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PERFECT ROADS FOR MULTIVALUED MAPS

Abstract

The main purpose of the paper is to extend Maximoff's theorem for real functions to the multivalued maps case.
Theorem (Maximoff [5]) *Let $f : \mathbb{R} \to \mathbb{R}$ be a first Baire class function. Then f has the Darboux property if and only if f has a perfect road at each point.*

1 Preliminaries.

In this paper, X denotes an Euclidean space and Y a separable metric space. For a non-empty set $A \subset Y$ and a number $\epsilon > 0$ we set

$$B_\epsilon(A) = \{x \in X;\ \text{there exists } y \in A \text{ such that } \rho(x, y) < \epsilon\}.$$

By $F : X \to Y$ we denote a multivalued map F which to each point $x \in X$ assigns a non-empty subset $F(x) \subset Y$. For sets $A \subset X$ and $B \subset Y$ we set

$$\begin{aligned} F(A) &= \bigcup\{F(x);\ x \in A\}, \\ F^+(B) &= \{x \in X;\ F(x) \subset B\}, \\ F^-(B) &= \{x \in X;\ F(x) \cap B \neq \emptyset\}. \end{aligned}$$

A multivalued map F is lower (upper) semicontinuous if for any open set $V \subset Y$ the set $F^-(V)$ (resp. $F^+(V)$) is open in X. F is lower (upper) first class if for any open set $V \subset Y$, $F^-(V)$ (resp. $F^+(V)$) is an F_σ-set.

F has the Darboux property if the image $F(E)$ is connected for any connected set $E \subset X$.

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We say that $g \in \mathbb{R}$ is a left (right) limit number of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x if for any open set $V \subset \mathbb{R}$ such that $g \in V$ and for any $\varepsilon > 0$

$$f^{-}(V) \cap (x - \varepsilon, x) \neq \emptyset \ \ (f^{-}(V) \cap (x, x + \varepsilon) \neq \emptyset).$$

If g is left and right limit number, then we say that it is a limit number. The set of limit numbers at a point x is denoted by $L(f, x)$.

2 Generalization of Maximoff Theorem.

Definition 1. A set $P \subset X$ is dense-in-itself (c-dense-in-itself) if for any $x \in P$ and any open, connected set $U \subset X$ with $x \in \overline{U}$, the set $P \cap U$ contains a point other than x ($P \cap U$ is uncountable).

Theorem 1. *Let $F : X \to Y$ be a lower first class multivalued map, with compact values. Then, the following conditions are equivalent:*

(i) for any open set $V \subset Y$ the counter image $F^{+}(V)$ is dense-in-itself,

(ii) for any open set $V \subset Y$ the counter image $F^{+}(V)$ is c-dense-in-itself.

PROOF. The implication $(ii) \to (i)$ is immediate.

$(i) \to (ii)$. Let $V \subset Y$ be an open set and let $x_0 \in F^{+}(V)$. Let $U \subset X$ be an open, connected set for which $x_0 \in \overline{U}$. Since $F^{+}(V)$ is dense-in-itself, there exists $x_1 \in U$ such that $F(x_1) \subset V$. Let V_1 be an open set such that $F(x_1) \subset V_1$ and $\overline{V}_1 \subset V$. Let us take an open, connected set W for which $x_1 \in W$ and $\overline{W} \subset U$, and define $D = W \cap F^{+}(V_1)$. Since $F^{+}(V_1)$ is dense-in-itself, the set D is dense-in-itself too, and hence $\overline{D}$ is a perfect set. Since F is lower first class, the set $C_l(F|_{\overline{D}})$ of lower semicontinuity points of $F|_{\overline{D}}$ is residual in $\overline{D}$ (Garg [3]). Thus it is uncountable and the inclusions $C_l(F|_{\overline{D}}) \subset \overline{W} \cap F^{+}(\overline{V}_1) \subset U \cap F^{+}(V)$ imply that $U \cap F^{+}(V)$ is also uncountable. □

Theorem 2. *Let $F : X \to Y$ be an upper first class multivalued map with compact values. The following conditions are equivalent:*

(i) for any open set $V \subset Y$ the counter image $F^{+}(V)$ is c-dense-in-itself,

(ii) if $U \subset X$ is an open, connected set and $x_0 \in \overline{U}$, then there exists a perfect set P such that $x_0 \in P$, $P \setminus \{x_0\} \subset U$ and $F|_P$ is upper semicontinuous at x_0.

PROOF. The implication $(ii) \to (i)$ is immediate.

$(i) \to (ii)$. Let $U \subset X$ be an open, connected set and let $x_0 \in \overline{U}$. Let U_1 be an open, connected set such that diameter $\rho(\overline{U}_1) < 1$, $\overline{U}_1 \setminus \{x_0\} \subset U$

and $x_0 \in \overline{U}_1$. Let $B_1 = B_1(F(x_0))$. The set $U_1 \cap F^+(B_1)$ is c-dense-in-itself and since F is upper first class, it is of F_σ type. Thus it contains a perfect set P_1 (Kuratowski [4]). We can assume that $x_0 \notin P_1$. We proceed by induction. Suppose we have disjoint sets $P_1, \dots, P_n$ such that for $i = 1, 2, \dots, n$, P_i is a perfect subset of U, $x_0 \notin P_i$, $\rho(x, x_0) < \frac{1}{i}$ if $x \in P_i$, and $P_i \subset F^+(B_i)$, where $B_i = B_{\frac{1}{i}}(F(x_0))$. The set $P_1 \cup \dots \cup P_n$ is then a perfect subset of U which has a positive distance d from x_0. Let U_{n+1} be an open, connected set whose diameter is less then $\min(d, \frac{1}{n+1})$ and such that $\overline{U}_{n+1} \setminus \{x_0\} \subset U$ and $x_0 \in \overline{U}_{n+1}$. The set $U_{n+1} \cap F^+(B_{n+1})$ is c-dense-in-itself and of F_σ type. As before, it contains a perfect subset P_{n+1} such that $x_0 \notin P_{n+1}$ and $\rho(x, x_0) < \frac{1}{n+1}$ if $x \in P_{n+1}$. Let $P = \bigcup_{n=1}^{\infty} P_n \cup \{x_0\}$. Then P is a perfect set such that $P \setminus \{x_0\} \subset U$ and $F|_P$ is upper semicontinuous at x_0. $\square$

The following theorem is an immediate consequence of Theorems 1 and 2.

Theorem 3. *Let $F : X \to Y$ be an upper and lower first class multivalued map with compact values. The following conditions are equivalent:*

- *(i) for any open set $V \subset Y$ the counter image $F^+(V)$ is dense-in-itself,*
- *(ii) if $U \subset X$ is an open, connected set and $x_0 \in \overline{U}$, then there exists a perfect set P such that $x_0 \in P$, $P \setminus \{x_0\} \subset U$ and $F|_P$ is upper semicontinuous at x_0.*

The following example shows that similar results for lower first class map and the sets $F^-(V)$ do not hold. We need a stronger condition (see Theorem 5).

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Darboux, Baire one function such that $L(f, 0) = [0, 1]$. Then the map $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} [0,1], & x = 0 \\ f(x), & x \neq 0 \end{cases}$$

is both upper and lower first class multivalued map with compact values and for any $V \subset R$ the set $F^-(V)$ is c-dense-in-itself.

Assume that P is a perfect set such that $0 \in P$ and $F|_P$ is lower semicontinuous at 0. Then there exists an open interval U such that $0 \in U$ and $U \cap P \subset F^-((\frac{1}{2}, 1))$. Since $F(x) = f(x)$ for $x \neq 0$, $F^-((0, \frac{1}{2})) \cap U \cap P = \{0\}$, which means that $F|_P$ is not lower semicontinuous at 0, a contradiction.

Therefore a perfect set P such that $F|_P$ is lower semicontinuous at 0, does not exist.

Theorem 4. *Let $F : X \to Y$ be an upper first class multivalued map. Then, the following conditions are equivalent:*

(i) the set $F^-(V_1) \cap \dots \cap F^-(V_n)$ is dense-in-itself, for any natural n and any open sets $V_1, \dots, V_n \subset Y$,

(ii) the set $F^-(V_1) \cap \dots \cap F^-(V_n)$ is c-dense-in-itself, for any natural n and any open sets $V_1, \dots, V_n \subset Y$.

PROOF. The implication $(ii) \to (i)$ is immediate.

$(i) \to (ii)$. Let $V_1, \dots, V_n \subset Y$ be open sets and let $x_0 \in F^-(V_1) \cap \dots \cap F^-(V_n)$. Let $U \subset X$ be an open, connected set for which $x_0 \in \overline{U}$. Since $F^-(V_1) \cap \dots \cap F^-(V_n)$ is dense-in-itself, there exists $x_1 \in U \cap F^-(V_1) \cap \dots \cap F^-(V_n)$. Let $G_1, \dots, G_n$ be open sets such that $\overline{G}_1 \subset V_1, \ \dots, \overline{G}_n \subset V_n$ and $x_1 \in F^-(G_1) \cap \dots \cap F^-(G_n)$. Let us take an open, connected set W for which $x_1 \in W$ and $\overline{W} \subset U$, and let $D = W \cap F^-(G_1) \cap \dots \cap F^-(G_n)$. The set D is dense-in-itself and hence $\overline{D}$ is a perfect set. Since F is an upper first class, the set $C_u(F|_{\overline{D}})$ of upper semicontinuity points of $F|_{\overline{D}}$ is residual in $\overline{D}$ (Garg [3]). Thus it has continuum cardinality and the inclusions

$$C_u(F|_{\overline{D}}) \subset \overline{W} \cap F^-(\overline{G}_1) \cap \dots \cap F^-(\overline{G}_n) \subset U \cap F^-(V_1) \cap \dots \cap F^-(V_n)$$

imply that $U \cap F^-(V_1) \cap \dots \cap F^-(V_n)$ is also uncountable. $\square$

Theorem 5. *Let $F : X \to Y$ be a lower first class multivalued map with compact values. The following conditions are equivalent:*

(i) the set $F^-(V_1) \cap \dots \cap F^-(V_n)$ is c-dense-in-itself, for any natural n and any open sets $V_1, \dots, V_n \subset Y$,

(ii) if $U \subset X$ is an open, connected set and $x_0 \in \overline{U}$, then there exists a perfect set P such that $x_0 \in P$, $P \setminus \{x_0\} \subset U$ and $F|_P$ is lower semicontinuous at x_0.

PROOF. $(i) \to (ii)$. Let $U \subset X$ be an open, connected set and let $x_0 \in \overline{U}$. Since $F(x_0)$ is a compact set, by induction we can construct a sequence of points $(y_1^k, \cdots, y_{n_k}^k)_{k=1}^\infty$, such that for any natural number k

(1) $\{y_1^k, \dots, y_{n_k}^k\} \subset F(x_0)$,

(2) $F(x_0) \subset \bigcup_{i=1}^{n_k} B_i^k$, where $B_i^k = B_{\frac{1}{k}}(y_i^k)$,

(3) $\{y_1^k, \cdots, y_{n_k}^k\} \subset \{y_1^{k+1}, \cdots, y_{n_{k+1}}^{k+1}\}$.

Since F is lower first class and $x_0 \in \bigcap_{i=1}^{n_k} F^-(B_i^k)$, by condition (i) the set $M_k = \bigcap_{i=1}^{n_k} F^-(B_i^k)$ is c-dense-in-itself and of type F_σ, for any natural k.

Let $U_1 \subset X$ be an open, connected set such that diameter $\rho(\overline{U}_1) < 1$, $\overline{U}_1 \setminus \{x_0\} \subset U$ and $x_0 \in \overline{U}_1$. The set $U_1 \cap M_1$ is c-dense-in-itself and of type F_σ, thus it contains a perfect set P_1. We can assume that $x_0 \notin P_1$. We proceed by induction. Suppose we have disjoint sets $P_1, \dots, P_n$ such that for $k = 1, 2, \dots, n$, P_k is a perfect subset of U, $x_0 \notin P_k$, $\rho(x, x_0) < \frac{1}{k}$ if $x \in P_k$, and $P_k \subset M_k$. The set $P_1 \cup \dots \cup P_n$ is then a perfect subset of U which has a positive distance d from x_0. Let $U_{n+1} \subset U$ be an open, connected set whose diameter is less then $\min(d, \frac{1}{n+1})$, such that $\overline{U}_{n+1} \setminus \{x_0\} \subset U$ and $x_0 \in \overline{U}_{n+1}$. The set $U_{n+1} \cap M_{k+1}$ is c-dense-in-itself and of type F_σ. As before, it contains a perfect subset P_{n+1} such that $x_0 \notin P_{n+1}$ and $\rho(x, x_0) < \frac{1}{n+1}$ if $x \in P_{n+1}$. Let $P = \bigcup_{n=1}^{\infty} P_n \cup \{x_0\}$. Then P is a perfect set such that $P \setminus \{x_0\} \subset U$ and $F|_P$ is lower semicontinuous at x_0.

(ii) $\rightarrow$ *(i)* Let $V_1, \cdots, V_n \subset Y$ be open sets such that $F^-(V_1) \cap \dots \cap F^-(V_n) \neq \emptyset$. Let $x_0 \in F^-(V_1) \cap \dots \cap F^-(V_n)$ and let $U \subset X$ be an open, connected set for which $x_0 \in \overline{U}$. Let us take a perfect set P such that $x_0 \in P$, $P \setminus \{x_0\} \subset U$ and $F|_P$ is lower semicontinuous at x_0. Then, there exists an open set $U_1 \subset X$ such that $x_0 \in U_1$ and $U_1 \cap P \subset F^-(V_1) \cap \dots \cap F^-(V_n)$. Since $U_1 \cap P$ is uncountable, $U \cap F^-(V_1) \cap \dots \cap F^-(V_n)$ is also uncountable. □

The following theorem is a consequence of Theorems 4 and 5.

Theorem 6. *Let $F : X \to Y$ be an upper and lower first class multivalued map with compact values. The following conditions are equivalent:*

- *(i) the set $F^-(V_1) \cap \dots \cap F^-(V_n)$ is dense-in-itself, for any natural n and any open sets $V_1, \cdots, V_n \subset Y$*
- *(ii) if $U \subset X$ is an open, connected set and $x_0 \in \overline{U}$, then there exists a perfect set P such that $x_0 \in P$, $P \setminus \{x_0\} \subset U$ and $F|_P$ is lower semicontinuous at x_0.*

We make some remarks for maps defined on the real line $F : \mathbb{R} \to \mathbb{R}$.

Definition 2. We say that a perfect set $P \subset \mathbb{R}$ is an upper (lower) perfect road of a multivalued map $F : \mathbb{R} \to \mathbb{R}$ at a point $x \in P$, if x is a point of bilateral accumulation of P and $F|_P$ is upper (lower) semicontinuous at x.

A perfect set P is a perfect road of a multivalued map F at a point x, if it is both an upper and lower perfect road at this point.

The following theorems are consequences of Theorems 3 and 6.

Theorem 7. *Let $F : \mathbb{R} \to \mathbb{R}$ be an upper and lower first class multivalued map with compact values. The following conditions are equivalent:*

(i) the set $F^+(V)$ is dense-in-itself for any open set $V \subset \mathbb{R}$,

(ii) F has an upper perfect road at each point.

Theorem 8. *Let $F : \mathbb{R} \to \mathbb{R}$ be an upper and lower first class multivalued map with compact values. The following conditions are equivalent:*

(i) the set $F^-(V_1) \cap \cdots \cap F^-(V_n)$ is dense-in-itself, for any natural n and any open sets $V_1, \cdots, V_n \subset \mathbb{R}$,

(ii) F has a lower perfect road at each point.

An open question is the following. For an upper and lower first class map $F : \mathbb{R} \to \mathbb{R}$ with compact values, are both conditions (i) from Theorems 7 and 8 enough for existence of a perfect road at each point?

Definition 3 (Czarnowska [2]). *A multivalued map $F : \mathbb{R} \to \mathbb{R}$ has the intermediate value property if for any distinct points x_1, $x_2 \in \mathbb{R}$ and every $y_1 \in F(x_1)$ there exists $y_2 \in F(x_2)$ such that $(y_1, y_2) \subset F((x_1, x_2))$.*

If F has connected values and it has the intermediate value property, then it has the Darboux property, too.

Theorem 9 (Czarnowska [2]). *Suppose $F : \mathbb{R} \to \mathbb{R}$ has closed values and it is both upper and lower first class. If for any open set $V \subset \mathbb{R}$ the counter images $F^+(V)$ and $F^-(V)$ are dense-in-itself, then F has the intermediate value property.*

As a consequence of Theorems 7, 8 and 9 we have the following theorem.

Theorem 10. *Let $F : \mathbb{R} \to \mathbb{R}$ be both an upper and a lower first class multivalued map with compact values. If F has a lower and an upper perfect road at each point, then it has the intermediate value property.*

Example 1 shows that an upper and lower first class map with the intermediate value property does not necessarily have a lower perfect road. The map

$$F(x) = \begin{cases} [0,1] & \text{for } x = 0 \\ [0,2] & \text{for } x \neq 0 \end{cases}$$

is both upper and lower first class and it has the intermediate value property, but it does not have an upper perfect road at zero.

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