m = s_k + h$, the sum s_m is not continuous at (x, y) and $C(s_m) = \mathbb{R}^2 \setminus A_m$.

Now we will prove that the sum s_m is symmetrically quasicontinuous. Evidently it is symmetrically quasicontinuous at all points of the set $C(s_m) = \mathbb{R}^2 \setminus A_m$. Let $(x, y) \in A_m$. Since the function $s_1 = \frac{f_1}{2}$ is symmetrically quasicontinuous, for the proof that s_m is symmetrically quasicontinuous at (x, y) (we will write $s_m \in \mathrm{Sqc}\,(x, y)$) it suffices to show that for $k < m$ the implication $s_k \in \mathrm{Sqc}\,(x, y) \Longrightarrow s_{k+1} \in \mathrm{Sqc}\,(x, y)$. So fix a positive integer $k < m$ and assume that s_k is symmetrically quasicontinuous at (x, y). Let $j \le m$ be the first integer such that $(x, y) \in A_j$. If $j > k$, then $(x, y) \in \mathbb{R}^2 \setminus A_k = C(s_k)$ and s_{k+1} is symmetrically quasicontinuous at (x, y) as the sum of the symmetrically quasicontinuous at (x, y) function f_{k+1} and continuous at this point s_k. Thus we can assume that $j \le k$. The function s_{j-1} is continuous at (x, y) and $g_j(x, y) = 0$. If for each integer $l \in \{j + 1, j + 2, \dots, k + 1\}$ the point $(x, y) \notin \mathrm{cl}(A_l \setminus A_{l-1})$, then the functions f_i, $j < i \le k + 1$, are continuous at (x, y), and consequently $s_{k+1} = \sum_{j\ne i\le k+1} f_i + f_j$ is symmetrically quasicontinuous at (x, y) as the sum of symmetrically quasicontinuous function f_j and a continuous function at this point (x, y). Now consider the case, where the family $\mathcal{A}$ of all integers l such that $j < l \le k + 1$ and $(x, y) \in \mathrm{cl}(A_l \setminus A_{l-1})$ is nonempty. Then for $i < j$ and for $j < i \notin \mathcal{A}$ the functions f_i are continuous at (x, y). Let $\psi = \sum_{i\in\mathcal{A}} \frac{f_i}{2^i}$ and let $h = s_{k+1} - \psi$. The function h is continuous at (x, y) and $\psi(x, y) = 0$. Let U and V be open intervals such that $(x, y) \in U \times V$. Since open intervals cannot be countable unions of pairwise disjoint closed sets ([6]), there is an open interval $J \subset V \setminus (A_{k+1})_x$ such that $(\{x\} \times J) \subset \psi^{-1}(0) \cap C(\psi)$. Consequently the function ψ is quasicontinuous at (x, y) with respect to x. Similarly we can prove that ψ is quasicontinuous at (x, y) with respect to y. Since ψ is symmetrically quasicontinuous at (x, y) and h is continuous at (x, y), the sum $s_{k+1} = h + \psi$ is also symmetrically quasicontinuous at (x, y). This proves that the function $f = \sum_{m=1}^{\infty} \frac{f_m}{2^m}$ as the limit of a uniformly convergent sequence of symmetrically quasicontinuous functions s_m is symmetrically quasicontinuous. Moreover $C(f) = \mathbb{R}^2 \setminus A$ and the proof is completed. □

Example 2. Let $X = Y = Z = \mathbb{R}$, let

$$T_X = T_Y = \{\emptyset\} \cup \{\mathbb{R} \setminus A; A \text{ is finite}\},$$

and let $T_Z = T_e$ be the natural topology in $\mathbb{R}$. Then each quasicontinuous (hence also symmetrically quasicontinuous) function $f : (X \times Y, T_X \times T_Y) \to$

(Z, T_Z) is constant. In fact, if a quasicontinuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is not constant, then there are different points (x_1, y_1) and (x_2, y_2) with $f(x_1, y_1) \neq f(x_2, y_2)$. Let

$$\eta = \frac{|f(x_1, y_1) - f(x_2, y_2)|}{2}.$$

Since f is quasicontinuous, there are nonempty sets $U_1, U_2, V_1, V_2 \in T_X = T_Y$ such that

$$f(U_1 \times V_1) \subset (f(x_1, y_1) - \eta, f(x_1, y_1) + \eta) \text{ and}$$
$$f(U_2 \times V_2) \subset (f(x_2, y_2) - \eta, f(x_1, y_1) + \eta).$$

Obviously there is a point $(u, v) \in (U_1 \times V_1) \cap (U_2 \times V_2)$. Thus,

$$2\eta = |f(x_1, y_1) - f(x_2, y_2)| \leq |f(x_1, y_1) - f(u, v)| + |f(u, v) - f(x_2, y_2)|$$
$$< \eta + \eta = 2\eta,$$

and the obtained contradiction shows that f is constant (so and continuous).

Thus if $A \subset X \times Y$ is a nonempty F_σ-set with of the first category sections A_x and A^y, $x, y \in \mathbb{R}$ (for example a nonempty finite set), then each symmetrically quasicontinuous function $f : X \times Y \to Z$ is continuous at all points of A.

Example 2 shows that an analogy of Theorem 6 in arbitrary topological spaces (X, T_X) and (Y, T_Y) is not true.

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