Real Analysis Exchange
ISSN:0147-1937
Vol. 31(1), 2005/2006, pp. 55–62

Michał Rams,* Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland. email: `rams@impan.gov.pl`

MEASURES OF MAXIMAL DIMENSION FOR LINEAR HORSESHOES

Abstract

We consider a linear Smale-William's' horseshoe with different contraction/dilatation coefficients and find equilibrium states of maximal Hausdorff dimension. We compute this dimension and show an example when the state of maximal dimension is non-unique.

1 Introduction.

In the one-dimensional dynamical systems one of most important properties of the hyperbolic (expanding) systems is the existence of an unique equilibrium state. It is an ergodic measure, invariant under the dynamics and equivalent to the geometric (Hausdorff) measure on the hyperbolic set. The Hausdorff dimension of this measure is equal to the dimension of the hyperbolic set.

Such a property is satisfied also for some of the higher-dimensional systems, in particular for linear horseshoes with all the contraction coefficients equal and all the dilatation coefficients equal [MM]. However, in more general situation, the supremum of Hausdorff dimensions of the invariant measures may be strictly smaller than the dimension of the invariant set.

The natural question then appears about the existence of measure for which this supremum is achieved. It was answered positively for topologically mixing surface diffeomorphisms by Barreira and Wolf in [BW].

In this paper we consider a very simple special case—the linear Smale-William's' horseshoe. We give an efficient way of computing this measure of maximal dimension and show that it need not to be unique.

The main results of the paper are:

Key Words: hyperbolic maps, Smale's horseshoe, ergodic measures, measures of maximal dimension
Mathematical Reviews subject classification: Primary: 37D35; Secondary: 28D20
Received by the editors October 20, 2004
Communicated by: Richard J. O'Malley
*supported by Polish KBN Grant No 2 P03A 025 12

Theorem 1.1. *Every local maximum of Hausdorff dimension over ergodic invariant probabilistic measures is achieved inside certain one-dimensional family $\vec{p}(s)$ of Bernoulli measures. The values of the parameter s for which the local maximum is achieved are attracting fixed points for certain monotone real map W.*

(proved in third section, the formulas for $\vec{p}(s)$ and W are given there) and

Example 1.2. The ergodic measure of maximal dimension need not to be unique.

(construction of an example given in fourth section).

2 Preliminaries.

For the definitions and results in one-dimensional fractal geometry we refer the reader to [F].

Let us define the dynamical system we will be working with. Let $\{I_i\}$ and $\{J_i\}$ be two k-element families ($k > 1$) of pairwise disjoint (inside the family) closed subintervals of $[0,1]$. Let f_i be an affine map, mapping $I_i \times [0,1]$ onto $[0,1] \times J_i$, preserving horizontal and vertical directions. Of course, such a map can be chosen in four ways; for each i we choose one of them.

Any point $(x,y) \in [0,1]^2$ can belong to the domain of only one map f_i (iff $x \in I_i$) and to the predomain of only one map f_j (iff $y \in J_j$). Hence, we can define f and f^{-1} on some subset of $[0,1]^2$.

We denote by Λ the hyperbolic set; i.e., the set of points for which both f^n and f^{-n} exist for all n. It is a product of two Cantor sets (attractors of two iterated function systems, one acting in the horizontal and one in vertical directions). We let $\lambda_i = |I_i|$ and $\mu_i = |J_i|$.

For any point $(x,y) \in \Lambda$ we define its symbolic expansion as an element of $\Sigma = \{1,\ldots,k\}^{\mathbb{Z}}$. We will denote it by $\omega(x,y) = \ldots\omega_{-1}\omega_0\omega_1\ldots$, ω_i such that $f^i(x,y) \in I_{\omega_i} \times [0,1]$. Such a symbolic expansion is uniquely defined for all points in Λ. For an infinite sequence ω we will denote by ω^n the finite sequence $\omega_0,\ldots,\omega_n$.

Let ν be an f-invariant ergodic probability measure on Λ. By [BPS], its dimension is given by

$$\dim_H(\nu) = \frac{S}{L} + \frac{S}{M}, \tag{2.1}$$

where S is a metric entropy of ν and L, M are absolute values of Lyapunov exponents:

$$
\begin{aligned}
S &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{\omega^n} -\nu(C_{\omega^n}) \log \nu(C_{\omega^n}),\\
L &= \sum_{i=1}^{k} -\nu(C_i) \log \lambda_i,\\
M &= \sum_{i=1}^{k} -\nu(C_i) \log \mu_i.
\end{aligned}
$$

Here C_i denotes the i-th cylinder; i.e., the set of points (x, y) such that $\omega_0(x, y) = i$. Similarly, C_{ω^n} is the n-th level cylinder; i.e., the set of points (x, y) such that $\omega_j(x, y) = \omega_j$ for all $j \in [0, n]$. The Lyapunov exponents can be written in such a simple form because the measure is invariant and the system is piecewise linear.

We can further restrict our attention thanks to the following easy lemma.

Lemma 2.1. *The metric entropy of ν is not greater than the metric entropy of the Bernoulli measure defined by probabilistic vector $\{\nu(C_i)\}$, with equality if and only if ν is Bernoulli.*

PROOF. Let

$$S_n = \frac{1}{n+1} \sum_{\omega^n} -\nu(C_{\omega^n}) \log \nu(C_{\omega^n}).$$

We can write

$$
\begin{aligned}
S_{n+1} &= \frac{1}{n+2} \sum_{\omega^{n+1}} -\nu(C_{\omega^{n+1}}) \log \nu(C_{\omega^{n+1}})\\
&= \frac{1}{n+2} \sum_{\omega^{n+1}} -\nu(C_{\omega^{n+1}}) \log \nu(C_{\omega^n}) + \sum_{\omega^{n+1}} -\nu(C_{\omega^{n+1}}) \log \frac{\nu(C_{\omega^{n+1}})}{\nu(C_{\omega^n})}\\
&= \frac{n+1}{n+2} S_n + \frac{1}{n+2} \tilde{S}_{n+1},
\end{aligned}
$$

where

$$\tilde{S}_n = \sum_{\omega^{n-1}} -\nu(C_{\omega^{n-1}}) \sum_{i=1}^{k} \frac{\nu(C_{\omega^{n-1}i})}{\nu(C_{\omega^{n-1}})} \log \frac{\nu(C_{\omega^{n-1}i})}{\nu(C_{\omega^{n-1}})}.$$

Of course, $\tilde{S}_n \le S_0$, with equality if and only if $\frac{\nu(C_{\omega^{n-1}i})}{\nu(C_{\omega^{n-1}})} = \text{const} = \nu(C_i)$. As the function $\sum -p_i \log p_i$ is concave on the simplex of probabilistic vectors,

$$\frac{\nu(C_{\omega^{n-1}i})}{\nu(C_{\omega^{n-1}})} = \sum_{j=1}^{k} \frac{\nu(C_{j\omega^{n-1}})}{\nu(C_{\omega^{n-1}})} \frac{\nu(C_{j\omega^{n-1}i})}{\nu(C_{j\omega^{n-1}})}$$

implies

$$\sum_{i=1}^{k} - \frac{\nu(C_{\omega^{n-1}i})}{\nu(C_{\omega^{n-1}})} \log \frac{\nu(C_{\omega^{n-1}i})}{\nu(C_{\omega^{n-1}})} \ge \sum_{j=1}^{k} - \frac{\nu(C_{j\omega^{n-1}})}{\nu(C_{\omega^{n-1}})} \sum_{i=1}^{k} \frac{\nu(C_{j\omega^{n-1}i})}{\nu(C_{j\omega^{n-1}})} \log \frac{\nu(C_{j\omega^{n-1}i})}{\nu(C_{j\omega^{n-1}})}.$$

Hence,

$$\begin{aligned}\tilde{S}_n &\ge \sum_{j=1}^{k} \sum_{\omega^{n-1}} - \frac{\nu(C_{j\omega^{n-1}})}{\nu(C_{\omega^{n-1}})} \nu(C_{\omega^{n-1}}) \sum_{i=1}^{k} \frac{\nu(C_{j\omega^{n-1}i})}{\nu(C_{j\omega^{n-1}})} \log \frac{\nu(C_{j\omega^{n-1}i})}{\nu(C_{j\omega^{n-1}})} \\ &= \sum_{\omega^n} -\nu(C_{\omega^n}) \sum_{i=1}^{k} \frac{\nu(C_{\omega^n i})}{\nu(C_{\omega^n})} \log \frac{\nu(C_{\omega^n i})}{\nu(C_{\omega^n})} = \tilde{S}_{n+1},\end{aligned}$$

$$S = \lim S_n = \liminf \tilde{S}_n \le S_0 = S(\tilde{\nu}),$$

where $\tilde{\nu}$ is the Bernoulli measure defined by the probability vector $\{\nu(C_i)\}$ and the equality is true if and only if $\nu = \tilde{\nu}$. $\square$

We can thus consider Bernoulli measures only (defined by a probabilistic vector $\vec{p} = \{p_i\}$) and write S, L, M in even simpler form

$$S = \sum -p_i \log p_i, \;\; L = \sum -p_i \log \lambda_i, \;\; \text{and} \;\; M = \sum -p_i \log \mu_i.$$

3 Proof of Theorem 1.1.

We are now to look for the maximum of $F(\vec{p}) = \dim_H(\mu_{\vec{p}})$ over the simplex Z of nonnegative probabilistic vectors. The function we are maximizing is continuous. Hence the maximum is achieved. If $\log \lambda_i$ and $\log \mu_i$ are proportional, specifically

$$\log \lambda_i = c \log \mu_i, \tag{3.1}$$

then the situation is trivial. The Moran measures $p_i = s \log \lambda_i$ and $\tilde{p}_i = \tilde{s} \log \mu_i$ ($s, \tilde{s}$ - normalizing constants) coincide. Hence the maximum is achieved for this measure. The dimension of this measure is equal to the dimension of Λ. This measure is proportional to the Hausdorff measure on Λ.

In what follows we assume that (3.1) does not hold.

Lemma 3.1. *There exists no local maximum of F on the boundary of Z.*

PROOF. Consider $\vec{p}$ on the boundary of Z. At least one of p_i's is equal to 0, without loss of generalization let $p_1 = p_2 = ... = p_n = 0$. Consider

$$\vec{p}(t) = (t, t, \dots, t, 0, \dots, 0) + (1 - nt)\vec{p},$$

where the first summand has n first coordinates equal to t. The image of $(0, \varepsilon]$ is an interval placed strictly inside Z while $\vec{p}(0) = \vec{p}$. We have $\frac{d}{dt}F(\vec{p}(t)) > 0$ for all t small enough (except $t = 0$, where the derivative does not exists). F is a strictly increasing function along the chosen interval; hence $\vec{p}$ is not a local maximum of F. □

All the maxima of F are thus strictly inside Z. Let $\vec{p}$ be such an maximum. We have

$$\frac{\partial F}{\partial p_i} = -\frac{(1 + \log p_i)L - S \log \lambda_i}{L^2} - \frac{(1 + \log p_i)M - S \log \mu_i}{M^2}.$$

Using p_i as Lagrange multipliers, we get (for all i)

$$\frac{\partial F}{\partial p_i}(\vec{p}) = \sum_{j=1}^{k} p_j \frac{\partial F}{\partial p_j}(\vec{p}) = -\frac{1}{L(\vec{p})} - \frac{1}{M(\vec{p})}.$$

Hence,

$$\begin{aligned} \log p_i\left(\frac{1}{L(\vec{p})} + \frac{1}{M(\vec{p})}\right) &= S(\vec{p})\left(\frac{\log \lambda_i}{L^2(\vec{p})} + \frac{\log \mu_i}{M^2(\vec{p})}\right) \\ \frac{\log p_i}{\log p_j} &= \frac{M^2(\vec{p}) \log \lambda_i + L^2(\vec{p}) \log \mu_i}{M^2(\vec{p}) \log \lambda_j + L^2(\vec{p}) \log \mu_j}, \end{aligned}$$

and finally

$$\log p_i = r(\vec{p})(\log \lambda_i + s^2(\vec{p}) \log \mu_i) \tag{3.2}$$

where

$$s(\vec{p}) = \frac{L(\vec{p})}{M(\vec{p})} \tag{3.3}$$

and r is a normalizing constant.

$$\sum \lambda_i^{r(\vec{p})} \mu_i^{s^2(\vec{p}) r(\vec{p})} = 1 \tag{3.4}$$

Note that the same formulas apply for all points where the derivative of F vanishes, not only for the local maxima of F. Note also the similarity between (3.2) and $\nu_{p,q}$ in [BW].

The formula (3.2) does not immediately give us $\vec{p}$, as both s and r themselves depend on $\vec{p}$. Let us consider the family of probability vectors $\vec{p}(s)$ given by (3.2), where s is any nonnegative parameter. It is an one-parameter family, uniquely defined by a parameter s. Let

$$W(s) = \frac{L(\vec{p}(s))}{M(\vec{p}(s))} \tag{3.5}$$

The derivative of F vanishes if and only if the equation (3.3) is satisfied (where L and M are both computed at $\vec{p}(s)$). Hence, all such points and only such points are fixed points of W.

We can now state our main result, which will be proved in the rest of this section.

Theorem 3.2. *If $\vec{p}(s)$ is a local maximum of F then s is an attracting fixed point of W.*

We start from two computations.

Proposition 3.3. *W is strictly increasing.*

PROOF. We can compute dr/ds from (3.4) and dp_i/ds from (3.2). Together, we get

$$\frac{dW}{ds} = -\frac{2sr}{SL^2M^2}(MSE_{LM} - M^2E_{LS} - LSE_{MM} + LME_{SM}), \tag{3.6}$$

where

$$E_{L^jM^lS^m} = \sum_{i=1}^{k} p_i(\log\lambda_i)^j(\log\mu_i)^l(\log p_i)^m.$$

Substituting (3.2) into (3.6) we get

$$\begin{aligned}\frac{dW}{ds} &= \frac{2sr}{M^2(Ms^2+L)}(L^2E_{MM} + M^2E_{LL} - 2LME_{LM})\\ &= \frac{2srL^2}{Ms^2+L}\sum_{i=1}^{k} p_i(\frac{\log\lambda_i}{L} - \frac{\log\mu_i}{M})^2 \geq 0,\end{aligned}$$

with equality possible only if (3.1) holds. □

Proposition 3.4. *$dF(\vec{p}(s))/ds$ is positive if and only if $W(s) < s$. It is negative if and only if $W(s) > s$.*

Proof. We compute

$$\frac{dF(\vec{p}(s))}{ds} = \sum_{i=1}^{k} \frac{\partial F}{\partial p_i}\frac{dp_i}{ds}$$

as in the proof of previous proposition (using (3.4) and (3.2)). We get

$$\begin{aligned}\frac{dF}{ds} =& \frac{2rs}{SL^2M^2}(M^2S^2E_{LM} + L^2S^2E_{MM} - M^3SE_{LS}\\ &+ LM^2(L+M)E_{SS} - LMS(2L+M)E_{MS})\end{aligned}$$

and after substituting (3.2)

$$\begin{aligned}\frac{dF}{ds} &= \frac{2r^2s}{L^2M^2(Ms^2+L)}(L^2 - M^2s^2)(L^2E_{MM} + M^2E_{LL} - 2LME_{LM})\\ &= \frac{2r^2sM^2}{Ms^2+L}((W(s))^2 - s^2)\sum_{i=1}^{k} p_i(\frac{\log\lambda_i}{L} - \frac{\log\mu_i}{M})^2\end{aligned}$$

and the sum on the right is strictly positive when (3.1) does not hold. □

Now the proof is easy. First, by Proposition 3.4 all the zeroes of dF/ds correspond to the fixed points of W. (We did know that the fixed points of W correspond to the zeroes of the derivative of F as the function on the $(k-1)$-dimensional simplex Z, but the restriction of F to the curve $\vec{p}(s)$ might have introduced new extremal points.)

As the map W is strictly increasing, $W(s_0) > s_0$ means that the closest fixed point of W on the right of s_0 is left-side attracting and the closest fixed point on the left is right-side repulsing. Similarly, if $W(s_0) < s_0$, then the closest fixed point on the right is left-side repulsing and the closest fixed point on the left is right-side attracting.

By Proposition 3.4, $W(s_0) > s_0$ implies dF/ds is positive (F is locally increasing) while if $W(s_0) < s_0$, then F is locally decreasing. Hence, the fixed point of W is attracting if and only if the corresponding point $\vec{p}(s)$ is a local maximum for F (restricted to the curve $\vec{p}(\cdot)$). The assertion follows. □

4 Construction of Example 1.2.

Let us consider $G = S/L$ for $k = 2$ (as $p_2 = 1 - p_1$, it is a function of one real variable). One computes

$$\frac{d^2}{dp_1^2}G = -\frac{(\frac{1}{1-p_1}\log\lambda_1 + \frac{1}{p_1}\log\lambda_2)(p_1\log\lambda_1 + (1-p_1)\log\lambda_2)}{(p_1\log\lambda_1 + (1-p_1)\log\lambda_2)^3} - \frac{2(\log\lambda_1 - \log\lambda_2)(\log(1-p_1)\log\lambda_1 - \log p_1 \log\lambda_2)}{(p_1\log\lambda_1 + (1-p_1)\log\lambda_2)^3}.$$

It is not easy to check for which pairs (λ_1, λ_2) this formula is positive for some p (so S/L is not concave). It leads to a polynomial-logarithmic equation. However, the necessary and sufficient condition for this formula to be positive at $p_1 = 1/2$ is easy to write. It is

$$\Big(\frac{\log\lambda_1}{\log\lambda_2} + 1\Big)^2 < 2\log 2 \cdot \Big(\frac{\log\lambda_1}{\log\lambda_2} - 1\Big)^2. \tag{4.1}$$

Now we can construct our example. It will be a horseshoe with $k = 2$. (One can easily construct similar examples for arbitrary k.) Let λ_1, λ_2 satisfy (4.1). Let $\mu_1 = \lambda_2$ and $\mu_2 = \lambda_1$. We look for the maximum of $F = S/L + S/M$. Because of the symmetry, it is either a point $p_1 = p_2 = 1/2$ or it is non-unique. However, at this point both S/L and S/M are locally convex. Hence this point is not a local maximum of F.

References

[BPS] L. Barreira, Y. Pesin, J. Schmelling *Dimension and product structure of hyperbolic measures*, Ann. of Math., **149** (1999), 755–783.

[BW] L. Barreira, C. Wolf, *Measures of maximal dimension for hyperbolic diffeomorphisms*, Comm. Math. Phys., **239** (2003), 93–113.

[F] K. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, John Wiley and Sons, Chichester, 1995.

[MM] A. Manning, H. McCluskey, *Hausdorff dimension for horseshoes*, Erg. Th. and Dyn. Sys., **3** (1983), 251–260.