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ON ADDITIVE ABSOLUTELY NONMEASURABLE SIERPIŃSKI-ZYGMUND FUNCTIONS

Abstract

Assuming Martin's Axiom, it is proved that there exists a Sierpiński-Zygmund function, which is additive (i.e., is a solution of the Cauchy functional equation) and is absolutely nonmeasurable with respect to the class of all nonzero σ-finite diffused measures on the real line $\mathbb{R}$.

Let E be a nonempty set, and let M be a class of measures on E (in general, the domains of measures from M may be different σ-algebras of subsets of E). Let $f : E \to \mathbb{R}$ be a function. We say that f is absolutely nonmeasurable with respect to M if f is nonmeasurable with respect to any measure from M.

For example, if $E = \mathbb{R}$ and M denotes the class of all translation invariant extensions of the Lebesgue measure λ on $\mathbb{R}$, then the characteristic function of a Vitali subset of $\mathbb{R}$ is absolutely nonmeasurable with respect to M.

Denote by $M(E)$ the class of all nonzero σ-finite diffused (i.e., vanishing on all singletons) measures on E. Let us stress once more that the domains of measures from this class are various σ-algebras of subsets of E. We say that a function $f : E \to \mathbb{R}$ is absolutely nonmeasurable if f is absolutely nonmeasurable with respect to $M(E)$.

Recall that a set $X \subset \mathbb{R}$ is universal measure zero if, for every σ-finite diffused Borel measure μ on $\mathbb{R}$, the equality $\mu^*(X) = 0$ is satisfied, where μ^* denotes, as usual, the outer measure associated with μ.

It is well known that there exist uncountable universal measure zero subsets of $\mathbb{R}$ (see, e.g., [7], [8]). A much stronger version of this result is contained in [10] (cf. also [13]).

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Let us formulate (without proof) a characterization of absolutely nonmeasurable functions in terms of universal measure zero sets and preimages of singletons.

Theorem 1. *Let $f : E \to \mathbb{R}$ be a function. The following two assertions are equivalent:*

1) f is absolutely nonmeasurable;

2) the range of f is a universal measure zero subset of $\mathbb{R}$ and $\mathrm{card}(f^{-1}(t)) \leq \omega$ for each $t \in \mathbb{R}$.

A detailed proof of this statement is given in [5]. In the same work, the question is considered whether any Vitali type function is absolutely nonmeasurable. Let $\mathbb{Q}$ denote the set of all rational numbers. We recall that $f : \mathbb{R} \to \mathbb{R}$ is a Vitali type function if the following conditions hold:

a) $f(x) - x \in \mathbb{Q}$ for every $x \in \mathbb{R}$;

b) $ran(f)$ is a selector of the partition $\mathbb{R}/\mathbb{Q}$.

It is shown in [5] that if $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary Vitali type function, then the Lebesgue measure λ can be extended to a measure μ on $\mathbb{R}$ such that f turns out to be measurable with respect to μ (in other words, f is not absolutely nonmeasurable with respect to the class of all possible extensions of λ).

Let $\mathbf{c}$ denote the cardinality of the continuum. The following three assertions are direct consequences of Theorem 1:

I. If $\mathrm{card}(E) > \mathbf{c}$, then there are no absolutely nonmeasurable functions f with $dom(f) = E$.

II. There exists an injective absolutely nonmeasurable function $f : E \to \mathbb{R}$, where $\mathrm{card}(E) = \omega_1$.

III. If the cardinality of any universal measure zero subset of $\mathbb{R}$ is strictly less than $\mathbf{c}$, then there are no absolutely nonmeasurable functions f with $dom(f) = \mathbb{R}$.

The latter consequence shows us that absolutely nonmeasurable functions $f : \mathbb{R} \to \mathbb{R}$ exist if and only if there exists a universal measure zero set $X \subset \mathbb{R}$ with $\mathrm{card}(X) = \mathbf{c}$. More generally, absolutely nonmeasurable functions $f : E \to \mathbb{R}$ exist if and only if there exists a universal measure zero subset X of $\mathbb{R}$ with $\mathrm{card}(X) = \mathrm{card}(E)$. This also yields that the existence of an absolutely nonmeasurable function acting from $\mathbb{R}$ into $\mathbb{R}$ cannot be established within the theory **ZFC**. Indeed, there are models of set theory in which the negation of the Continuum Hypothesis holds and the cardinality of each universal measure zero subset of $\mathbb{R}$ does not exceed the first uncountable cardinal ω_1 (see, e.g., [8]). Clearly, in such a model we do not have absolutely nonmeasurable functions acting from $\mathbb{R}$ into $\mathbb{R}$. Note, in this context, that the nonexistence of absolutely nonmeasurable functions acting from $\mathbb{R}$ into $\mathbb{R}$ can be deduced by using some other set-theoretical assumptions. For instance, it is easy to see that if $\mathbf{c}$ is a real-valued measurable cardinal, then no function acting from $\mathbb{R}$ into $\mathbb{R}$ is absolutely nonmeasurable.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is additive (satisfies the Cauchy functional equation) if

$$f(x+y) = f(x) + f(y) \qquad (x \in \mathbb{R},\ y \in \mathbb{R}).$$

In other words, f is a solution of the Cauchy functional equation if and only if f is a homomorphism of the additive group $\mathbb{R}$ into itself. There are many works devoted to the Cauchy functional equation (see, e.g., [6] and references therein). Obviously, all continuous solutions of the Cauchy functional equation are representable in the form

$$x \to ax \qquad (x \in \mathbb{R}),$$

where $a \in \mathbb{R}$. Any such function is called a trivial solution of the Cauchy equation. By using the technique of Hamel bases, one can get many nontrivial solutions of this equation. All of them are nonmeasurable in the Lebesgue sense (for more details, see [6] or [3]). As mentioned above, it is impossible to establish (within **ZFC**) the existence of absolutely nonmeasurable solutions of the Cauchy functional equation. On the other hand, by assuming Martin's Axiom and applying some properties of so-called generalized Luzin subsets of $\mathbb{R}$, it can be proved that there are additive absolutely nonmeasurable functions acting from $\mathbb{R}$ into $\mathbb{R}$. In other words, under Martin's Axiom, there exist absolutely nonmeasurable solutions of the Cauchy functional equation (see, e.g., [5]).

Sierpiński and Zygmund constructed in their remarkable work [12] a function $f : \mathbb{R} \to \mathbb{R}$ having the following property: for each subset Y of $\mathbb{R}$ with $\mathrm{card}(Y) = \mathbf{c}$, the restriction $f|Y$ is not continuous on Y. This classical result of Sierpiński and Zygmund was essentially motivated by a theorem of Blumberg [2] stating that, for any function $g : \mathbb{R} \to \mathbb{R}$, there exists an everywhere dense subset D of $\mathbb{R}$ such that the restriction $g|D$ is continuous on D. Obviously, the set D being dense in $\mathbb{R}$ is infinite. The existence of Sierpiński-Zygmund function $f : \mathbb{R} \to \mathbb{R}$ shows that one cannot assert (within **ZFC**) the uncountability of D.

In the sequel, for any Sierpiński-Zygmund function, we use the abbreviation SZ-function.

Various works are devoted to SZ-functions (see, e.g., [1], [9], [11]). In those works, different constructions are presented which yield further examples of SZ-functions with additional properties important from the viewpoint of real analysis.

Let M denote the class of all completions of nonzero σ-finite diffused Borel measures on the real line $\mathbb{R}$. It is not difficult to prove that every SZ-function $f : \mathbb{R} \to \mathbb{R}$ turns out to be nonmeasurable with respect to M (see, for instance, Chapter 6 in [3]). In particular, f is nonmeasurable with respect to the Lebesgue measure λ on $\mathbb{R}$. At the same time, as said above, one cannot assert that f is absolutely nonmeasurable. Moreover, it may happen that the graph of f is a $(\lambda \times \lambda)$-thick subset of the plane $\mathbb{R}^2$, and, in this case, f becomes measurable with respect to a suitable extension of λ. A much stronger result was obtained in paper [4]. Namely, it was demonstrated in [4] that there exists an SZ-function measurable with respect to some translation invariant extension of λ.

The following statement shows that, under an appropriate set-theoretical assumption, there are additive absolutely nonmeasurable functions acting from $\mathbb{R}$ into $\mathbb{R}$ which are not SZ-functions.

Theorem 2. *Assume Martin's Axiom. There exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:*

1) f is injective and additive;

2) f is absolutely nonmeasurable;

3) the restriction of f to some set $K \subset \mathbb{R}$ with $\mathrm{card}(K) = \mathbf{c}$ is the identical transformation of K (consequently, f is not an SZ-function).

Proof. We recall that a set $L \subset \mathbb{R}$ is a generalized Luzin subset of $\mathbb{R}$ if $\mathrm{card}(L) = \mathbf{c}$ and, for every first category set $P \subset \mathbb{R}$, we have $\mathrm{card}(P \cap L) < \mathbf{c}$. It is well known that, under Martin's Axiom, there exists a generalized Luzin subset L of the real line, being simultaneously a vector space over the field $\mathbb{Q}$ of all rational numbers (see, e.g., [8], [6], [3]). Let H_L denote a Hamel basis for L. Obviously, we have $\mathrm{card}(H_L) = \mathbf{c}$. Extend H_L to a Hamel basis H of $\mathbb{R}$. Let $f_0 : H \to H_L$ be a bijection such that, for some set $K \subset H_L$ with $\mathrm{card}(K) = \mathbf{c}$, the restriction of f_0 to K is the identical transformation of K. The bijection f_0 can be extended to an isomorphism f (linear over $\mathbb{Q}$) between $\mathbb{R}$ and L. At the same time, we may consider f as an injective group homomorphism from $\mathbb{R}$ into $\mathbb{R}$ whose range coincides with L. Since L is a universal measure zero subset of $\mathbb{R}$ (see again [8], [6], or [3]), we infer from Theorem 1 that the function f is absolutely nonmeasurable. At the same time, the restriction $f|K$ coincides with the restriction $f_0|K$. Consequently, $f|K$ is the identical transformation of K, which immediately implies that f cannot be an SZ-function. □

The next statement establishes (within **ZFC**) that, for any $G \subset \mathbb{R}$ which is a vector space over $\mathbb{Q}$ and has the cardinality $\mathbf{c}$, there is an injective additive SZ-function acting from $\mathbb{R}$ into G.

Theorem 3. *Let G be as above. There exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:*

1) f is an injection;

2) f is a nontrivial solution of the Cauchy functional equation;

3) $ran(f) \subset G$;

4) f is an SZ-function.

PROOF. We shall construct the required function f by using the method of transfinite recursion.

Let $\preceq$ be a well-ordering of $\mathbb{R}$ isomorphic to the natural well-ordering of $\mathbf{c}$, which is denoted by $\leq$.

Let $\{h_\xi : \xi < \mathbf{c}\}$ be an enumeration of all Borel mappings acting from uncountable Borel subsets of $\mathbb{R}$ into $\mathbb{R}$.

Under this notation, we are going to define three $\mathbf{c}$-sequences

$$\{x_\xi : \xi < \mathbf{c}\}, \quad \{V_\xi : \xi < \mathbf{c}\}, \ \ and \ \ \{f_\xi : \xi < \mathbf{c}\}$$

satisfying the following relations:

(a) $\{x_\xi : \xi < \mathbf{c}\}$ is a Hamel basis of $\mathbb{R}$;

(b) for each ordinal $\xi < \mathbf{c}$, the set V_ξ is the vector subspace of $\mathbb{R}$ over the field $\mathbb{Q}$ of all rationals generated by $\{x_\zeta : \zeta \leq \xi\}$;

(c) for each ordinal $\xi < \mathbf{c}$, the function f_ξ is a group monomorphism acting from V_ξ into G;

(d) if $\zeta < \xi < \mathbf{c}$, then f_ξ extends f_ζ;

(e) if $\xi < \mathbf{c}$, then we have

$$f_\xi(qx_\xi + v) \neq h_\zeta(qx_\xi + v)$$

for all $\zeta < \xi$, $q \in \mathbb{Q} \setminus \{0\}$, $v \in \cup\{V_\zeta : \zeta < \xi\}$, $qx_\xi + v \in dom(h_\zeta)$.

Suppose that, for an ordinal $\xi < \mathbf{c}$, the partial ξ-sequences

$$\{x_\zeta : \zeta < \xi\}, \quad \{V_\zeta : \zeta < \xi\}, \text{ and } \{f_\zeta : \zeta < \xi\}$$

have already been constructed. Let us put:

$$V' = \cup\{V_\zeta : \zeta < \xi\}, \quad f' = \cup\{f_\zeta : \zeta < \xi\}.$$

Applying (c) and (d), we claim that f' is a group monomorphism from V' into G and

$$\text{card}(V') \leq \text{card}(\xi) + \omega < \mathbf{c}.$$

Let x denote the least element (with respect to $\preceq$) of the set $\mathbb{R} \setminus V'$. We put $x_\xi = x$ and denote by V_ξ the vector space over $\mathbb{Q}$ generated by $V' \cup \{x\}$. Further, we define

$$D = \{(1/q)(h_\zeta(qx+v) - f'(v)) : q \in \mathbb{Q} \setminus \{0\}, \zeta < \xi, v \in V', qx+v \in dom(h_\zeta)\}.$$

Since $\text{card}(D) < \mathbf{c}$ and $\text{card}(ran(f')) < \mathbf{c}$, we may choose an element $y \in G \setminus (D \cup ran(f'))$. Clearly, there exists a unique group monomorphism

$$f_\xi : V_\xi \to G$$

extending f' and such that $f_\xi(x) = y$.

Continuing in this manner, we get the required families

$$\{x_\xi : \xi < \mathbf{c}\}, \quad \{V_\xi : \xi < \mathbf{c}\}, \text{ and } \{f_\xi : \xi < \mathbf{c}\}.$$

Finally, denote $f = \cup\{f_\xi : \xi < \mathbf{c}\}$. It easily follows from our construction that $\{x_\xi : \xi < \mathbf{c}\}$ is a Hamel basis of $\mathbb{R}$ and, therefore,

$$dom(f) = \cup\{V_\xi : \xi < \mathbf{c}\} = \mathbb{R}.$$

Since all f_ξ ($\xi < \mathbf{c}$) are group monomorphisms, f is an injective homomorphism from $\mathbb{R}$ into G. Also, it can readily be verified that, for any Borel function h_ξ ($\xi < \mathbf{c}$), we have

$$\text{card}(\{z \in dom(h_\xi) : f(z) = h_\xi(z)\}) \leq \text{card}(\xi) + \omega < \mathbf{c}.$$

This shows that f is an SZ-function (cf. [7], [12]) and completes the proof. □

From Theorem 3, we easily obtain the following result.

Theorem 4. *Assuming Martin's Axiom, there exists an injective additive absolutely nonmeasurable SZ-function.*

Proof. Let L denote again a generalized Luzin subset of $\mathbb{R}$ which simultaneously is a vector space over $\mathbb{Q}$. Putting in Theorem 3 $G = L$, we get a group monomorphism $f : \mathbb{R} \to L$ such that f is also an SZ-function. Now, keeping in mind the fact that L is universal measure zero and taking into account Theorem 1, we see that f is an absolutely nonmeasurable SZ-function. □

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