ON THE DARBOUX PROPERTY

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A function $f(x)$ with a finite real value for each x in the closed interval (a, b) is said to have the Darboux property if $f(x)$ assumes on every sub-interval (c, d) all values between $f(c)$ and $f(d)$. This note discusses *local* conditions which are necessary and sufficient in order that f have the Darboux property (and corresponding conditions for a generalization of the Darboux property).

For each x in (a, b) let $I_r(x)$ denote the open interval with end points

$$f^r(x)=\limsup\{f(t)\,;\,t\geqslant x,\ t\to x\} \text{ and } f_r(x)=\liminf\{f(t)\,;\,t\geqslant x,\ t\to x\};$$

let $I_l(x)$, $f^l(x)$, $f_l(x)$ be defined similarly, using $t\leqslant x$, $t\to x$. Let $\mathcal{N}$ be any family of N-sets with the properties:

(a) Whenever an open interval is an N-set, its closure is also an N-set.

(b) Every subset of an N-set is an N-set.

(c) The union of a countable number of N-sets is an N-set.

We shall say that f is $\mathcal{N}$-Darboux on (a, b) if $f(x)$ assumes on every sub-interval (c, d) all values between $f(c)$ and $f(d)$ with the exception of an N-set. We shall say that f is $\mathcal{N}$-Darboux at x if for every $h>0$:

(i) the values assumed by $f(t)$ for $x<t<x+h$ include all of $I_r(x)$ with the exception of an N-set;

(ii) the values assumed by $f(t)$ for $x-h<t<x$ include all of $I_l(x)$ with the exception of an N-set, (i) to be omitted when $x=b$, (ii) to be omitted when $x=a$.

We shall prove the theorem:

THEOREM. *f is $\mathcal{N}$-Darboux on (a, b) if and only if f is $\mathcal{N}$-Darboux at every x in the closed interval (a, b).*

The theorem was suggested by a paper by Akos Csaszar [1] who established the theorem for the two special cases: Case 1: the only N-set is the empty set, giving the usual Darboux property; and Case 2: (iii) also holds, every set consisting of a single point is an N-set.

We use the following modification of a lemma of Csaszar:

LEMMA. *If E is not an N-set then E contains a point y_0 such that IE fails to be an N-set for every open interval I containing y_0, and I*

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fails to be an N-set for every open interval I which has y_0 as one of its end points.

To prove the lemma let E_1 be the set of x in E for which $I(x)E$ is an N-set for some open interval $I(x)$ containing x, let E_2 be the set of x in $E-E_1$ such that x is the right end point of some open interval $J(x)$ which is an N-set and let E_3 be the set of x in $E-E_1$ such that x is the left end point of some open interval which is an N-set. Then

$$\begin{aligned} E_1 &= E_1 \sum \{I(x), \text{ all } x \text{ in } E_1\} \\ &= E_1 \sum \{I(x_n\}, \text{ for a suitable sequence of } x_n\} \\ &= \sum (E_1 I(x_n)) = \text{union of a countable collection of } N\text{-sets.} \end{aligned}$$

By (c), E_1 is an N-set. Since the $J(y)$ are clearly disjoint for different y in E_2, they form a countable collection; the closure of $J(y)$ includes y and is an N-set because of (a); it follows that E_2 and similarly E_3, are N-sets. Hence $E_1+E_2+E_3$ is an N-set, thus not identical with E which must therefore contain some y_0 not in $E_1+E_2+E_3$. This proves the lemma.

To prove the theorem, we note that the 'only if' part is an easy consequence of (b) and (c). To prove the 'if' part it is sufficient to assume that the set E of real numbers which lie between $f(a)$ and $f(b)$ but are not assumed by $f(t)$ is *not* an N-set, that y_0 is a point of E as described in the preceding lemma and obtain a contradiction. For this purpose we shall prove:

(*) *For every sub-interval (a_1, b_1) of (a, b) with y_0 between $f(a_1)$ and $f(b_1)$ and for every $m>0$ there is a sub-interval (a_2, b_2) of (a_1, b_1) such that y_0 is between $f(a_2)$ and $f(b_2)$ and*

$$|f(t)-y_0|<1/m \text{ for all } a_2 \leqslant t \leqslant b_2 .$$

Successive application of (*) with $m\to\infty$ will give a nested sequence of closed intervals such that at any of their common points $f(t)-y_0=0$, a contradiction since y_0 is in E, the set of omitted values.

Thus we need only prove (*). Since y_0 is in E, we have $f(x)\neq y_0$ for all x. It is easily seen that if $f(x)>y_0$ then $f_r(y)\geqslant y_0$ and $f_l(x)\geqslant y_0$ (because of the particular properties of y_0) and hence x lies in some open interval $I(x)$ on which $f(t)-y_0>-1/m$. Similarly if $f(x)<y_0$ then x lies in some open interval $J(x)$ on which $f(t)-y_0<1/m$. By the Heine-Borel theorem, a finite number of $I(x)$ and $J(x)$ cover (a_1, b_1) and hence it follows that some $I(x_1)$ and some $J(x_2)$ must contain a common open interval (u, v) say. We may suppose $x_1<u<v<x_2$. If y_0 is between $f(u)$ and $f(v)$ we can choose (u, v) to be the (a_2, b_2) required by (*). Otherwise we may suppose $f(u)>y_0$, $f(x_2)<y_0$. Let a_2 be sup t with $f(x)>y_0$ on $u\geqslant x>t$. Then $f(a_2)<y_0$ is impossible; for if $f(a_2)<y_0$ held, the open interval $(f(a_2), y_0)$ would be contained in $I_l(a_2)$ and yet

omitted from the values of f on (u, a_2), implying that $(f(a_2), y_0)$ is an N-set and thus contradicting the particular properties of y_0. Thus $f(a_2)>y_0$ and $u\leqslant a_2<x_2$. It now follows easily that $f_r(a_2)=y_0$ and that a_2 is the limit of a sequence of t_n with $t_n>a_2$ and $f(t_n)<y_0$. Hence, for sufficiently large n, t_n may be selected as b_2 to give (a_2, b_2) with the properties required by (*).

The example $f(x)=x$ for $x<0$ and $f(x)=1$ for $x\geqslant 0$ with the open subsets of (0, 1) as the class $\mathcal{N}$ shows that the condition (a) cannot be omitted.

References

1. Akos Csaszar, *Sur la propriété de Darboux,* C.R. Premier Congres des Mathematiciens Hongrois, Akademiai Kiado, Budapest, (1952), 551-560.

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