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APPROXIMATION OF CARATHÉODORY FUNCTIONS AND MULTIFUNCTIONS

Abstract

In this paper we present four results of approximation of Carathéodory functions by the sequence of continuous functions. We obtain almost everywhere pointwise convergence with respect to the first variable, and discrete or uniform convergence on compact sets with respect to the second variable.

1 Introduction.

Let us start from the definition of Carathéodory functions.

Definition 1.1. Let X, Y be topological spaces and $(T, \mathcal{M}, \mu)$ be a measurable space. We say that $f : T \times X \to Y$ is a *Carathéodory function* if

(i) $f(\cdot, u)$ is measurable for each u,

(ii) $f(t, \cdot)$ is continuous for each t.

The presented theorems say that, for a given Carathéodory function $f : T \times X \to Y$, under suitable assumptions we can find a set $S \subset \mathcal{M}$ with $\mu(S) = 0$ and a sequence of continuous functions $f_n : T \times X \to Y$ such that, for every $t \in T \setminus S$, the sequence $f_n(t, \cdot)$ converges to $f(t, \cdot)$ in the discrete way or uniformly on compact sets.
We give two groups of theorems in this paper. The first one is obtained on the basis of Scorza Dragoni type theorems and is presented in Section 2. In Section 3 we give a theorem proved by means of a Fréchet type theorem.

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In this paper we assume that $(T, \mathcal{M}, \mu)$ is a measurable, topological space such that $\mathcal{B}(T) \subset \mathcal{M}$, where $\mathcal{B}(T)$ stands for a σ-algebra of Borel sets. We also assume that all considered topological spaces are at least Hausdorff. For the sake of clearness, we say that μ is *regular* if for each $M \in \mathcal{M}$ and $\varepsilon > 0$, there exists a closed set K such that $K \subset M$ and $\mu(M \backslash K) < \varepsilon$.

2 The Approach Based on Scorza Dragoni Type Theorem.

In this section we present three results obtained by the use of suitable Scorza Dragoni type theorems and extension theorems. In these cases, we get the discrete convergence of a sequence of continuous functions $f_n : T \times X \to Y$ to the given Carathéodory function $f : T \times X \to Y$, that is, for almost all t, there exists an index n_t such that for all $n > n_t$, $f_n(t, \cdot) = f(t, \cdot)$ (see [5]).
Before formulating a suitable Scorza Dragoni type theorem, we introduce a definition of projection property. Let $(T, \mathcal{M})$ be a measurable space and X be a topological space. Let p denote the projection from $T \times X$ onto T, and $\mathcal{M} \otimes \mathcal{B}(X)$ the product $\sigma-$algebra on $T \times X$. We say that $(T, \mathcal{M}; X)$ has the *projection property* if for each $A \in \mathcal{M} \otimes \mathcal{B}(X)$ its projection $p(A)$ belongs to $\mathcal{M}$. In particular, the projection property is satisfied when $(T, \mathcal{M})$ is complete with respect to a $\sigma-$finite measure and X is a Polish space.
We shall use the following Scorza Dragoni theorem (see e.g. [11], Theorem 1, statement (a), p. 198):

Theorem 2.1. *Let T be a topological space, $\mathcal{M}$ a $\sigma-$algebra on T, and μ a regular measure on $\mathcal{M}$. Assume that X, Y are second-countable topological spaces and $(T, \mathcal{M}; X)$ has the projection property. Let f be a function from $T \times X$ to Y that is $\mathcal{M} \otimes \mathcal{B}(X)-$ measurable. Assume that for all $t \in T$ the function $f(t, \cdot)$ is continuous. Then for each $\varepsilon > 0$, there exists a closed set $K \subset T$ such that $\mu(T \backslash K) < \varepsilon$, and the restriction $f|_{K \times X}$ is continuous.*

We also need the extension theorem (see [7]).

Theorem 2.2. (The Dugundji Extension Theorem) *Let T be a metrizable topological space, Y be a locally convex linear topological space and A be a closed subset of T. Then for every continuous function $f_A : A \to Y$, there exists a continuous function $f : T \to Y$ such that $f|_A = f_A$.*

We can now formulate and prove our first theorem.

Theorem 2.3. *Let T be a metrizable topological space, $\mathcal{M}$ a $\sigma-$algebra on T, and μ a regular measure on $\mathcal{M}$. Assume that X is a metrizable, separable topological space and $(T, \mathcal{M}; X)$ has the projection property. Let Y be* *a locally convex, second-countable topological space, and* $f : T \times X \to Y$ *be a Carathéodory function. Then there exists a set* $S \in \mathcal{M}$, $\mu(S) = 0$ *and a sequence of continuous functions* $f_n : T \times X \to Y$ *such that for every* $t \in T \backslash S$, $\{f_n(t, \cdot)\}_{n \in I\!N}$ *converges to* $f(t, \cdot)$ *in the discrete way.*

Proof. The Carathéodory function f fulfils the assumptions of Scorza Dragoni theorem 2.1. By applying this theorem, we obtain for every $n \in N$, a closed set $K_n \subset T$ such that $\mu(T \backslash K_n) < \frac{1}{n}$ and the restriction $f|_{K_n \times X}$ is continuous. Let us consider for every n the set $F_n := K_1 \cup K_2 \cup ... \cup K_n$. The sets $F_n \times X$ are closed and the restriction of the Carathéodory function $f|_{F_n \times X}$ is continuous. By an application of Dugundji extension theorem 2.2, for every n we obtain a continuous function $f_n : T \times X \to Y$ such that $f_n|_{F_n \times X} = f$. We define the set S as $S := \cap_{n=1}^{\infty} (T \backslash F_n)$. The measure of the set S is equal to 0, which follows from simple calculations in which we use the fact that the family $\{T \backslash F_n\}_{n=1}^{\infty}$ is decreasing. We have $\mu(S) = \lim_{n \to \infty} \mu(T \backslash F_n) \leq \lim_{n \to \infty} \mu(T \backslash K_n) \leq \lim_{n \to \infty} \frac{1}{n} = 0$. Moreover, for every $t \in T \backslash S$, there exists n_0 such that $t \notin T \backslash F_{n_0}$, so $t \in F_{n_0}$. The family $\{F_n\}_{n=1}^{\infty}$ is increasing and hence it follows that for every $n \geq n_0$, $t \in F_n$ and $f_n(t, \cdot) = f(t, \cdot)$, so we obtain the discrete convergence. This completes the proof. □

It is worth mentioning that we have obtained the characterization of Carathéodory functions. When the thesis of theorem 2.3 is fulfilled and additionally we assume the completeness of $\sigma-$algebra $\mathcal{M}$, then the limit function is a function such that $f(\cdot, u)$ is measurable for each u and $f(t, \cdot)$ is continuous for almost all t.

In the second theorem, we consider the Carathéodory multifunction F with values in $CL(Y)$, where $CL(Y)$ is the hyperspace of all nonempty closed subsets of the metric space (Y, d) with the Wijsman topology t_{W_d}. The Wijsman topology is the least topology on $CL(Y)$ such that for every $y \in Y$, the function $dist(y, \cdot) : CL(Y) \to [0, +\infty)$ is continuous (see [3]). In this case the continuity of a multifunction $F : X \to CL(Y)$ is equivalent to the continuity of function $dist(y, F(\cdot))$ for every $y \in Y$. We say that a multifunction $F : X \to CL(Y)$ is measurable if for each open set $A \subset Y$, the set $\{x \in X : F(x) \cap A \neq \emptyset\}$ is measurable. In order to consider the measurability of the multifunction F with respect to the Wijsman topology as a measurability of the function $dist(y, F(\cdot))$, we have to add an assumption that Y is a separable space.

Now we recall the Scorza Dragoni theorem proved by A. Kucia (see e.g. [11], Theorem 1, statement (e), p. 198).

Theorem 2.4. *Let* T *be a topological space,* $\mathcal{M}$ *a* $\sigma-$*algebra on* T*, and* μ *a regular measure on* $\mathcal{M}$*. Assume that* X *is second-countable topological space,*

(Y,d) is a metric space and $(T,\mathcal{M};X)$ has the projection property. Let F be a multifunction from $T\times X$ to $CL(Y)$ that is $\mathcal{M}\otimes\mathcal{B}(X)-$ measurable. Assume that for all $t\in T$, the multifunction $F(t,\cdot)$ is continuous with respect to the Wijsman topology. Then for each $\varepsilon>0$, there exists a closed set $K\subset T$ such that $\mu(T\backslash K)<\varepsilon$, and the restriction $f\,|_{K\times X}$ is continuous with respect to the Wijsman topology.

Before formulating a suitable extension theorem, we introduce a result of W. Kubiś (see [10]). Following Kubiś' approach, we say that a metrizable space X is an *absolute retract* if it is a retract of a normed linear space.

Theorem 2.5. *If $(Y,\|\cdot\|)$ is a separable Banach space, then $(CL(Y),t_{W_d})$ is an absolute retract.*

By applying Kubiś' result, we easily obtain the needed extension theorem.

Corollary 2.6. *If $(Y,\|\cdot\|)$ is a separable Banach space, then $(CL(Y),t_{W_d})$ has the extension property; i.e. if A is a closed subset of a metrizable space T and $f_A : A\to CL(Y)$ is a continuous function, then there exists a continuous function $f : T\to CL(Y)$ such that $f\,|_A = f_A$.*

We can give now the result concerning Carathéodory multifunctions.

Theorem 2.7. *Let T be a metrizable topological space, $\mathcal{M}$ a $\sigma-$algebra on T, and μ a regular measure on $\mathcal{M}$. Assume that X is a metrizable, separable topological space and $(T,\mathcal{M};X)$ has the projection property. Let Y be a separable Banach space and let us consider the hyperspace $CL(Y)$ with the Wijsman topology. Let $F : T\times X\to CL(Y)$ be a Carathéodory multifunction. Then there exist a set $S\subset T$, $\mu(S)=0$ and a sequence of continuous multifunctions $F_n : T\times X\to CL(Y)$ such that for every $t\in T\backslash S$, $\{F_n(t,\cdot)\}_{n\in I\!N}$ converges to $F(t,\cdot)$ in the discrete way.*

PROOF. The proof is analogous to the one of Theorem 2.3. The thesis follows from Theorem 2.4 and Corollary 2.6. □

The last result based on Scorza Dragoni and extension theorems is obtained for multifunctions taking values in $CLB(Y)$, the space of all nonempty bounded closed subsets of the metric space (Y,d). We consider $CLB(Y)$ with the topology induced by the Hausdorff distance d_H:

$$d_H(A,B) = max\{e(A,B), e(B,A)\},$$

where e is an excess given by $e(A,B)=\sup_{a\in A} dist(a,B)$ for every $A,B\in CLB(Y)$. The topology induced by the Hausdorff distance is stronger than the Wijsman topology on $CLB(Y)$.

Now let us specify the continuity and the measurability of a multifunction F from T to $CLB(Y)$, where T is a measurable and a topological space. The continuity is meant with respect to the topology induced by Hausdorff distance. We say that a multifunction F is *h-measurable* if for every set $C \in CLB\,(Y)$, the functions $t \to e(F(t), C)$ and $t \to e(C, F(t))$ are measurable on T (see [6], Definition 1, p. 3).

The following Scorza Dragoni type theorem is due to F. De Blasi and G. Pianigiani (see [6], Theorem 2, p. 12).

Theorem 2.8. *Let T be a complete separable metric space. Let μ be a non-negative finite measure on the completion $\mathcal{B}_T^*$ of the Borel $\sigma-$algebra $\mathcal{B}_T$. Let X be a complete separable metric space with Borel $\sigma-$algebra $\mathcal{B}_X$, and let Y be a metric space. Suppose that $F : T \times X \to CLB(Y)$ is a multifunction such that:*

(a) *F is $\mathcal{B}_T^* \otimes \mathcal{B}_X$ h-measurable;*

(b) *for each $t \in T, x \to F(t, x)$ is continuous;*

(c) *$F\,(T \times X)$ is a separable subset of $CLB(Y)$.*

Then for each $\varepsilon > 0$, there exists a compact set $K \subset T$ such that $\mu(T \backslash K) < \varepsilon$, and the restriction $F\,\big|_{(K \times X)}$ is continuous.

We find the extension theorem in a paper by H. A. Antosiewicz and A. Cellina, (see also [1], Theorem 1, p.107])

Theorem 2.9. *Let T be a metric space with distance d, let Y be a normed space, and let $CLB(Y)$ be the metric space of non-empty bounded closed subsets of Y with Hausdorff distance d_H. Given any non-empty closed set $A \subset T$ and any continuous mapping $F : A \to CLB(Y)$, there exists a continuous mapping $G : T \to CLB(Y)$ such that $G\,(a) = F\,(a)$ for every $a \in A$.*

We now formulate the theorem for Carathéodory multifunctions, which take values in $CLB(Y)$.

Theorem 2.10. *Let T be a complete separable metric space. Let μ be a non-negative finite measure on the completion $\mathcal{B}_T^*$ of the Borel $\sigma-$algebra $\mathcal{B}_T$. Let X be a complete separable metric space with Borel $\sigma-$algebra $\mathcal{B}_X$, and let Y be a normed space. Suppose that $F : T \times X \to CLB(Y)$ is a multifunction h-measurable in t, continuous in x such that $F\,(T \times X)$ is a separable subset of $CLB(Y)$. Then there exists a set $S \subset T$ with $\mu(S) = 0$ and a sequence of continuous multifunctions $F_n : T \times X \to CLB(Y)$ such that for every $t \in T \backslash S$, $\{F_n(t, \cdot)\}_{n \in I\!N}$ converges to $F(t, \cdot)$ in the discrete way.*

Proof. The scheme of the proof is similar to the ones of Theorems 2.3 and 2.7. We use the Scrorza Dragoni type theorem proved by F. De Blasi and G. Pianigiani and the extension theorem for multifunctions proved by H. A. Antosiewicz and A. Cellina. □

3 The Approach Based on Fréchet Type Theorem

In this section, we establish one result concerning the approximation of Carathéodory multifunctions, obtained by using a Fréchet type theorem. One of the characteristics of this approach is that in the thesis we get uniform convergence on compact sets. This is a consequence of the fact that in the space $C(X,Y)$ of continuous functions $f : X \to Y$, we deal with the topology of uniform convergence on compact sets. If X is a locally compact, second countable space and (Y,d) is a metric space, then the topology of $C(X,Y)$ is metrizable by the metric

$$\rho\left(x,y\right)=\sum_{n=1}^{\infty}2^{-n}k\left(\sup_{u\in U_n}d\left(x\left(u\right),y\left(u\right)\right)\right),$$

where $k\left(t\right)=\frac{t}{1+t}$ and U_n is a sequence of compact subsets such that $X = \bigcup_{n=1}^{\infty} U_n$ and for every n: $U_n \subset \operatorname{int} U_{n+1}$.
Now we formulate two lemmas. The first one is a characterization of Carathéodory functions. In a paper by J. Appell and M. Văth we can find an analogous result proved in the context of Bochner measurability (see [2], Theorem 1, p. 40). In our approach, the measurability is defined by the fact that preimages of open sets are measurable.

Lemma 3.1. *Let T be a measurable space. Assume that X is a locally compact, second countable space and that (Y,d) is a separable, metric space. Then the following conditions are equivalent:*

(i) *$f : T\times X \to Y$ is a Carathéodory function,*

(ii) *$F : T \to C(X,Y)$ defined by $F(t)(\cdot) = f(t,\cdot)$ is a measurable function.*

Proof. (i) $\Longrightarrow$ (ii) Let us consider the space $C(X,Y)$, which is metrizable by a metric ρ. The space $C(X,Y)$ is also separable because Y is separable. Now, in order to prove the measurability of F, it is enough to show that for every ball $B\left(g,r\right)\subset C\left(X,Y\right)$ the preimage $F^{-1}\left(B\left(g,r\right)\right)$ is measurable. We can write the set $F^{-1}\left(B\left(g,r\right)\right)$ in the form:

$$F^{-1}\left(B\left(g,r\right)\right)=\left\{t\in T: F\left(t\right)\in B\left(g,r\right)\right\}$$

$$=\left\{t\in T:\sum_{n=1}^{\infty}2^{-n}k\left(\sup_{u\in U_n}d\left(F\left(t\right)\left(u\right),g\left(u\right)\right)\right)<r\right\}.$$

First let us consider for every fixed u the function $h_1 : t \to d\left(F\left(t\right)\left(u\right), g\left(u\right)\right)$. The function h_1 is measurable because it is a composite of two functions: the inner one $t \to F\left(t\right)\left(u\right)$, which is equal to $t \to f\left(t, u\right)$, is measurable from assumption (i) and the outer one $d\left(\cdot, g\left(u\right)\right)$ which is continuous. Now let us look at the function $v_n = \sup_{u \in U_n} d\left(F\left(t\right)\left(u\right), g\left(u\right)\right)$. The space X is separable, so there exists a countable and dense subset $E \subset X$. The function v_n is measurable because $v_n = \sup_{u \in U_n \cap E} d\left(F\left(t\right)\left(u\right), g\left(u\right)\right)$. Then $2^{-n} k\left(v_n\left(t\right)\right)$ is measurable because $2^{-n} k\left(\cdot\right)$ is continuous. In order to prove that $\rho\left(F\left(t\right), g\right) = \sum_{n=1}^{\infty} 2^{-n} k\left(v_n\left(t\right)\right)$ is a measurable function of t, it is enough to notice that the series of measurable functions converges to a measurable function. It gives us that $F^{-1}\left(B\left(g, r\right)\right) = \left\{t \in T : \rho\left(F\left(t\right), g\right) < r\right\}$ is measurable.

(ii) $\Longrightarrow$ (i) The continuity of the function $f\left(t, \cdot\right)$ is obvious. In order to prove the measurability of the function $h_u : t \to f\left(t, u\right)$, for every fixed u, let us consider the function $w_u : C\left(X, Y\right) \ni f \to f\left(u\right) \in Y$. Then $h_u\left(t\right) = w_u\left(F\left(t\right)\right)$ is measurable because it is the composition of the measurable function F and the continuous function w_u. $\square$

The other lemma, having a straightforward proof, is the following.

Lemma 3.2. *Let us assume that T is a topological space, X is a locally compact, second countable space and (Y, d) is a metric space. If $G : T \to C(X, Y)$ is a continuous function, then $g : T \times X \to Y$ defined as $g\left(t, x\right) = G\left(t\right)\left(x\right)$ is continuous too.*

Before giving the Fréchet theorem, we introduce some necessary definitions. A topological space Y is called *almost arcwise connected,* if there exists a dense set $D \subset Y$ such that for every two points $a, b \in D$, there exists a continuous map $\gamma : [0, 1] \to Y$ with $\gamma\left(0\right) = a$ and $\gamma\left(1\right) = b$. A map $f : T \to Y$ is *almost separably-valued* if there exists a set $M \in \mathcal{M}$ of measure zero such that $f\left(T \backslash M\right)$ is separable.
Now we give a Fréchet type theorem proved by B. Kubiś and W. Kubiś (see [9], Theorem 2, p. 167).

Theorem 3.3. *Let T be a normal space with a regular $\sigma-$finite measure and let Y be an almost arcwise connected metrizable space. Assume that $f : T \to Y$ is a measurable almost separably-valued map. Then there exists a sequence of continuous maps $f_n : T \to Y$ almost everywhere pointwise convergent to f.*

We can now formulate and prove the next theorem concerning the Carathéodory multifunctions taking values in the space $K\left(Y\right)$ of all nonempty compact subsets of the normed space Y with the topology induced by the Hausdorff distance d_H. It is known that the measurability of compact-valued multifunction from T to Y is equivalent to the measurability of a function from T to

$K(Y)$ endowed with the Hausdorff metric. Thanks to this fact, we can apply lemma 3.1 in the proof of the following theorem.

Theorem 3.4. *Let T be a normal space with a regular $\sigma-$finite measure. Assume that X is a metric, locally compact, second countable space and that Y is a normed, separable space. Let $F : T \times X \to K(Y)$ be a Carathéodory multifunction. Then there exists a set $S \in \mathcal{M}$, $\mu(S) = 0$ and a sequence of continuous multifunctions $F_n : T \times X \to K(Y)$ such that for every $t \in T \backslash S$, $\{F_n(t, \cdot)\}_{n\in \mathbb{N}}$ converges uniformly on the compact sets to $F(t, \cdot)$.*

Proof. The space $K(Y)$ of all nonempty compact subsets of Y is arcwise connected. We define an arc γ by $\gamma(\lambda)(x) = (1-\lambda) A + \lambda B$ for $\lambda \in [0, 1]$ and $A, B \in K(Y)$. The continuity of γ follows from the estimation:

$$d_H(\gamma(\lambda), \gamma(\delta)) < M|\lambda - \delta|,$$

where $M = sup_{a\in A, b\in B}\|a - b\|$ (cf. [9], Proposition 7, p.169). It is also well known that $K(Y)$ with the topology induced by the Hausdorff distance is a separable space. Let us consider the function $G : T \to C(X, K(Y))$ given by the formula $G(t)(x) = F(t, x)$. By applying Lemma 3.1 we obtain the measurability of G. The space $C(X, K(Y))$ is metrizable and separable (see also [2], Lemma 1, p. 40). In order to prove that $C(X, K(Y))$ is arcwise connected, let us define the arc Γ for two multifunctions $f, g \in C(X, K(Y))$ by the formula

$$\Gamma(\lambda)(x) = (1-\lambda) f(x) + \lambda g(x), \lambda \in [0, 1].$$

We can easily prove that Γ is well defined, that is for every $\lambda : \Gamma(\lambda) \in C(X, K(Y))$, and that Γ is continuous, by using the Hausdorff distance properties. From all previous considerations, we can deduce that the assumptions of the Fréchet type theorem 3.3 are satisfied. Hence, we obtain a sequence of continuous functions $G_n : T \to C(X, K(Y))$, almost everywhere pointwise convergent to G, which means that there exists a set $S \subset T$, with $\mu(S) = 0$, such that for every $t \in T\backslash S$, $G_n(t)(\cdot)$ converges to $G(t)(\cdot)$ in the topology of $C(X, K(Y))$. By using Lemma 3.2 for every function G_n, we obtain a sequence of continuous multifunctions $F_n : T \times X \to K(Y)$ such that $F_n(t, x) = G_n(t)(x)$. In this way we obtained a sequence of continuous multifunctions F_n, such that for almost all $t \in T$, $\{F_n(t, \cdot)\}_{n\in \mathbb{N}}$ converges to $F(t, \cdot)$ uniformly on the compact sets. $\square$

References

[1] H. A. Antosiewicz and A. Cellina, *Continuous extensions of multifunctions*, Ann. Pol. Math., **34** (1977), 107–111.

[2] J. Appell and M. Văth, *The space of Carathéodory functions*, Trudy Inst. Mat. Akad. Nauk Belarusi, **2** (1999), 39–43.

[3] G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer Academic Publishers, 1993.

[4] A. Cellina and S. Solimini, *Continuous extension of selections*, Bull. Pol. Ac. Math., **35** (1987), 573–581.

[5] Á. Császár and M. Laczkovicz, *Discrete and equal convergence*, Studia Sci. Math. Hung., **10** (1975), 463–472.

[6] F. De Blasi and G. Pianigiani, *Hausdorff measurable multifunctions*, J. Math. Anal. Appl., **228** (1998), 1–15.

[7] J. Dugundji, *An extension of Tietze's Theorem*, Pacific J. Math., **1** (1951), 353–369.

[8] J. Dugundji and A. Granas, *Fixed Point Theory*, Vol. I, PWN - Polish Scientific Publishers, 1982.

[9] B. Kubiś and W. Kubiś, *Fréchet type theorem and its applications to multifunctions*, Bull. Pol. Ac. Math., **48** (2000), 165–170.

[10] W. Kubiś, *Completely metrizable absolute neighborhood retracts*, Topology Appl. (to appear).

[11] A. Kucia, *Scorza Dragoni type theorems*, Fund. Math., **138** (1991), 197–203.

[12] A. Nowak, *A note on the Fréchet theorem*, Ann. Math. Sil., **9** (1995), 43–45.