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TURBULENCE PHENOMENA IN ELEMENTARY REAL ANALYSIS

Abstract

The purpose of this note is to show that if $-\infty < \alpha < \beta < \infty$ and E_α^β is the equivalence relation, which is defined on the Polish group $C([\alpha,\beta),\mathbb{R}_+^*)$ by $fE_\alpha^\beta g \iff \lim_{x\to\beta^-}\frac{f(x)}{g(x)} = 1$, where f, g are in $C([\alpha,\beta),\mathbb{R}_+^*)$, then E_α^β is induced by a turbulent Polish group action. Hence if L is any countable language and $\mathcal{A} : C([\alpha,\beta),\mathbb{R}_+^*) \to X_L$ is any Baire measurable function from the Polish group $C([\alpha,\beta),\mathbb{R}_+^*)$ to the Polish space X_L of countably infinite structures for L with the property that $fE_\alpha^\beta g \Rightarrow \mathcal{A}(f) \cong \mathcal{A}(g)$, whenever f, g are in $C([\alpha,\beta),\mathbb{R}_+^*)$, then there exists a E_α^β-invariant comeager subset S of $C([\alpha,\beta),\mathbb{R}_+^*)$ for which all countable structures in $\mathcal{A}[S]$ are isomorphic.

1 Introduction

An equivalence relation which is introduced in elementary calculus (see, for example, page 124 of Nikolsky [1977]) is the following. Given any real numbers α and β such that $\alpha < \beta$, two continuous functions $f : [\alpha,\beta) \to \mathbb{R}_+^*$ and $g : [\alpha,\beta) \to \mathbb{R}_+^*$ are said to be *equivalent* or *asymptotically equal* as the argument tends to β from the left, in symbols $fE_\alpha^\beta g$, if

$$\lim_{x\to\beta^-}\frac{f(x)}{g(x)} = 1.$$

Even though this equivalence relation seems to be very simple (for it is introduced in elementary calculus!), our purpose in this paper is to show that it is complicated.

A way to measure the complexity of an equivalence relation E defined on some Polish space X is to determine whether there exists a countable language

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L and a non-trivial Baire measurable function $f : X \to X_L$ with the property that

$$(\forall (x,y) \in X^2)(xEy \Rightarrow f(x) \cong f(y)) \qquad (\star)$$

Here X_L is the *Polish space of countably infinite structures for* L (see, for example, 16.5 on page 96 of Kechris [1995]) and $\cong$ stands for isomorphism of structures, while $f : X \to X_L$ is said to be *trivial* if there exists a E-invariant comeager subset A of X for which all countable structures in $f[A]$ are isomorphic. When such a language L and such a non-trivial function $f : X \to X_L$ exist, we say that E is *classifiable by countable structures* and E is considered to be "less complicated" than the equivalence relation of isomorphism between countable structures. But if for any countable language L, every Baire measurable function $f : X \to X_L$ with property ($\star$) is trivial, then we say that E is not classifiable by countable structures and E is considered to be "more complicated" than the equivalence relation of isomorphism between countable structures.

A method to prove that an equivalence relation E defined on some Polish space X is not classifiable by countable structures is to show that there exists a Polish group G acting continuously on X with the following properties:

- E is *induced by the action of* G *on* X; that is, we have $E = E_G^X$, where E_G^X is the corresponding *orbit equivalence relation*; namely,

  $$xE_G^X y \iff (\exists g \in G)(g \cdot x = y),$$

  whenever x, y are in X.

- The action of G on X is *generically turbulent*.

We explain what we mean below. (See, for example, Chapter 3 on pages 37-58 of Hjorth [2000] or pages 1461-1462 of Kechris-Sofronidis [2001].)

Definition. (Hjorth) Let G be any Polish group acting continuously on a Polish space X and let $x \in X$. For any open neighborhood U of x in X and for any symmetric open neighborhood V of 1^G in G, the (U,V)-local orbit $O(x,U,V)$ of x in X is defined as follows:

> $y \in O(x,U,V)$ if there exist $g_0, \ldots, g_k$ in V ($k \in \mathbb{N}$) such that if $x_0 = x$ and $x_{i+1} = g_i \cdot x_i$ for every $i \in \{0,\ldots,k\}$, then all the x_i are in U and $x_{k+1} = y$.

The action of G on X is said to be turbulent at the point x, in symbols $x \in T_G^X$, if for any such U and V, there exists an open neighborhood U' of x in X such that $U' \subseteq U$ and $O(x,U,V)$ is dense in U'.

Theorem. *(Hjorth) Let G be any Polish group acting continuously on a Polish space X in such a way that the orbits of the action are meager and at least one orbit is dense. Then the following are equivalent:*

- *The action of G on X is generically turbulent, in the sense that T_G^X is comeager in X.*
- *For any countable language L and for any Baire measurable function $f : X \to X_L$ with the property that*

$$(\forall (x,y) \in X^2)(xE_G^X y \Rightarrow f(x) \cong f(y)),$$

there exists a E_G^X-invariant comeager subset A of X for which all countable structures in $f[A]$ are isomorphic.

We are finally in position to state our result.

Theorem. *If $-\infty < \alpha < \beta < \infty$, then the relation E_α^β of asymptotic equality of functions in the Polish group $C([\alpha,\beta),\mathbb{R}_+^*)$ as the argument tends to β from the left is induced by a turbulent Polish group action.*

Hence if L is any countable language and $\mathcal{A} : C([\alpha,\beta),\mathbb{R}_+^*) \to X_L$ is any Baire measurable function with the property that

$$(\forall (f,g) \in C([\alpha,\beta),\mathbb{R}_+^*)^2)(fE_\alpha^\beta g \Rightarrow \mathcal{A}(f) \cong \mathcal{A}(g)),$$

then there is a E_α^β-invariant comeager subset S of $C([\alpha,\beta),\mathbb{R}_+^*)$ for which all countable structures in $\mathcal{A}[S]$ are isomorphic. In other words, the equivalence relation of asymptotic equality of functions in the Polish group $C([\alpha,\beta),\mathbb{R}_+^*)$ as the argument tends to β from the left is "more complicated" than the equivalence relation of isomorphism between countable structures.

2 The Proof of the Theorem

In what follows let α and β be two arbitrary but fixed real numbers such that $\alpha < \beta$.

Proposition 1. *$C([\alpha,\beta),\mathbb{R}_+^*)$ equipped with the compact-open topology constitutes a commutative Polish group under the operation of point-wise multiplication.*

PROOF. Since obviously $[\alpha,\beta) = \bigcap_{n\in\mathbb{N}} \left(\alpha - \frac{1}{n+1}, \beta\right)$ constitutes a G_δ subset of $\mathbb{R}$ and $\mathbb{R}_+^* = (0,\infty)$ is open and therefore G_δ in $\mathbb{R}$, by virtue of 3.11 on page

17 of Kechris [1995], we deduce that $[\alpha, \beta)$ and $\mathbb{R}_+^*$ are Polish in the relative topology. Moreover $[\alpha, \beta)$ is easily seen to be locally compact. Therefore, Theorem 1 on page 93 and Theorem 3 on page 94 of Kuratowski [1968] imply that the compact-open topology on $C([\alpha, \beta), \mathbb{R}_+^*)$ is Polish. The commutative group operation on $C([\alpha, \beta), \mathbb{R}_+^*)$ is point-wise multiplication. Thus, in order to prove that $C([\alpha, \beta), \mathbb{R}_+^*)$ is a commutative Polish group, by virtue of 9.15 on page 62 of Kechris [1995], it is enough to show that the mapping

$$\Phi : C([\alpha, \beta), \mathbb{R}_+^*) \ni f \mapsto \frac{1}{f} \in C([\alpha, \beta), \mathbb{R}_+^*)$$

is continuous and given any $g \in C([\alpha, \beta), \mathbb{R}_+^*)$, the mapping

$$\Psi_g : C([\alpha, \beta), \mathbb{R}_+^*) \ni f \mapsto fg \in C([\alpha, \beta), \mathbb{R}_+^*)$$

is also continuous. So let $f \in C([\alpha, \beta), \mathbb{R}_+^*)$, $\epsilon > 0$ and $0 < \eta < \beta - \alpha$. If $M = \max\limits_{\alpha \leq x \leq \alpha + \eta} g(x) > 0$ and $h \in C([\alpha, \beta), \mathbb{R}_+^*)$ is such that $\max\limits_{\alpha \leq x \leq \alpha + \eta} |f(x) - h(x)| < \frac{\epsilon}{M}$, then given $\alpha \leq x \leq \alpha + \eta$, we have

$$|f(x)g(x) - h(x)g(x)| \leq M|f(x) - h(x)|.$$

Hence $\max\limits_{\alpha \leq x \leq \alpha + \eta} |f(x)g(x) - h(x)g(x)| \leq M \cdot \max\limits_{\alpha \leq x \leq \alpha + \eta} |f(x) - h(x)| < \epsilon$. Therefore Ψ_g is continuous at f. The proof that Φ is also continuous at f is left as an exercise.

$\square$

The following proposition constitutes an immediate consequence of Proposition 5.6 (ii) on pages 1470-1471 of Kechris-Sofronidis [2001].

Proposition 2. *$C([\alpha, \beta], \mathbb{R}_+^*)$ equipped with the topology of uniform convergence constitutes a Polish group under the operation of point-wise multiplication.*

A corollary of Proposition 2 is the following.

Proposition 3. *$\mathcal{G} = \{f \in C([\alpha, \beta], \mathbb{R}_+^*) : f(\beta) = 1\}$ constitutes a closed subgroup of $C([\alpha, \beta], \mathbb{R}_+^*)$ and consequently it constitutes a Polish group with respect to the operation of point-wise multiplication and the topology of uniform convergence.*

Proposition 4. *For any $f \in C([\alpha, \beta), \mathbb{R}_+^*)$, the equivalence class $[f]_{E_\alpha^\beta}$ of f is dense in $C([\alpha, \beta), \mathbb{R}_+^*)$.*

Proof. Let $g \in C([\alpha,\beta),\mathbb{R}_+^*)$ and let $0<\eta<\beta-\alpha$. It is enough to construct $h \in [f]_{E_\alpha^\beta}$ such that $h=g$ on $[\alpha,\alpha+\eta]$. So let $\theta>0$ be such that $\alpha+\eta+\theta<\beta$. We set

$$h(x)=\begin{cases} g(x) & \text{if } x\in[\alpha,\alpha+\eta) \\ g(\alpha+\eta)+\frac{f(\alpha+\eta+\theta)-g(\alpha+\eta)}{\theta}(x-\alpha-\eta) & \text{if } x\in[\alpha+\eta,\alpha+\eta+\theta) \\ f(x) & \text{if } x\in[\alpha+\eta+\theta,\beta) \end{cases}$$

It is not difficult to verify that $h \in C([\alpha,\beta),\mathbb{R}_+^*)$ and $h\in[f]_{E_\alpha^\beta}$, while by definition $h=g$ on $[\alpha,\alpha+\eta]$. □

Proposition 5. *For any $f \in C([\alpha,\beta),\mathbb{R}_+^*)$, the equivalence class $[f]_{E_\alpha^\beta}$ of f is meager in $C([\alpha,\beta),\mathbb{R}_+^*)$.*

Proof. If $g\in[f]_{E_\alpha^\beta}$, then $\lim\limits_{x\to\beta^-}\frac{g(x)}{f(x)}=1$. Hence there exists $r\in\mathbb{Q}\cap(\alpha,\beta)$ such that for any $x\in[r,\beta)$, we have $\left|\frac{g(x)}{f(x)}-1\right|<\frac{1}{2}\Rightarrow g(x)<\frac{3}{2}f(x)$. Therefore, we have $[f]_{E_\alpha^\beta}\subseteq M$, where

$$M=\bigcup_{r\in\mathbb{Q}\cap(\alpha,\beta)}\bigcap_{x\in[r,\beta)}\left\{g\in C([\alpha,\beta),\mathbb{R}_+^*): g(x)\leq\frac{3}{2}f(x)\right\}$$

is easily seen to be F_σ in $C([\alpha,\beta),\mathbb{R}_+^*)$. Hence it is enough to show that M is meager or (equivalently) that $G=C([\alpha,\beta),\mathbb{R}_+^*)\setminus M$ is dense in $C([\alpha,\beta),\mathbb{R}_+^*)$. But this can be shown by similar methods as the ones employed in the proof of Proposition 4. □

Proposition 6. *The mapping*

$$\mathcal{A}:\mathcal{G}\times C([\alpha,\beta),\mathbb{R}_+^*)\ni(g,f)\mapsto g\cdot f=(g|[\alpha,\beta))f\in C([\alpha,\beta),\mathbb{R}_+^*)$$

constitutes a continuous action of $\mathcal{G}$ on $C([\alpha,\beta),\mathbb{R}_+^*)$.

Proof. It is not difficult to verify that the mapping in question constitutes an action of $\mathcal{G}$ on $C([\alpha,\beta),\mathbb{R}_+^*)$. Since

$$C([\alpha,\beta],\mathbb{R}_+^*)\ni g\mapsto g|[\alpha,\beta)\in C([\alpha,\beta),\mathbb{R}_+^*)$$

is easily seen to be continuous, it follows that so is

$$\mathcal{G}\ni g\mapsto g|[\alpha,\beta)\in C([\alpha,\beta),\mathbb{R}_+^*)$$

and by virtue of Proposition 1 we deduce that $\mathcal{A}$ is continuous because so is

$$C([\alpha,\beta),\mathbb{R}_+^*)^2\ni(f,g)\mapsto fg\in C([\alpha,\beta),\mathbb{R}_+^*).$$ □

Proposition 7. *E_α^β is induced by the action of $\mathcal{G}$ on $C([\alpha,\beta),\mathbb{R}_+^*)$.*

PROOF. What we need to show is that given any u, v in $C([\alpha,\beta),\mathbb{R}_+^*)$, we have

$$uE_\alpha^\beta v \iff (\exists g \in \mathcal{G})(v = g \cdot u).$$

Indeed, if there exists $g \in \mathcal{G}$ such that $v = g \cdot u$, then

$$\lim_{x\to\beta^-} \frac{v(x)}{u(x)} = \lim_{x\to\beta^-} g(x) = 1,$$

which implies that $vE_\alpha^\beta u$. Conversely, if $vE_\alpha^\beta u$, then $\lim\limits_{x\to\beta^-} \frac{v(x)}{u(x)} = 1$. Hence it is not difficult to verify that the function $g : [\alpha,\beta] \to \mathbb{R}_+^*$ defined by the relation

$$g(x) = \begin{cases} \frac{v(x)}{u(x)} & \text{if } x \in [\alpha,\beta) \\ 1 & \text{if } x = \beta \end{cases}$$

is in $\mathcal{G}$ and moreover $v = g \cdot u$. □

In what follows let

$$U(f;\epsilon,\eta) = \left\{ g \in C([\alpha,\beta),\mathbb{R}_+^*) : \max_{\alpha\le x\le\alpha+\eta} |f(x) - g(x)| < \epsilon \right\},$$

whenever $f \in C([\alpha,\beta),\mathbb{R}_+^*)$, $\epsilon > 0$ and $0 < \eta < \beta - \alpha$. It is not difficult to see that the $U(f;\epsilon,\eta)$ form a base of open neighborhoods of f in $C([\alpha,\beta),\mathbb{R}_+^*)$.

Proposition 8. *If $f \in C([\alpha,\beta),\mathbb{R}_+^*)$, $\epsilon > 0$, $0 < \eta < \beta - \alpha$ and $g \in \mathcal{G}$ are such that $g \cdot f \in U(f;\epsilon,\eta)$, then there exists a continuous path*

$$[0,1] \ni t \mapsto h_t \in \mathcal{G}$$

such that $h_0 = 1^{\mathcal{G}}$, $h_1 = g$ and $h_t \cdot f \in U(f;\epsilon,\eta)$ for every $t \in [0,1]$.

PROOF. Given $t \in [0,1]$, we set $h_t = 1 - t + tg$ and it is not difficult to verify that $h_t \in \mathcal{G}$, while obviously $h_0 = 1^{\mathcal{G}}$ and $h_1 = g$. Moreover, if s, t are in $[0,1]$, then for any $x \in [\alpha,\beta]$, we have

$$|h_s(x) - h_t(x)| = |g(x) - 1| \cdot |s - t|.$$

Hence

$$\max_{\alpha\le x\le\beta} |h_s(x) - h_t(x)| \le C|s - t|,$$

where

$$C = \max_{\alpha\le x\le\beta} |g(x) - 1| \in \mathbb{R}_+.$$

Therefore, we deduce that the path

$$[0,1] \ni t \mapsto h_t \in \mathcal{G}$$

is continuous. What is left to show is that $h_t \cdot f \in U(f;\epsilon,\eta)$ for every $t \in [0,1]$. So let $t \in [0,1]$ and let $x \in [\alpha, \alpha+\eta]$. Then

$$|(h_t \cdot f)(x) - f(x)| = t|g(x)f(x) - f(x)| \leq |g(x)f(x) - f(x)| = |(g \cdot f)(x) - f(x)|,$$

hence

$$\max_{\alpha \leq x \leq \alpha+\eta} |(h_t \cdot f)(x) - f(x)| \leq \max_{\alpha \leq x \leq \alpha+\eta} |(g \cdot f)(x) - f(x)| < \epsilon,$$

since $g \cdot f \in U(f;\epsilon,\eta)$, and consequently $h_t \cdot f \in U(f;\epsilon,\eta)$. □

The following proposition is Lemma 5.7 on page 1472 of Kechris-Sofronidis [2001].

Proposition 9. *Let G be any Polish group acting continuously on a Polish space X and let $x \in X$. Suppose $G \cdot x$ is dense in X and there exists a basis of open neighborhoods U of x in X with the property that for any $g \in G$ for which $g \cdot x \in U$, there exists $h \in G$ and a continuous path*

$$[0,1] \ni t \mapsto h_t \in G$$

such that $g \cdot x = h \cdot x$, $h_0 = 1^G$, $h_1 = h$ and $h_t \cdot x \in U$, whenever $t \in [0,1]$. Then the action of G on X is turbulent at the point x.

Proof. Let V be any open neighborhood of x in X and let W be any symmetric open neighborhood of the identity in G. Then there exists an open neighborhood U of x in X which is contained in V and satisfies the condition stated in the formulation of Proposition 9. We need only prove that

$$O(x,U,W) = U \cap (G \cdot x).$$

So let $g \in G$ be such that $g \cdot x \in U \cap (G \cdot x)$ and let $h \in G$ and

$$[0,1] \ni t \mapsto h_t \in G$$

be as in the statement of Proposition 9. Then there exists a positive integer N such that for any s, t in $[0,1]$, we have

$$|s-t| \leq N^{-1} \Rightarrow h_s \cdot h_t^{-1} \in W.$$

Hence, setting $t_0 = 0$, $t_k = t_{k-1} + N^{-1}$ and $g_k = h_{t_k} \cdot h_{t_{k-1}}^{-1}$, whenever $k \in \{1, \ldots, N\}$, it follows immediately that $g_k \in W$ and

$$g_k \ldots g_1 \cdot x = h_{t_k} \cdot x \in U,$$

whenever $k \in \{1, \ldots, N\}$, while

$$g_N \ldots g_1 \cdot x = g \cdot x.$$

We have thus proved that $O(x, U, W) = U \cap (G \cdot x)$. $\square$

Thus, Propositions 4–9 imply that E_α^β is induced by a turbulent Polish group action and the proof of the Theorem is complete.

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