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MAXIMUMS OF ALMOST CONTINUOUS FUNCTIONS

Abstract

It is shown that each function which can be written as the maximum of two Darboux functions can be written as the maximum of two almost continuous functions as well.

The letter $\mathbb{R}$ denotes the real line. If $P \subset \mathbb{R}^2$, then the symbol $\operatorname{dom} P$ denotes the x-projection of P. For each $A \subset \mathbb{R}$ the symbol $\operatorname{card} A$ stands for the cardinality of A. We write $\mathfrak{c} = \operatorname{card} \mathbb{R}$. We consider cardinals as ordinals not in one-to-one correspondence with the smaller ordinals.

Let $f\colon \mathbb{R} \to \mathbb{R}$. For each nondegenerate interval $I \subset \mathbb{R}$ we define

$$\mathfrak{c}\text{-}\sup(f, I) = \sup\bigl\{y \in \mathbb{R}\colon \operatorname{card}\{x \in I\colon f(x) > y\} = \mathfrak{c}\bigr\}.$$

For each $x \in \mathbb{R}$ we denote

$$\mathfrak{c}\text{-}\overline{\lim}_{t\to x^-} f(t) = \lim_{\delta\to 0^+} \mathfrak{c}\text{-}\sup\bigl(f, (x-\delta, x)\bigr),$$

and similarly we define the symbol $\mathfrak{c}\text{-}\overline{\lim}_{t\to x^+} f(t)$. We say that f is *Darboux* if it maps intervals onto connected sets. We say that f is *connected*, if it is a connected subset of $\mathbb{R}^2$. (We make no distinction between a function and its graph.) We say that f is *almost continuous* in the sense of Stallings [10] if for every open set $U \subset \mathbb{R}^2$ containing f there is a continuous function $h\colon \mathbb{R} \to \mathbb{R}$ with $h \subset U$. Recall that almost continuous functions are connected and connected functions possess the Darboux property, and that the converse is not true [10]. Moreover in Baire class one these three notions coincide [1].

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In 1974 A. M. Bruckner, J. G. Ceder, and T. L. Pearson proved that a function f is the maximum of Darboux functions g_0 and g_1 if and only if

$$\min\bigl\{\mathfrak{c}\text{-}\overline{\lim}_{t\to x^-} f(t), \mathfrak{c}\text{-}\overline{\lim}_{t\to x^+} f(t)\bigr\} \geq f(x) \quad \text{for each } x \in \mathbb{R}, \tag{1}$$

and we can make sure that g_0 and g_1 are Lebesgue measurable (belong to Baire class α, $\alpha \geq 2$) provided that f is so [2, Theorem 3]. It is well-known that the algebraic properties of Darboux functions and almost continuous functions are very similar. (See, e.g., [8].) In 1992 T. Natkaniec proved that if $\mathfrak{c}\text{-}\sup(f, [a, b]) = \infty$ for all $a < b$ (in particular, if f is a Darboux function whose graph is dense in $\mathbb{R}^2$), then f is the maximum of two almost continuous functions [8, Theorem 6.10 and Corollary 6.9]. Clearly this sufficient condition is much stronger than (1). So, it is natural to ask whether every function fulfilling condition (1) is the maximum of two almost continuous functions. (See [8, Problem 6.4] or [3, Question 9.33].) We will show that the answer to this question is affirmative.

Theorem 1. *Assume that a function $f\colon \mathbb{R} \to \mathbb{R}$ fulfills condition* (1). *Then f is the maximum of two almost continuous functions g_0 and g_1. If moreover f is Lebesgue measurable* (*has the Baire property*), *then we can make sure that both g_0 and g_1 are Lebesgue measurable* (*have the Baire property*) *as well.*

PROOF. Let $\mathcal{P}$ be the family of all sets P of the form $P = (p_0, p_1) \times (p_2, \infty)$ for some rationals p_0, p_1, and p_2, such that $P \cap f \neq \emptyset$. For each $P \in \mathcal{P}$, notice that by (1), $\operatorname{card} \operatorname{dom}(P \cap f) = \mathfrak{c}$, and choose a subset $A_P \subset \operatorname{dom}(P \cap f)$ with $\operatorname{card} A_P = \mathfrak{c}$. (We do this to prove the measurability of g_0 and g_1. In the general case we can define $A_P = \operatorname{dom}(P \cap f)$.) Set

$$A = \bigcup\nolimits_{P\in\mathcal{P}} A_P, \qquad Q = \bigcup\nolimits_{x\in A} \bigl(\{x\} \times (-\infty, f(x))\bigr).$$

Let $\mathcal{K}$ be the family of all closed sets $K \subset \mathbb{R}^2$ such that $\operatorname{card} \operatorname{dom}(K \cap Q) = \mathfrak{c}$. Arrange the elements of $\mathcal{K}$ in a transfinite sequence, $\langle K_\xi \colon \xi < \mathfrak{c}\rangle$. Using transfinite induction pick for each $\xi < \mathfrak{c}$ points $\langle x_{\xi,0}, y_{\xi,0}\rangle, \langle x_{\xi,1}, y_{\xi,1}\rangle \in K_\xi \cap Q$ such that $x_{\xi,0}, x_{\xi,1} \notin \{x_{\zeta,i} \colon \zeta < \xi, i < 2\}$ and $x_{\xi,0} \neq x_{\xi,1}$. For $i < 2$ define

$$g_i(x) = \begin{cases} y_{\xi,i} & \text{if } x = x_{\xi,i},\ \xi < \mathfrak{c}, \\ f(x) & \text{otherwise.} \end{cases}$$

One can easily verify that $f = \max\{g_0, g_1\}$ on $\mathbb{R}$. We will prove that g_0 and g_1 are almost continuous.

Let $i < 2$. By [4] or [8, Corollary 2.2], it suffices to show that $g_i{\restriction}[\alpha, \beta]$ is almost continuous whenever $\alpha < \beta$. Fix $\alpha < \beta$ and let $U \subset \mathbb{R}^2$ be an open set such that $g_i\restriction[\alpha,\beta] \subset U$. Denote by S the set of all $x \in [\alpha,\beta]$ for which there exists a continuous function $h\colon [x,\beta] \to \mathbb{R}$ with $h \subset U$ such that $h = g_i$ on $\{x,\beta\}$, and notice that $\beta \in S$. Define $\bar\alpha = \inf S$ and

$$B = \big\{x \in A\colon \{x\} \times (-\infty, f(x)) \not\subset U\big\} = \operatorname{dom}(Q \setminus U).$$

The rest of the proof of the theorem consists of three auxiliary claims. The end of the proof of each claim will be marked with $\lhd$.

Claim 1. $\operatorname{card} B < \mathfrak{c}$.

Indeed, otherwise $\mathbb{R}^2 \setminus U \in \mathcal{K}$. But $g_i \cap K \neq \emptyset$ for each $K \in \mathcal{K}$, whence $g_i \not\subset U$, an impossibility. $\lhd$

Claim 2. $\bar\alpha \in S$.

Suppose this is not the case. Then $\bar\alpha < \beta$. Let $\delta \in (0, \beta - \bar\alpha)$ be such that

$$(\bar\alpha - \delta, \bar\alpha + \delta) \times (g_i(\bar\alpha) - 2\delta, g_i(\bar\alpha) + \delta) \subset U. \tag{2}$$

(Recall that U is open and $g_i \subset U$.) Put $R = (\bar\alpha, \bar\alpha+\delta) \times (g_i(\bar\alpha) - \delta, \infty)$. Then by (1), $R \cap f \neq \emptyset$, and consequently, there is a $P \in \mathcal{P}$ with $P \subset R$. Pick an $x_0 \in A_P \setminus B$ (cf. Claim 1) and an $x_1 \in S \cap (\bar\alpha, x_0)$. Then $\langle x_0, f(x_0)\rangle \in P \subset R$, whence $f(x_0) > g_i(\bar\alpha) - \delta$. Let h_0 correspond to $x_1 \in S$. We consider three cases.

Case 1. If $h_0(x_0) \leq g_i(\bar\alpha) - \delta$, then let $I = [h_0(x_0), g_i(\bar\alpha) - \delta]$. Recall that $x_0 \notin B$, and use the compactness of the set $\{x_0\} \times I \subset U$ to find an $\eta \in (0, x_0 - \bar\alpha)$ such that $(x_0 - \eta, x_0 + \eta) \times I \subset U$. Extend $h_0\restriction[x_0,\beta]$ to $h\colon [\bar\alpha,\beta] \to \mathbb{R}$ by connecting the following pairs of points by straight line segments: $\langle \bar\alpha, g_i(\bar\alpha)\rangle$ with $\langle x_0 - \eta, g_i(\bar\alpha) - \delta\rangle$ and $\langle x_0 - \eta, g_i(\bar\alpha) - \delta\rangle$ with $\langle x_0, h_0(x_0)\rangle$. Clearly this function proves $\bar\alpha \in S$. *Case* 2. If $g_i(\bar\alpha) - \delta < g_i(x_1)$, then let $\tau \in (0, x_1 - \bar\alpha)$ be such that $g_i(x_1) - \tau > g_i(\bar\alpha) - \delta$ and $(x_1 - \tau, x_1 + \tau) \times (g_i(x_1) - 2\tau, g_i(x_1) + \tau) \subset U$. Put $R' = (x_1 - \tau, x_1) \times (g_i(x_1) - \tau, \infty)$. There is a $P' \in \mathcal{P}$ with $P' \subset R'$. Pick an $x_2 \in A_{P'} \setminus B$. Let $I' = [g_i(\bar\alpha) - \delta, g_i(x_1) - \tau]$. Use the compactness of the set $\{x_2\} \times I' \subset U$ to find an $\eta \in (0, x_2 - \bar\alpha)$ such that $(x_2 - \eta, x_2 + \eta) \times I' \subset U$. Extend h_0 to $h\colon [\bar\alpha,\beta] \to \mathbb{R}$ by connecting the following pairs of points by straight line segments: $\langle \bar\alpha, g_i(\bar\alpha)\rangle$ with $\langle x_2 - \eta, g_i(\bar\alpha) - \delta\rangle$, $\langle x_2 - \eta, g_i(\bar\alpha) - \delta\rangle$ with $\langle x_2, g_i(x_1) - \tau\rangle$, and $\langle x_2, g_i(x_1) - \tau\rangle$ with $\langle x_1, g_i(x_1)\rangle$. Clearly this function proves $\bar\alpha \in S$.

Case 3. Finally assume that $h_0(x_0) > g_i(\bar\alpha) - \delta \geq g_i(x_1) = h_0(x_1)$. Then there is an $x_2 \in [x_1, x_0)$ such that $h_0(x_2) = g_i(\bar\alpha) - \delta$. Extend $h_0\restriction[x_2,\beta]$ to $h\colon [\bar\alpha,\beta] \to \mathbb{R}$ by connecting the following pair of points by a straight line segment: $\langle \bar\alpha, g_i(\bar\alpha)\rangle$ with $\langle x_2, h_0(x_2)\rangle$. Clearly this function proves $\bar\alpha \in S$. $\lhd$

Claim 3. $\bar{\alpha} = \alpha$.

Indeed, suppose $\bar{\alpha} > \alpha$. Let $\delta \in (0, \bar{\alpha} - \alpha)$ be such that condition (2) holds. Put $R = (\bar{\alpha} - \delta, \bar{\alpha}) \times (g_i(\bar{\alpha}) - \delta, \infty)$. There is a $P \in \mathcal{P}$ with $P \subset R$. Pick an $x_0 \in A_P \setminus B$. Let I be the closed interval (maybe a singleton) with endpoints $g_i(\bar{\alpha}) - \delta$ and $g_i(x_0)$. Use the compactness of the set $\{x_0\} \times I \subset U$ to find an $\eta \in (0, \bar{\alpha} - x_0)$ such that $(x_0 - \eta, x_0 + \eta) \times I \subset U$. Let h_0 correspond to $\bar{\alpha} \in S$. (See Claim 2.) Extend h_0 to $h \colon [x_0, \beta] \to \mathbb{R}$ by connecting the following pairs of points by straight line segments: $\langle x_0, g_i(x_0) \rangle$ with $\langle x_0 + \eta, g_i(\bar{\alpha}) - \delta \rangle$ and $\langle x_0 + \eta, g_i(\bar{\alpha}) - \delta \rangle$ with $\langle \bar{\alpha}, g_i(\bar{\alpha}) \rangle$. Clearly this function proves $x_0 \in S$. But $x_0 < \bar{\alpha} = \inf S$, an impossibility. $\triangleleft$

By Claim 3, there is a continuous function $h \colon [\alpha, \beta] \to \mathbb{R}$ with $h \subset U$. Since U was an arbitrary open neighborhood of $g_i \restriction [\alpha, \beta]$, we conclude that $g_i \restriction [\alpha, \beta]$ is almost continuous. Since $\alpha < \beta$ were arbitrary, g_i is almost continuous as well.

Finally observe that $f = g_0 = g_1$ outside of A. So, if f is Lebesgue measurable (has the Baire property) and we require that each A_P be a null-set (a meager set), then both g_0 and g_1 are Lebesgue measurable (have the Baire property) as well. The proof is complete. $\square$

Corollary 2. *Let $f \colon \mathbb{R} \to \mathbb{R}$. The following are equivalent*:

(i) *the function f is the maximum of two almost continuous functions,*

(ii) *the function f is the maximum of two connected functions,*

(iii) *the function f is the maximum of two Darboux functions,*

(iv) *the function f fulfills condition* (1).

PROOF. The implications '(i) $\Rightarrow$ (ii)' and '(ii) $\Rightarrow$ (iii)' follow by [10], the implication '(iii) $\Rightarrow$ (iv)' follows by [2, Theorem 3], and the implication '(iv) $\Rightarrow$ (i)' follows by Theorem 1. $\square$

In connection with [2, Theorem 3], we can ask the following question (see also [6, p. 552] or [7]):

> Suppose $f \colon \mathbb{R} \to \mathbb{R}$ belongs to Baire class α, $\alpha \geq 2$, and condition (1) holds. Can we find almost continuous functions g_0, g_1 in Baire class α such that $f = \max\{g_0, g_1\}$ on $\mathbb{R}$?

The affirmative answer to this question was given by D. Preiss [9]. Recall also that in Baire class one almost continuity and Darboux property coincide, and that each Baire one function $f \colon \mathbb{R} \to \mathbb{R}$ which fulfills condition (1) is the maximum of two Darboux Baire one functions [5].

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