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DEFINING FUNCTIONS FOR OPEN SETS IN $\mathbb{R}^n$

Abstract

In this note we give, for any open subset in $\mathbb{R}^n$, a function describing the boundary of this set with exact regularity and being, globally, as regular as possible.

1 Introduction.

Let Ω be an open subset of $\mathbb{R}^n$.

For many purposes it is convenient to describe Ω by means of a function r whose zero level set is $\partial\Omega$, and is negative exactly in Ω. (See for example [2], [4]). Let us call such a function a defining function for Ω.

This approach is very useful in the case where Ω is a bounded domain and $\partial\Omega$ is $\mathcal{C}^2$ at every point, because then r can be chosen to agree with the (signed) distance function to $\partial\Omega$ in a neighborhood of $\partial\Omega$. Then the geometry of $\partial\Omega$ can be understood in terms of the derivatives up to the order two of r. (See [2], Appendix). In [3], Krantz and Parks showed that the distance function has the same regularity as the boundary, whenever it is $\mathcal{C}^k$ for $k > 1$, and also studied the validity of this assertion in the case $k = 1$. (See also [5]).

The focus of this note is on the non regular open subsets of $\mathbb{R}^n$, so is, open subsets whose boundary is not an embedded $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$ (although it can have eventually regular pieces). For any given open subset $\Omega \subset \mathbb{R}^n$, we construct, for each degree of regularity k, a defining function for Ω that is $\mathcal{C}^k$ in the $\mathcal{C}^k$ part of the boundary with non vanishing gradient there, and smooth away of this part. In some (weak) sense, this function is $\mathcal{C}^k$ equivalent to the distance function to the $\mathcal{C}^k$ part of the boundary.

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Let us notice that by means of this function is possible to obtain (applying Sard's theorem) a family of smooth domains whose boundaries approximate $\partial\Omega$ in a controlled way.

Now we give precise statements of the concepts and assertions in the previous paragraphs. First we recall the standard notion of regular point of $\partial\Omega$.

Definition. Fix $k \in \mathbb{N}$ or $k = \infty$. For a given point $P_0 \in \partial\Omega$, and $k \in \mathbb{N}$, $\partial\Omega$ is said to be $\mathcal{C}^k$ at P_0 if and only if there exist a neighborhood $V_{P_0} \subset \mathbb{R}^n$ of P_0, a neighborhood $U_0 \subset \mathbb{R}^n$ of 0 and a $\mathcal{C}^k$ diffeomorphism $\Phi : U_0 \to V_{P_0}$, such that if $t_1, \dots, t_n$ are the standard coordinates of $\mathbb{R}^n$ in U_0, then $V_{P_0} \cap \partial\Omega = \Phi(U_0 \cap \{t_n = 0\})$ and $V_{P_0} \cap \Omega = \Phi(U_0 \cap \{t_n < 0\})$.

(This is the description of $\partial\Omega$ as a locally embedded submanifold of $\mathbb{R}^n$. See [4] for equivalent definitions and the relationship among these.)

For $k \in \mathbb{N} \setminus \{0\}$ or $k = \infty$ let R_k stand for the set of points in $\partial\Omega$ where $\partial\Omega$ is $\mathcal{C}^k$ and $S_k = \partial\Omega \setminus R_k$. So $\partial\Omega$ is the disjoint union of R_k and S_k. Note that one of the two sets (or both) can be empty.

This note is devoted to the proof of the following fact.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, open, and $k \in \mathbb{N} \setminus \{0\}$ or $k = \infty$. There exists a function $r \in \mathcal{C}^k(\mathbb{R}^n)$ such that $\Omega = \{r < 0\}$ and $\partial\Omega = \{r = 0\}$. Moreover $\nabla r \neq 0$ in R_k and $r \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \bar{R}_k)$.*

In the particular case of Ω bounded domain and $\partial\Omega \in \mathcal{C}^\infty$, the result follows from an elementary application of the implicit function theorem and a finite partition of the unit.

On the other extreme, if $S_k = \partial\Omega$, the result is a consequence of the following theorem due to Whitney (See [1], [4].):

Theorem (Whitney)**.** *Let $F \subset \mathbb{R}^n$ closed. There exists a positive function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $F = \{x \in \mathbb{R}^n : \varphi(x) = 0\}$.*

Remark. The proof is given by constructing the function φ in a completely general setting. That includes the case of F being the complement of an open set Ω whose boundary has no regular points, (the bounded part of the complement of the Von Koch's snowflake in $\mathbb{R}^2$ is an example of such domains, as can be deduced from its properties of self-similarity and non rectifiability. Cf. [7]), so $\varphi = 0$ in F at the infinite order.

Corollary (of Whitney's theorem)**.** *For any open set Ω there exits a real valued function $\kappa \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\Omega = \{\kappa < 0\}$ and $\partial\Omega = \{\kappa = 0\}$.*

Proof. Applying Whitney's theorem to $F = \bar{\Omega}$ we obtain a function $\varphi^+ \in \mathcal{C}^\infty(\mathbb{R}^n)$, non negative, such that $F = \{\varphi^+ = 0\}$. Moreover, φ^+ vanishes at F up to the infinite order. By the same procedure one has a function $\varphi^- \in \mathcal{C}^\infty(\mathbb{R}^n)$, non negative, vanishing at Ω^c up to the infinite order. Finally take $\kappa = \varphi^+ - \varphi^-$. □

Remark. As observed above, $\varphi = 0$ in $\partial\Omega$ at the infinite order. The main property of the function r constructed in Theorem 1 is that r defines globally Ω and $\nabla r \neq 0$ in R_k, so it defines Ω near R_k as a sub-manifold of $\mathbb{R}^n$ with $\mathcal{C}^k$ boundary, and is $\mathcal{C}^\infty$ in the complement of $\overline{R_k}$.

The proof of Theorem 1 is a non trivial combination of these facts, and is developed in the forthcoming sections.

2 The Regular Part.

The set R_k is a relatively open subset of $\partial\Omega$. If it is nonempty we have, from the definition of R_k, a defining function in a neighborhood of each point of R_k. The construction of a global defining function in a neighborhood of R_k makes use of a suitable covering, supporting a partition of the unit.

The precise results are contained in the next two lemmas.

2.1 The Auxiliary Tools.

The following lemma is a variant of the classical Besicovitch covering lemma for open balls, after the comments in [6].

Lemma 1 (Covering lemma)**.** *Let $A \subset \mathbb{R}^n$, and a function $r_0 : A \to \mathbb{R}_+$. Consider the family of open balls*

$$\mathcal{B} = \{B_r(x) : x \in A,\ r \leq r_0(x)\}.$$

There exists a countable family $\mathcal{B}' \subset \mathcal{B}$ covering A, such that every point $x \in \bigcup_{B\in\mathcal{B}'} B$ has a neighborhood intersecting only a finite number of balls in the family $\mathcal{B}'$.

Next, we consider a partition of unity adapted to this covering. It is a well known construction, and we just recall it to stress the properties of the constructed functions used later.

Lemma 2 (Partition of unity)**.** *Let A be a subset of $\mathbb{R}^n$ and $\mathcal{B}'$ the covering of A given by the previous lemma. There is a family of $\mathcal{C}^\infty$ functions $\{\chi_l; l \in \mathbb{N}\}$, positive in B_l such that* $\operatorname{spt}\chi_l \subset \overline{B_l}$ *and $\sum_l \chi_l(x) = 1$, for any $x \in A$.*

2.2 The Defining Function for R_k.

Now we can construct a defining function for Ω near R_k.

Lemma 3. *There exists an open set $V = V_{R_k} \subset \mathbb{R}^n$, containing R_k, and a $\mathcal{C}^k$ function $r : V \to \mathbb{R}$ such that $\{r < 0\} = V \cap \Omega$, $\{r = 0\} = R_k$, $\{r > 0\} = V \cap \bar{\Omega}^c$ and $0 < \|\nabla r(x)\| \leq 1$ for all $x \in V$.*

Proof. For any $P \in R_k$ there exists $\rho_0(P) > 0$ such that for any $\rho \in (0, \rho_0(P))$, we can find $r \in \mathcal{C}^k(B_\rho(P))$ satisfying that $\{r < 0\} = B_\rho(P) \cap \Omega$, $\{r = 0\} = B_\rho(P) \cap \partial\Omega$ and $dr(x) \neq 0$ in $B_\rho(P)$.

Take $\mathcal{B}_0$ the family of such balls, for all $P \in R_k$ and all $\rho \in (0, \rho_0(P))$, and let $\mathcal{B}$ be the countable subfamily of $\mathcal{B}_0$ covering R_k given by the Covering lemma and call $B_j = B_{\rho_j}(P_j)$ a typical ball in this family.

Take $\{\chi_j\}$ the $\mathcal{C}^\infty$ partition of the unit for R_k relative to $\mathcal{B}$, provided by Lemma 2.

For any j, pick r_j a $\mathcal{C}^k$ function defining Ω in B_j, and put

$$\alpha_j = \max\{1, \sup\{\|\nabla r_j(x)\| : \ x \in B_j\}\}.$$

If $V = \cup_{B_j \in \mathcal{B}} B_j$ and $r = \sum_j \alpha_j^{-1} r_j \chi_j$, since every point in V has a neighborhood where the function $\rho(x) = \sharp\{B \in \mathcal{B}' : x \in B\}$ is locally bounded, the function r is in $\mathcal{C}^k(V)$. Moreover, since for $x \in B_j$ we have $r_j(x) = 0$ if $x \in R_k$, $r_j(x) < 0$ if $x \in \Omega$ and $r_j(x) > 0$ if $x \in \bar{\Omega}^c$, and also

$$\nabla r(x) = \sum{}' \ \alpha_j^{-1}(\chi_j(x) \nabla r_j(x) + r_j(x) \nabla \chi_j(x)),$$

then for $x \in R_k$, $\nabla r(x) = \sum' \alpha_j^{-1} \chi_j(x) \nabla r_j(x)$. Also, as $\nabla r_j(x) = c_j \eta(x)$ with $c_j > 0$ and η the exterior normal unit vector, we have $\nabla r(x) \neq 0$. And we have this for any $x \in R_k$. Then shrink V if necessary. $\square$

In order to extend r to a neighborhood of S_k, we need to modify the function near $\bar{R}_k \setminus R_k$:

Lemma 4. *Let V the neighborhood of R_k obtained in Lemma 3. There exists a function $\phi \in \mathcal{C}^k(\mathbb{R}^n)$ such that:*

1. *$\phi \equiv 0$ in V^c.*
2. *$\{\phi < 0\} = V \cap \Omega$ and $\{\phi > 0\} = V \cap \overset{\circ}{\widehat{\Omega^c}}$.*
3. *$\nabla \phi(x) \neq 0$, for any $x \in R_k$.*

PROOF. Take r and V as in Lemma 3 and choose a family of compact sets $\{K_l; l \in \mathbb{N}\}$ such that $K_0 = \emptyset$, $K_l \subset \overset{\circ}{K}_{l+1}$, $\cup K_l = V$. Take $\psi_l \in \mathcal{C}^\infty(\mathbb{R}^n)$, such that $\psi_l \geq 0$, and $\psi_l \equiv 1$ in K_l and $\psi_l \equiv 0$ in K_{l+1}^c. Let $A_{l,s} = 1 + \|r\|_{\mathcal{C}^s(K_l)}$, $C_{l,s} = \|\psi_l\|_{\mathcal{C}^s(\mathbb{R}^n)}$ and

$$\beta_l = \frac{1}{2^l \gamma_l C_{l-1,l} c(n,l)}$$

where $c(n,l) = \sum_{j \leq l} \sum_{|\alpha|=j} \sum_{\beta \subset \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}$, and $\gamma_l \geq 1$. Define $\psi = \sum_l \beta_l \psi_l$. The series converges uniformly in $\mathbb{R}^n$ (because of the choice of β_l) and ψ is a continuous function supported in $\bar{V}$. Also, for $l \geq 1$, if $x \in K_l \setminus K_{l-1}$, we have that for any $j > l$, $\psi_j \equiv 1$ in a neighborhood of x, and for any $j < l-1$, $\psi_j \equiv 0$ in a neighborhood of x. Then

$$\psi(x) = \beta_{l-1}\psi_{l-1}(x) + \beta_l \psi_l(x) + \sum_{j \neq l, l-1} \beta_j \psi_j(x),$$

and for $|\alpha| > 0$

$$D^\alpha \psi(x) = \beta_{l-1} D^\alpha \psi_{l-1}(x) + \beta_l D^\alpha \psi_l(x) = \beta_{l-1} D^\alpha \psi_{l-1}(x),$$

if we assume that $\overline{\overset{\circ}{K_l}} = K_l$, for then $\beta_l D^\alpha \psi_l(x) = 0$ on K_l. This implies that $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ and for $x \in K_l \setminus K_{l-1}$ and any α,

$$|D^\alpha \psi(x)| \leq \beta_{l-1} C_{l-1,|\alpha|}.$$

Now, for any fixed $k > 0$, since for $|\alpha| \leq k$ and $x \in K_l \setminus K_{l-1}$,

$$\begin{aligned}
|D^\alpha(r\psi)(x)| &\leq \sum_{\beta \preceq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} |D^{\alpha-\beta} r(x)| |D^\beta \psi(x)| \\
&\leq \sum_{\beta \preceq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} |D^{\alpha-\beta} r(x)| \beta_{l-1} C_{l-1,|\beta|} \\
&\leq \sum_{\beta \preceq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} A_{l,|\alpha-\beta|} \beta_{l-1} C_{l-1,|\beta|},
\end{aligned}$$

For $l \geq k$, we have that

$$\begin{aligned}
\|r\psi\|_{\mathcal{C}^k(\overline{K_l \setminus K_{l-1}})} &\leq A_{l,k} \sum_{j \leq l} \sum_{|\alpha|=j} \sum_{\beta \preceq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \beta_{l-1} C_{l-1,|\beta|} \\
&\leq A_{l,k} \beta_{l-1} C_{l-1,l} \sum_{j \leq l} \sum_{|\alpha|=j} \sum_{\beta \preceq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \\
&\leq A_{l,k} \beta_{l-1} C_{l-1,l} c(n,l) \leq \frac{A_{l,k}}{2^l \gamma_l}.
\end{aligned}$$

Then, the choice of $\gamma_l = A_{l,k}$ if $r \in \mathcal{C}^k$, $k \in \mathbb{N}$, or $\gamma_l = A_{l,l}$ if $r \in \mathcal{C}^\infty$ implies that for $l \geq k$ and any $k \in \mathbb{N}$,

$$\|r\psi\|_{\mathcal{C}^k(\overline{K_l \setminus K_{l-1}})} \leq \frac{1}{2^{l-1}}$$

Now define

$$\phi(x) = \begin{cases} 0 & \text{if } x \in V^c \\ r(x)\psi(x) & \text{if } x \in V. \end{cases}$$

The estimates above imply that $\phi \in \mathcal{C}^k(\mathbb{R}^n)$. Also, since $\psi > 0$ in V, we have that $\{\phi < 0\} \cap V = \{r < 0\}$ and $\{\phi = 0\} \cap V = \{r = 0\}$. And since $r = 0$ and $\nabla r \neq 0$ on R_k, we have that

$$\nabla\phi(x) = \nabla(r\psi)(x) = \nabla r(x)\psi(x) + r(x)\nabla\psi(x) = \nabla r(x)\psi(x) \neq 0,$$

for any $x \in R_k$. □

3 The Defining Function for Ω.

Now the proof of the Theorem 1 amounts to modifying ϕ to a function in $\mathbb{R}^n$ defining Ω. To do so, we will distinguish the $\mathcal{C}^\infty$ case from the others.

3.1 The $\mathcal{C}^\infty$ Case.

This case is a direct consequence of the constructions in the previous sections. Once we have a function ϕ as in Lemma 4, we take a defining function κ for Ω as in Whitney's theorem above. The function $r = \kappa + \phi$ is strictly positive outside Ω, because κ is, and ϕ is non negative there, and strictly negative inside, for analogous reasons. Moreover, if $x \in R_\infty$ then $\nabla\phi(x) \neq 0$ and $\nabla\kappa(x) = 0$, and if $x \in S_\infty$ then both terms are 0.

3.2 The Case $k < \infty$.

The construction made in section 3.1 provides a function satisfying the requirements of Theorem 1, except for the fact that the resulting function is at least $\mathcal{C}^k$ in a neighborhood of R_k and we want it to be in $\mathcal{C}^\infty(\mathbb{R}^n \setminus \bar{R}_k)$.

Since the function $\tilde{r} = \kappa + \phi \in \mathcal{C}^k(\mathbb{R}^n)$, the k-jet $(f_\alpha(x))_{|\alpha| \leq k}$, where $f_\alpha(x) = D^\alpha \tilde{r}(x)$, is a Whitney jet in $F = \Omega^c$. This means that for any $x_0 \in F$

and any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in B_\delta(x_0) \cap F$, and $|\alpha| \leq k$,

$$\left| f_\alpha(y) - \sum_{0 \leq |\beta| \leq k-|\alpha|} \frac{f_{\alpha+\beta}(x)}{\beta!}(y-x)^\beta \right| \leq \epsilon \|y-x\|^{k-|\alpha|}.$$

Whitney's extension of this jet to Ω provides a function ρ which is $\mathcal{C}^\infty$ in Ω, but the property $\rho < 0$ is not guaranteed now. So we need a modification of the method.

Let us recall the main features of the Whitney's method, as given in [8], Chap VI. Let $F \subset \mathbb{R}^n$ be a closed set. F^c can be covered by a countable family $\mathcal{F}^* = \{Q_l^* : l \in \mathbb{Z}\}$ of open cubes with their sides parallel to the axes, such that

$$\frac{9}{64}\operatorname{diam}(Q_l^*) \leq d(F, Q_l^*) \leq \frac{4}{3}\operatorname{diam}(Q_l^*)$$

and every $x_0 \in F^c$ belongs to, at most, $(12)^n$ of these cubes. There is a $\mathcal{C}^\infty$ partition of unity for F^c, namely $\Phi = \{\varphi_l^*\}$, such that $\varphi_l^* \geq 0$, $\operatorname{spt}(\varphi_l^*) \subset Q_l^*$, and for any $x \in \mathbb{R}^n$, $l \in \mathbb{Z}$ and α n-index

$$|D^\alpha \varphi_l^*(x)| \leq A_\alpha \operatorname{diam}(Q_l^*)^{-|\alpha|},$$

where A_α is a constant depending only on α and n. Finally, one chooses, for any $l \in \mathbb{Z}$, a point $p_l \in F$ such that $d(p_l, Q_l^*) = d(F, Q_l^*)$.

Also, in general, if $h \in \mathcal{C}^k$, for any x_0 there is a ball $B_\lambda(x_0)$ such that for $x, a \in B_\lambda(x_0)$,

$$h(x) = \sum_{|\alpha| \leq k} \frac{D^\alpha h(a)}{\alpha!}(x-a)^\alpha + R_k^h(x, a)$$

where $R_k^h(x, a)$ is the k'th Taylor remainder term and satisfies

$$|R_k^h(x, a)| \leq \omega_k^h(a, \|x-a\|)\|x-a\|^k$$

and $\omega_k^h(a, \delta) = \sup\{|D^\alpha h(\zeta) - D^\alpha h(a)| : \|\zeta - a\| \leq \delta,\ |\alpha| = k\}$, for $x \in F^c$.

Back to our case, if F is Ω^c and $f_\alpha(x) = D^\alpha \tilde{r}(x)$, define, as usual,

$$g_k(x) = \begin{cases} \sum_l (\sum_{|\alpha| \leq k} \frac{f_\alpha(p_l)}{\alpha!}(x-p_l)^\alpha)\varphi_l^*(x) & \text{if } x \in F^c \\ f(x) & \text{if } x \in F. \end{cases}$$

Since any given $x \in F^c$ has a neighborhood contained in, at most, $(12)^n$ fixed cubes, the sum converges at every point and $g_k \in \mathcal{C}^\infty(F^c)$, and as the classical proof of Whitney shows, g_k is a $\mathcal{C}^k$ extension of $\tilde{r}_{|F^c}$. Since for $x \in F^c$,

$$\begin{aligned} g_k(x) &= \sum_l \Big(\sum_{|\alpha| \le k} \frac{D^\alpha \phi(p_l)}{\alpha!}(x-p_l)^\alpha\Big)\varphi_l^*(x) \\ &= \phi(x) - \sum_{l: p_l \in R_k} R_\alpha^\phi(x,p_l)\varphi_l^*(x), \end{aligned}$$

the function g_k is $\mathcal{C}^\infty$ in Ω because every φ_l^* is, and the sum refers only to finitely many functions at every point. Moreover, since $|R_\alpha^\phi(x,p_l)| \le \omega_k^\phi(p_l, \frac{7}{3}\operatorname{diam}(Q_l^*))(\frac{7}{3}\operatorname{diam}(Q_l^*))^k$ whenever $x \in Q_l^*$, if we define

$$\varphi(x) = \begin{cases} \sum_l \omega_k^\phi(p_l, 3\operatorname{diam}(Q_l^*))(3\operatorname{diam}(Q_l^*))^k \varphi_l^*(x) & \text{if } x \in F^c \\ 0 & \text{if } x \in F, \end{cases}$$

the function φ is clearly $\mathcal{C}^\infty$ in Ω.

Moreover, since for $x \in \Omega$ we have

$$D^\alpha \varphi(x) = \sum_l \omega_k^\phi(p_l, 3\operatorname{diam}(Q_l^*))(3\operatorname{diam}(Q_l^*))^k D^\alpha \varphi_l^*(x)$$

and the estimates above imply that

$$\begin{aligned} |D^\alpha \varphi(x)| &\le \sum_l \omega_k^\phi(p_l, 3\operatorname{diam}(Q_l^*))(3\operatorname{diam}(Q_l^*))^k A_\alpha \operatorname{diam}(Q_l^*)^{-|\alpha|} \\ &\le 3^k A_\alpha (8\ d(x,F))^{k-|\alpha|} \sum_l \omega_k^\phi(p_l, 8d(x,p_l)), \end{aligned}$$

and since for any $x_0 \in F$ and $\delta > 0$ we have, for $x \in B_\delta(x_0) \cap \Omega$, that $\|x - p_l\| \le 9\|x - x_0\|$, and $\omega_k^\phi(p_l, 8d(x,p_l)) \le 2\omega_k^\phi(x_0, 72d(x,x_0))$, then

$$|D^\alpha \varphi(x)| \le (12)^n 3^k A_\alpha (8\|x-x_0\|))^{k-|\alpha|} 2\omega_k^\phi(x_0, 72\|x-x_0\|).$$

This implies that $D^\alpha \varphi(x) \to_{x \to x_0 \in F} 0$. So φ is a $\mathcal{C}^k$ function in $\mathbb{R}^n$ and $D^\alpha \varphi(x_0) = 0$ for any $|\alpha| \le k$ and $x_0 \in F$. In fact, if $x_0 \in S_k \setminus \bar{R}_k$, then there is $\lambda > 0$ such that $B_\lambda(x_0) \cap \bar{R}_k = \emptyset$. Since for $x \in B_{\frac{\lambda}{9}}(x_0) \cap \Omega$, any Whitney cube Q^* containing x is in $B_\lambda(x_0) \cap \Omega$, then $\varphi(x) = 0$. So $\varphi \equiv 0$ in $B_{\frac{\lambda}{9}}(x_0)$ and $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \bar{R}_k)$.

Also, by construction, for any $x \in \Omega$,

$$\Big| \sum_{l: p_l \in R_k} R_\alpha^\phi(x,p_l)\varphi_l^*(x) \Big| \le \varphi(x).$$

Then the function $G = g_k - \varphi$ is in $\mathcal{C}^k(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n \setminus \bar{R}_k)$ and for any $x \in \Omega$

$$G(x) = g_k(x) - \varphi(x) = \phi(x) - \sum_{l: p_l \in R_k} R_\alpha^\phi(x, p_l)\varphi_l^*(x) - \varphi(x)$$
$$\leq \phi(x) + \left| \sum_{l: p_l \in R_k} R_\alpha^\phi(x, p_l)\varphi_l^*(x) \right| - \varphi(x) < 0.$$

Moreover $G(x) = 0$ in $\partial\Omega$ and

$$\nabla G(x) = \nabla g_k(x) - \nabla\varphi(x) = \nabla g_k(x) = \nabla\phi(x) \neq 0$$

in R_k. Finally we can play an identical game for G in $\bar{\Omega}$ and obtain another function r extending G to $\bar{\Omega}^c$, in such a way that r is $\mathcal{C}^\infty$ and strictly positive in $\bar{\Omega}^c$. This finishes the construction and provides the proof of Theorem 1. $\square$

References

[1] Th. Bröcker, *Differentiable Germs and Catastrophes*, Lecture Notes LMS, Cambridge Univ. Press, Cambridge, 1975.

[2] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd. edition, Springer-Verlag, New York, 1983.

[3] S. G. Krantz, H. R. Parks, *Distance to $\mathcal{C}^k$ Hypersurfaces*, Journal of Differential Equations, **40** (1981), 116–120.

[4] S. G. Krantz, H. R. Parks, *The Geometry of Domains in Space*, Birhkhäuser Boston Inc., Boston, MA, 1999.

[5] S. G. Krantz, H. R. Parks, *The implicit function theorem. History, theory, and applications*, Birhkhäuser Boston Inc., Boston, MA, 2002.

[6] M. de Guzman, *Real variable methods in Fourier analysis*, North-Holland Mathematics Studies, 46, North-Holland Publishing Co., Amsterdam-New York, 1981.

[7] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Univ. Press, Cambridge, 1995.

[8] E. M. Stein, *Singular Integrals and Differentiable Propperties of Functions*, Princeton Univ. Press, Princeton, New Jersey, 1970.