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SUM AND DIFFERENCE FREE SETS

Abstract

In this paper we prove that if X is an uncountable subset of the reals and κ is a cardinal smaller than the cardinality of the set X, then the algebraic difference $X - X$ of the set X is not a finite union of κ sum free or κ difference free sets. An application of the above result is that for any function $f : \mathbb{R} \to \{1, 2, ..., n\}$ and for each cardinal $\lambda < 2^\omega$, the set of all x such that $|\{h > 0 : f(x - h) = f(x + h)\}| \geq \lambda$ is of the size of the continuum. Among other things, we show that a finite union of countably many translates of 2^ω difference free subsets of the reals is not residual in an interval. In the above statement, "countably many" can be replaced by "fewer than continuum many" provided that 2^ω is a regular cardinal.

1 Introduction

Throughout this paper all sets are subsets of the reals. A set S is said to be κ sum free if, for every real number r, the equation $x + y = r$ has less than κ solutions in S. A set S is said to be κ difference free if, for every nonzero real number r, the equation $x - y = r$ has less than κ solutions in S. A countable partition $P = \{P_n : n < \omega\}$ of $\mathbb{R}$ is called κ difference free partition if, for every $0 \neq r \in \mathbb{R}$, the equation $x - y = r$ has less than κ solutions, where x and y belong to the same set P_n for some n. Theorems 2.3 and 3.2 in [2] say that, for any vector space V over the rationals, if $|V| \leq \omega_1$, then there is a countable ω difference free partition of V, but if $|V| \geq \omega_1$, then there is no countable κ difference free partition of V for any $\kappa < \omega$. A subset C of $\mathbb{R}$ is called κ covering for $\mathbb{R}$ if the set C contains a translated copy of every subset of $\mathbb{R}$ of size κ. For subsets A and B of $\mathbb{R}$, $A + B$ is defined to be the set $\{x + y : x \in A \text{ and } y \in B\}$.

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In this paper we prove two theorems about covering sets which imply that (i) for any cardinal $\kappa < 2^\omega$, if P_i is a κ^+ difference free set and G_i is a set of size smaller than 2^ω, then a finite union of sets of the form $P_i + G_i$ is not residual in an interval; (ii) under the assumption of $2^\omega \geq \omega_2$, for any cardinal $\kappa < 2^\omega$, no ω_1 covering set for $\mathbb{R}$, in particular $\mathbb{R}$, is a countable union of κ^+ difference free sets and, hence there is no countable κ^+ difference free partition of $\mathbb{R}$; (iii) a countable union of 2^ω difference free sets is not residual in an interval provided that $\mathbb{R}$ is not a union of ω_1 many meager sets and $cf(2^\omega) > \omega_1$. We also prove that if κ is an infinite cardinal smaller than 2^ω and V is a subset of the reals of size κ^+, then V - V is not a finite union of κ sum free or κ difference free sets. An application of the above result is that if $f : \mathbb{R} \to \{1, 2, ..., n\}$ is a function and $S_x = \{h > 0 : f(x - h) = f(x + h)\}$, then for each cardinal $\lambda < 2^\omega$, the cardinality of the set $\{x : |S_x| \geq \lambda\}$ is 2^ω.

2 Notation

The set of all real numbers and the set of all rational numbers are denoted by $\mathbb{R}$ and $\mathbb{Q}$, respectively. For subsets A and B of $\mathbb{R}$, the symbols $A - B$ and $A + B$ stand for the sets $\{x - y : x \in A \text{ and } y \in B\}$ and $\{x + y : x \in A \text{ and } y \in B\}$, respectively. $A \backslash B$ is the set theoretic difference of sets A and B. For $r \in \mathbb{R}$, $A + r = \{x + r : x \in A\}$. The cardinality of a set A is denoted by $|A|$. κ stands for a positive cardinal.

Definition 2.1. *A subset S of $\mathbb{R}$ is said to be κ sum free in a subset T of $\mathbb{R}$ if, for every $t \in T$, the equation $x + y = t$ has less than κ solutions in S. That is, for each t in T, $|\{\{x, y\} : x, y \in S \text{ and } x + y = t\}| < \kappa$. A subset S of $\mathbb{R}$ is said to be κ sum free if it is κ sum free in $\mathbb{R}$. A subset S of $\mathbb{R}$ is said to be κ difference free in a subset T of $\mathbb{R}$ if, for every $0 \neq t \in T$, the equation $x - y = t$ has less than κ solutions in S. That is, for each nonzero t in T, $|\{\{x, y\} : x, y \in S \text{ and } x - y = t\}| < \kappa$. A subset S of $\mathbb{R}$ is said to be κ difference free if it is κ difference free in $\mathbb{R}$. A countable partition $P = \{P_n : n < \omega\}$ of $\mathbb{R}$ is called κ difference free partition if, for every $0 \neq r \in \mathbb{R}$, the equation $x - y = r$ has less than κ solutions, where x and y belong to the same set P_n for some n. That is, $| \cup_{n<\omega} \{\{x, y\} : x, y \in P_n \text{ and } x - y = r\}| < \kappa$.*

Note that (i) if S and T are subsets of $\mathbb{R}$ and S is κ sum free or κ difference free (in $\mathbb{R}$), then S is κ sum free or κ difference free in T;

(ii) if $P = \{P_n : n < \omega\}$ is κ difference free partition of $\mathbb{R}$, then $\mathbb{R}$ is a countable union of κ difference free sets P_n.

3 Results

Theorem 3.1. *Let κ be an infinite cardinal smaller than 2^ω. If V is a subset of the reals of size κ^+, then $V-V$ is not a finite union of κ sum free or κ difference free sets in $V-V$ and hence, $V-V$ is not a finite union of κ sum free or κ difference free sets in $\mathbb{R}$. In fact, $\left|(V-V)\setminus \cup_{i=1}^{i=n} S_i\right| = \kappa^+$, where each S_i is a κ sum free or κ difference free set in $V-V$.*

First, we prove the following lemmas.

Lemma 3.2. *Let S be a κ sum free set in $V-V$. If $|S\cap(V-v)| = \kappa^+$ or $|S\cap(v-V)| = \kappa^+$ for some $v\in V$, then there exists a set W such that $|W| = \kappa^+$ and $W\subseteq W-W\subseteq((V-V)\setminus S)\cup\{0\}$.*

PROOF. Suppose that $|S\cap(V-v)| = \kappa^+$ for some $v\in V$. Define a set mapping F from $S\cap(V-v)$ into the power set of $S\cap(V-v)$ by $F(x) = (-S+x)\cap S\cap(V-v)$. Since the set S is κ sum free in $V-V$, for every $x\in V-V$, we have $|(-S+x)\cap S| < \kappa$. (A classical result of Hajnal [3] states that if F is a set mapping from an uncountable set A into the power set of A and $|F(x)| < \kappa \forall x\in A$, where $\kappa < |A|$, then there exists a subset B of A such that $|B| = |A|$ and $x\notin F(y)$ for all distinct x,y in B.) By Hajnal's set mapping theorem, there exists a subset X of $S\cap(V-v)$ such that $|X| = \kappa^+$ and $x\notin F(y)$ for all distinct x,y in X, that is, $x\notin -S+y$. Hence $(X-X)\cap S\subseteq\{0\}$. Let a be a fixed element of X and let $W = X-a$. Then $|W| = \kappa^+$ and $W\subseteq W-W = X-X\subseteq((V-v-(V-v))\setminus S)\cup\{0\} = ((V-V)\setminus S)\cup\{0\}$. The proof is essentially the same for the case $|S\cap(v-V)| = \kappa^+$. $\square$

Lemma 3.3. *Let S be a κ difference free set in $V-V$. If $|S\cap(V-v)| = \kappa^+$ or $|S\cap(v-V)| = \kappa^+$ for some $v\in V$, then there exists a set W such that $|W| = \kappa^+$ and $W\subseteq W-W\subseteq((V-V)\setminus S)\cup\{0\}$.*

PROOF. Define a set mapping F from $S\cap(V-v)\setminus\{0\}$ into the power set of $S\cap(V-v)\setminus\{0\}$ by $F(x) = (S+x)\cap S\cap(V-v)\setminus\{0\}$. Note that $|(S+x)\cap S| < \kappa$ for every nonzero $x\in V-V$. By following the proof of Lemma 3.2, we get the required result. $\square$

Lemma 3.4. *Let V be a subset of the reals of size κ^+. If S is a set such that $|S\cap(V-v)|\le\kappa$ and $|S\cap(v-V)|\le\kappa\forall v\in V$, then there exists a set W such that $|W| = \kappa^+$ and $W\subseteq W-W\subseteq((V-V)\setminus S)\cup\{0\}$.*

PROOF. Let $v_1\in V$. Since $|(S+v_1)\cap V| = |S\cap(V-v_1)|\le\kappa$ and $|(-S+v_1)\cap V| = |-S\cap(V-v_1)| = |S\cap(v_1-V)|\le\kappa$, we have $|((S+v_1)\cup(-S+v_1))\cap V|\le\kappa$. Hence, because of $|V| = \kappa^+$, there exists an element $v_2\in V$ such that $v_2\ne v_1$

and $v_2 - v_1 \notin S \cup -S$. Since $|((S+v_1)\cup(S+v_2)\cup(-S+v_1)\cup(-S+v_2))\cap V| \leq \kappa$ and $|V| = \kappa^+$, there exists an element $v_3 \in V$ such that v_1, v_2, v_3 are distinct and $v_3 - v_1$, $v_3 - v_2$, $v_2 - v_1 \notin S \cup -S$. Continuing in this way, by transfinite induction, we get a set $Y = \{v_i : i < \kappa^+\}$ such that $v_i - v_j \notin S$ for all v_i, $v_j \in Y$ with $i \neq j$. Let $W = Y - v_1$. Then the set W is of size κ^+ and $W \subseteq W - W = Y - Y \subseteq ((V - V)\backslash S) \cup \{0\}$. $\square$

PROOF of Theorem 3.1. Let $(S_i)_{1 \leq i \leq n}$ be a finite collection of κ sum free or κ difference free sets in $V - V$. By Lemma 3.2, 3.3, or 3.4, there exists a set W such that $|W| = \kappa^+$ and $W \subseteq W - W \subseteq ((V - V)\backslash S_1) \cup \{0\}$. Again by applying Lemma 3.2, 3.3, or 3.4 to W, we obtain that there exists a set Y such that $|Y| = \kappa^+$ and $Y \subseteq Y - Y \subseteq ((W - W)\backslash S_2) \cup \{0\}$. Hence $Y - Y \subseteq ((W - W)\backslash S_2) \cup \{0\} \subseteq ((V - V)\backslash (S_1 \cup S_2)) \cup \{0\}$. Continuing in this way, we obtain that $((V - V)\backslash \cup_{i=1}^{i=n} S_i) \cup \{0\}$ contains a set of size κ^+ . Hence $|(V - V)\backslash \cup_{i=1}^{i=n} S_i| = \kappa^+$, and $V - V$ is not a finite union of k sum free or k difference free sets in $V - V$. By definition, any k sum free or k difference free set in $\mathbb{R}$ is k sum free or k difference free set in $V - V$. Thus $V - V$ is not a finite union of k sum free or k difference free sets in $\mathbb{R}$. $\square$

Corollary 3.5. *If G is an additive subgroup of the reals of size κ^+ and $\kappa < 2^\omega$, then G is not a finite union of κ sum free or κ difference free sets in G. In fact, $\left|G\backslash \cup_{i=1}^{i=n} S_i\right| = \kappa^+$, where each S_i is a κ sum free or κ difference free set in G.*

Corollary 3.6. *Let κ be a cardinal smaller than 2^ω. Then the cardinality of the complement of a finite union of κ sum free or κ difference free subsets of the reals is 2^ω.*

PROOF. To see this, suppose that $\left|\mathbb{R}\backslash \cup_{i=1}^{i=n} S_i\right| = \lambda < 2^\omega$, where each S_i is a κ sum free or κ difference free subset of the reals. Let $\mu = \max\{\kappa, \lambda\}$ and let G be an additive subgroup of the reals of size μ^+. Note that any κ sum free or κ difference free set is μ sum free or μ difference free set and any μ sum free or μ difference free set (in $\mathbb{R}$) is μ sum free or μ difference free set in G. It follows from Corollary 3.5 that $\left|G\backslash \cup_{i=1}^{i=n} S_i\right| = \mu^+$, where each S_i is μ sum free or μ difference free in G. This contradicts our assumption that $\left|\mathbb{R}\backslash \cup_{i=1}^{i=n} S_i\right| = \lambda$ and $\lambda \leq \mu$. $\square$

Corollary 3.7. *Let $f : \mathbb{R} \to \{1, 2, ..., n\}$ be a function and, for each $x \in \mathbb{R}$, let $S_x = \{h > 0 : f(x - h) = f(x + h)\}$. Then, for each cardinal $\lambda < 2^\omega$ the cardinality of the set $\{x : |S_x| \geq \lambda\}$ is 2^ω.*

PROOF. Let $\lambda < 2^\omega$ and $X = \{x : |S_x| \geq \lambda\}$. Assume, to the contrary, that $|X| = \mu < 2^\omega$. Hence $|S_x| < \lambda$ for some real number x. By translating the function f if necessary, we may assume that $|S_0| < \lambda$. Let $\kappa = \max\{\lambda, \mu\}$. In Lemma 3.4, choose $S = 2X$ and V to be an additive subgroup of the reals of size κ^+. Now let $Y = V \setminus 2X$. Then, by Lemma 3.4 together with the fact that $V - V = V$ and $0 \notin X$, the set Y contains a set W such that $|W| = \kappa^+$ and $W - W \subseteq Y$. We will show that, for each $1 \leq i \leq n$, the set $f^{-1}(i)$ is λ sum free in the set Y. For, let $y \in Y$. If $y = a + b$ for some $a, b \in f^{-1}(i)$ with $a \neq b$, then $a = \frac{y}{2} - \frac{b-a}{2}$ and $b = \frac{y}{2} + \frac{b-a}{2}$ and hence

$$\left|a - \frac{y}{2}\right| = \left|\frac{b-a}{2}\right| \in S_{\frac{y}{2}}. \tag{1}$$

By the choice of y, $\frac{y}{2} \notin X$. Since $|S_{\frac{y}{2}}| < \lambda$, it follows from (1) that the set $f^{-1}(i)$ is λ sum free in Y. Hence $\mathbb{R}$, in particular $W - W$, is a finite union of λ sum free sets in Y. Since $W - W \subseteq Y$, any λ sum free set in Y is λ sum free in $W - W$. Consequently, since $\lambda \leq \kappa$, $W - W$ is a finite union of κ sum free sets in $W - W$, which contradicts Theorem 3.1. $\square$

Definition 3.8. *For any two subsets X and C of $\mathbb{R}$, we define $Tr(X, C) = \{r \in \mathbb{R} : X + r \subseteq C\}$. A subset C of $\mathbb{R}$ is called κ covering for $\mathbb{R}$, if $Tr(X, C) \neq \emptyset$ for each subset X of $\mathbb{R}$ of size κ.*

It follows from the following theorem that a finite union of countably many translates of 2^ω difference free sets is not residual in an interval. In the above statement, "countably many"can be replaced by "fewer than continuum many"provided that 2^ω is a regular cardinal.

Theorem 3.9. *Let G be a set such that $|G| < 2^\omega$, μ an infinite cardinal smaller than 2^ω, and C a μ covering for $\mathbb{R}$.*

(i) *If κ is a cardinal smaller than 2^ω and a set S is κ^+ difference free, then the set $C \setminus (S + G)$ is μ covering for $\mathbb{R}$.*

(ii) *If a set S is 2^ω difference free and $cf(2^\omega) > Max\{|G|, \mu\}$, then the set $C \setminus (S + G)$ is μ covering for $\mathbb{R}$. In particular, if a set S is 2^ω difference free and $cf(2^\omega) > |G|$, then the set $\mathbb{R} \setminus (S + G)$ is ω covering for $\mathbb{R}$.*

Lemma 3.10. *If a set C is μ covering for $\mathbb{R}$, $\omega \leq \mu < 2^\omega$, and X is a set of size μ, then $|Tr(X, C)| = 2^\omega$.*

PROOF. Let $r \in \mathbb{R}$. Since $|X + \{0, r\}| = \mu$ and the set C is μ covering, $X + \{0, r\} + y \subseteq C$ for some real number y. Hence, $0 + y$ and $r + y$ are elements of $Tr(X, C)$. This implies that $\forall r \in \mathbb{R}$, $r = (r+y) - (0+y) \in Tr(X, C) - Tr(X, C)$. Consequently, $\mathbb{R} = Tr(X, C) - Tr(X, C)$ and $|Tr(X, C)| = 2^\omega$. $\square$

PROOF of Theorem 3.9(i). Assume, to the contrary, that $C\backslash(S+G)$ is not μ covering for $\mathbb{R}$. Then there exists a subset X of $\mathbb{R}$ of size μ such that $(X+r)\cap(S+G)\neq\emptyset\ \forall r\in Tr(X,C)$. Let H be the additive group generated by the set $G-X$. Then $|H|<2^\omega$ and $Tr(X,C)\subseteq S+H$. Pick real numbers a and b such that $a-b\notin H$ and denote the set $X+\{a,b\}$ by Y. Then, since the cardinality of the set Y is μ and the set C is μ covering, $Y+r\subseteq C\ \forall r\in Tr(Y,C)$. Note that $\forall r\in Tr(Y,C), a+r,\ b+r\in Tr(X,C)\subseteq S+H$, $|Tr(Y,C)|=2^\omega$ and $|H|\leq\lambda<2^\omega$, where $\lambda=\max\{|G|,\mu,\kappa\}$. Hence, there exist $h\in H$ and a subset P of $Tr(Y,C)$ of size λ^+ such that $a+p\in S+h\forall p\in P$. Since $b+p\in Tr(X,C)\subseteq S+H\forall p\in P$, there exist $h_1\in H$ and a subset T of P such that $|T|=\lambda^+$ and $b+t\in S+h_1\forall t\in T$. Consequently, $\forall t\in T$, $a+t-h$ and $b+t-h_1$ belong to the set S and $(a+t-h)-(b+t-h_1)$ is constant. Also, note that $(a+t-h)-(b+t-h_1)$ is nonzero; otherwise $a-b$ would belong to the additive group H, contradicting the choice of a and b. Thus, we obtain that the set S is not λ^+ free and, since $\lambda\geq\kappa, S$ is not κ^+ free. This contradicts the hypothesis of the theorem that the set S is κ^+ free. □

PROOF of Theorem 3.9(ii). In the proof of Theorem 3.9(i), we obtained that $a+r,\ b+r\in S+H\ \forall r\in Tr(Y,C)$. Since $|H|=|G|\mu=Max\{|G|,\mu\}<2^\omega$, $|Tr(Y,C)|=2^\omega$, and $cf(2^\omega)>Max\{|G|,\mu\}$, by a slight modification of the proof of (i), we obtain that there exist $h,\ h_1\in H$ and a set W of size 2^ω such that $a+w\in S+h$ and $b+w\in S+h_1\forall w\in W$. This contradicts that the set S is 2^ω difference free. □

Corollary 3.11. *Let κ be a cardinal smaller than 2^ω. Then a finite union of fewer than continuum many translates of κ^+ difference free sets is not residual in an interval.*

PROOF. Assume, to the contrary, that $I\backslash F\subseteq\cup_{i=1}^{i=n}(S_i+H)$, where I is a nonempty open interval, F is a meager set, each set S_i is κ^+ difference free, and H is any set of size less than 2^ω. Then $\mathbb{R}\backslash(F+\mathbb{Q})\subseteq\cup_{i=1}^{i=n}(S_i+H+\mathbb{Q})$. Since $F+\mathbb{Q}$ is meager, it is easy to see that $\mathbb{R}\backslash(F+\mathbb{Q})$ is ω covering for $\mathbb{R}$. By applying part (i) of Theorem 3.9, we obtain that the set

$$(\mathbb{R}\backslash(F+\mathbb{Q}))\backslash\bigcup_{i=1}^{n}(S_i+H+\mathbb{Q})$$

is ω covering and empty, which is impossible. □

Corollary 3.12.

(i) *Let κ be a cardinal smaller than 2^ω. If $cf(2^\omega)>\kappa$, then $\mathbb{R}$ is not a finite union of κ many translates of 2^ω difference free sets. In fact, if*

$cf(2^\omega) > \kappa$, a finite union of κ many translates of 2^ω difference free sets is not residual in an interval.

(ii) *A finite union of countably many translates of 2^ω difference free sets is not residual in an interval.*

The proof of this corollary follows from part (ii) of Theorem 3.9 and the idea of the proof of Corollary 3.11.

Corollary 3.16 of the following theorem implies that, under the assumption of Martin's axiom, a countable union of 2^ω difference free sets is not residual in an interval.

Theorem 3.13. *Let a set C be ω_1 covering for $\mathbb{R}$.*

(i) *Let κ be a cardinal smaller than 2^ω. If $2^\omega \geq \omega_2$ and S_i is κ^+ difference free set for each i, then $C \setminus \cup_{1\leq i<\omega} S_i$ is ω covering for $\mathbb{R}$.*

(ii) *If $2^\omega \geq \omega_2$, S_i is 2^ω difference free set for each i, and $cf(2^\omega) > \omega_1$, then $C \setminus \cup_{1\leq i<\omega} S_i$ is ω covering for $\mathbb{R}$.*

Proof of Theorem 3.13(i). Suppose that $C\setminus \cup_{1\leq i<\omega} S_i$ is not ω covering for $\mathbb{R}$. Then there exists a countable subset X of $\mathbb{R}$ such that $(X+r)\cap(\cup_{1\leq i<\omega} S_i) \neq \emptyset \forall r \in Tr(X,C)$. For simplicity, denote $Tr(X,C)$ by T. Hence, since X is countable and $T \subseteq \cup_{1\leq i<\omega} S_i - X$, the set T is a subset of a countable union of countably many translates of κ^+ difference free sets S_i. Let $T \subseteq \cup_{1\leq i<\omega} H_i$, where each H_i is κ^+ difference free set. Let $Y \subseteq \mathbb{R}$ and $|Y| = \omega_1$. Then $T + y \subseteq \cup_{1\leq i<\omega}(H_i + y)\ \forall y \in Y$.

(1) Hence $\cap_{y\in Y}(T+y) \subseteq \cap_{y\in Y} \cup_{1\leq i<\omega} (H_i + y)$.

(2) Because H_i is κ^+ difference free set, $|(H_i+u)\cap(H_i+v)| \leq \kappa$ for distinct elements u and v of the set Y.

Since $|Y| = \omega_1$ and $\cap_{y\in Y} \cup_{1\leq i<\omega} (H_i + y) \subseteq \cup_{1\leq i<\omega} \cup_{u,v\in Y, u\neq v} (H_i + u) \cap (H_i + v)$, (1) and (2) imply that

(3) $|\cap_{y\in Y} (T+y)| \leq \kappa\omega_1 < 2^\omega$.

Let G be the additive group generated by the set $\cap_{y\in Y}(T+y)$. By (3), $|G| < 2^\omega$. Let $a \in \mathbb{R}\setminus G$. Since $|X - Y + \{a, 0\}| = \omega_1$ and C is ω_1 covering, we have $X - Y + \{a, 0\} + r \subseteq C$ for some real number r. This implies that $-y+a+r, -y+0+r \in T \forall y \in Y$ and hence $a+r, r \in \cap_{y\in Y}(T+y) \subseteq G$. Now $a = a+r-r \in G - G = G$, which contradicts the choice that $a \notin G$. Thus the proof of (i) is complete.

PROOF of Theorem 3.13(ii). By the hypothesis, $cf(2^\omega) > \omega_1$. It follows from the proof of (i) that $|(H_i + u) \cap (H_i + v)| < 2^\omega$ and $|\cap_{y \in Y} (T + y)| < 2^\omega$(see (2) and (3) in proof of (i)). The rest of the proof is identical to the proof of (i). $\square$

An immediate consequence of Theorem 3.13 is the following.

Corollary 3.14.

(i) *If* $2^\omega \geq \omega_2$ *and* $\kappa < 2^\omega$*, then the complement of a countable union of* κ^+ *difference free sets is* ω *covering for* $\mathbb{R}$ *and hence it is of the size of the continuum.*

(ii) *If* $2^\omega \geq \omega_2$ *and* $cf(2^\omega) > \omega_1$*, then the complement of a countable union of* 2^ω *difference free sets is* ω *covering for* $\mathbb{R}$ *and hence it is of the size of the continuum. In particular, if* 2^ω *is a regular cardinal and* $2^\omega \geq \omega_2$*, then the complement of a countable union of* 2^ω *difference free sets is* ω *covering for* $\mathbb{R}$.

Theorem 2.3 in [2] says that if V is a vector space over the rationals and $|V| \leq \omega_1$, then there exists a countable ω difference free partition of V. It follows from the following corollary that the converse of the above statement is true for $\mathbb{R}$.

Theorem 3.2 in [2] says that if V is a vector space over the rationals and $|V| \geq \omega_1$, then there is no countable κ difference free partition of V for any $\kappa < \omega$. A generalization of this theorem is given below.

Corollary 3.15. *If* $2^\omega \geq \omega_2$ *and* $\kappa < 2^\omega$*, then* $\mathbb{R}$ *is not a countable union of* κ^+ *difference free sets and hence, there is no countable* κ^+ *difference free partition of* $\mathbb{R}$.

Corollary 3.16. *If* $\mathbb{R}$ *is not a union of* ω_1 *many meager sets and* $cf(2^\omega) > \omega_1$*, in particular if* $MA(\omega_1)$ *holds, then a countable union of* 2^ω *difference free sets is not residual in an interval.*

PROOF. Assume, to the contrary, that $I \setminus F \subseteq \cup_{1 \leq i < \omega} S_i$, where I is a nonempty open interval, F is a meager set, and each set S_i is 2^ω difference free. Then $\mathbb{R} \setminus (F + \mathbb{Q}) \subseteq \cup_{1 \leq i < \omega}(S_i + \mathbb{Q})$. First, note that $\mathbb{R} \setminus (F + \mathbb{Q})$ is ω_1 covering for $\mathbb{R}$; otherwise for some set X of size ω_1, $(X + r) \cap (F + \mathbb{Q}) \neq \emptyset \ \forall r \in \mathbb{R}$ and hence $\mathbb{R}$ is a union of ω_1 many meager sets, which is a contradiction. By part (ii) of Theorem 3.13, $\mathbb{R} \setminus (F + \mathbb{Q}) \setminus \cup_{1 \leq i < \omega} (S_i + \mathbb{Q}) = \emptyset$ is ω covering, which is impossible. $\square$

References

[1] K. Ciesielski and L. Larson, *Uniformly antisymmetric functions*, Real Anal. Exchange, **19(1)** (1993-94), 226–235.

[2] K. Ciesielski, *Sum and difference free partitions of vector spaces*, Colloquium Mathematicum, **71** (1996), 263–271.

[3] A. Hajnal, *Proof of a conjecture of S. Ruziewicz*, Fund. Math., **50** (1961), 123–128.

[4] P. Komjath, *Vector sets with no repeated differences*, Colloquium Mathematicum, **64** (1993), 129–134.

[5] K. Muthuvel and A. Miller, *Everywhere of second category sets*, Real Anal. Exchange, **24(2)** (1998-99), 607–613.